Some generalisations and extensions of a remarkable geometry puzzle

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1. *Introduction*

There is a very interesting mathematical puzzle involving the geometrical configuration in the book *Mathematical Curiosities* [1, 2] by Alfred Posamentier and Ingmar Lehmann. It is shown in Figure 1.

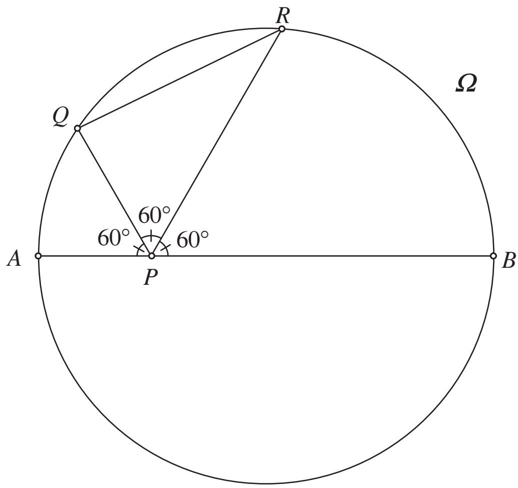

FIGURE 1: A geometric puzzle by Alfred Posamentier and Ingmar Lehmann

Theorem 1 (A geometric puzzle by Alfred Posamentier and Ingmar Lehmann): Let AB be a fixed diameter of a fixed circle Ω. Point P lies on the segment AB; points Q and R lie on the semicircle such that

$$\angle APQ = \angle QPR = \angle RPB = 60^\circ.$$

Then the length of the segment QR is a constant when P, Q and R change. (See Figure 1.)

There are numerous proofs of this nice theorem in [2]. In this paper, we introduce some generalisations and extensions for the theorem. In Theorem 2, we show that angle 60° may be replaced by any angle, in Theorem 3 that the diameter AB may be replaced by two diameters, and in Theorem 4 that these two diameters may be replaced by two chords of equal length. Theorem 5 extends Theorem 2. The proofs are given in the next section.

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2. *General theorems and proofs*

Theorem 2 (A generalisation of Theorem 1 with constant angle): Let AB be a fixed diameter of a fixed circle Ω. Point P lies on the segment AB; points Q and R lie on the semicircle such that $\angle APQ = \angle RPB = \alpha$, with α being a constant acute angle. Then the length of the segment QR is a constant when P, Q and R change.

Lemma 1: Let ABC be a triangle. Then the external bisector of $\angle BAC$ and the perpendicular bisector of BC meet on the circumcircle of triangle ABC.

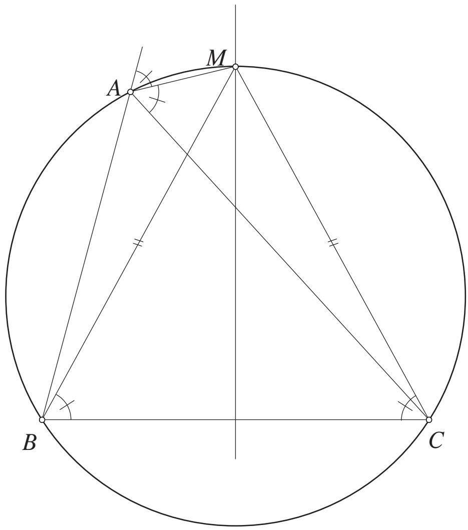

FIGURE 2: Proof of Lemma 1

Proof of Lemma 1: (See Figure 2).

Let M be the second intersection of the external bisector of $\angle BAC$ with circumcircle of triangle ABC. Since AM is the external bisector of $\angle BAC$, M is the midpoint of arc BC containing A so $\angle MBC = \angle MCB$, hence $MB = MC$ and therefore M lies on the perpendicular bisector of BC. In other words, M also lies on the perpendicular bisector of BC. So M is the intersection of the external bisector of $\angle BAC$ and the perpendicular bisector BC. Since two lines intersect at only one point, M is unique. Thus the intersection of the external bisector of $\angle BAC$ and the perpendicular bisector of BC is obviously M lying on the circumcircle of ABC.

Continuing with the above Lemma, we introduce a simple proof for Theorem 2:

Proof of Theorem 2: (See Figure 3.) Let O be the centre of Ω. From the assumption of the Theorem, PO is the external bisector of $\angle QPR$. Also, $OQ = OR$ (because Q and R lie on circle Ω). Therefore O is the intersection of the external bisector of $\angle QPR$ and the perpendicular bisector of QR.

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Using Lemma 1, O lies on circumcircle of triangle PQR. Thus

$$\angle QOR = \angle QPR = 180^\circ - 2\alpha$$

which is a constant angle. Since QR is a chord of Ω and $\angle QOR$ is constant, the length of the segment QR must be invariant. This completes the proof.

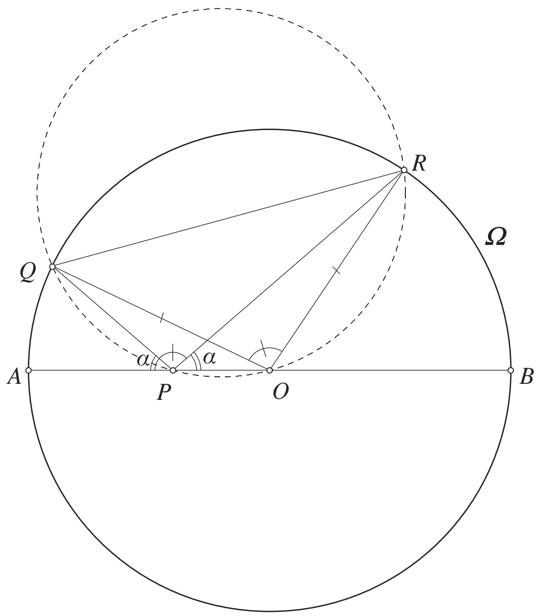

FIGURE 3: Proof of Theorem 2

Theorem 3 (The first further generalisation of Theorem 1): Let AB and CD be two fixed diameters of a fixed circle Ω. Points M and N lie on the segment AB and CD, respectively; points Q and R lie on the minor arc AD such that $MN \,//\, AC$ and $\angle AMQ = \angle RND = \alpha$, with α a constant acute angle. Then the length of the segment QR is a constant when the points M, N, Q and R change.

Proof of Theorem 3: (See Figure 4.) Let O be the centre of Ω. Since $MN \,//\, AC$, triangle OMN isosceles at O. This means

$$\angle OMN = \angle ONM. \tag{1}$$

Let P be the point where the line QM extended meets RN. Also from the assumption, we get

$$\angle PMO = \angle AMQ = \angle DNR = \alpha. \tag{2}$$

From (1) and (2), we deduce that

$$\angle PMN = \angle PNM. \tag{3}$$

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From this equality of angle, OP is the perpendicular bisector of MN and also the perpendicular bisector of AC (because $MN \parallel AC$). Let OP meet Ω at EF then EF is a fixed diameter of Ω. Since $MN \perp EF$, we deduce that

$$\angle QPE = 90^\circ - \angle PMN = 90^\circ - \angle PNM = \angle RPF. \qquad (4)$$

Using (4) and Theorem 2, we have the length of the segment QR is a constant. This completes the proof.

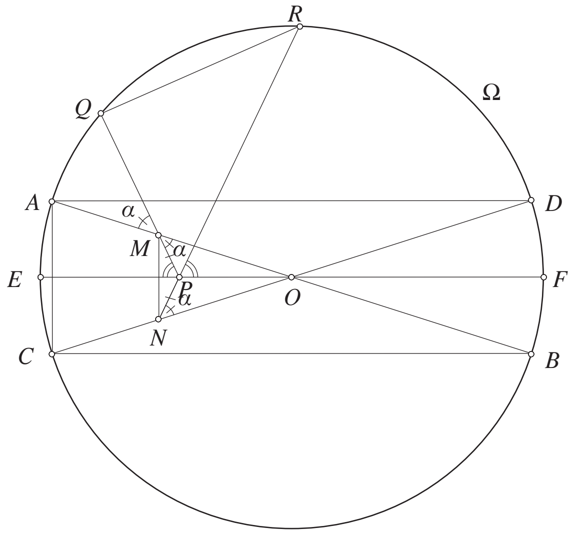

FIGURE 4: Proof of Theorem 3

Theorem 4 (The second further generalisation of Theorem 1): Let AB be a fixed chord of a fixed circle Ω. A chord $A'B'$ changes on the major arc AB (of Ω) such that $A'B' = AB$ and the intersection P of AA' and BB' lies inside Ω. Points Q and R lie on the minor arc AB such that $\angle APQ = \angle BPR = \alpha$, with α being a constant acute angle. Then the length of the segment QR is a constant when the chord $A'B'$ changes.

Proof of Theorem 4: (See Figure 5.) Let O be the centre of Ω. Since AB and $A'B'$ are equal chords and P lies inside Ω, AB' is parallel to BA'. This implies that PO is bisector of $\angle BPA'$ or PO is the external bisector of $\angle APB$. Combining with $\angle APQ = \angle BPR = \alpha$, we deduce that PO is the external bisector of $\angle QPR$. Also $OQ = OR$ because O is the centre of Ω. Hence, using Lemma 1, the four points Q, R, O and P are concyclic.

Hence PO is the external bisector of $\angle APB$ and $OA = OB$. By Lemma 1, we also get that the four points A, B, O and P are concyclic. This means that $\angle APB = \angle AOB$.

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From four concyclic points Q, R, O and P (as above), we have

$$\angle QOR = \angle QPR = \angle APB - \angle APQ - \angle BPR = \angle AOB - 2\alpha \quad (5)$$

which is a constant angle. This means the length of the segment QR is a constant. This completes proof.

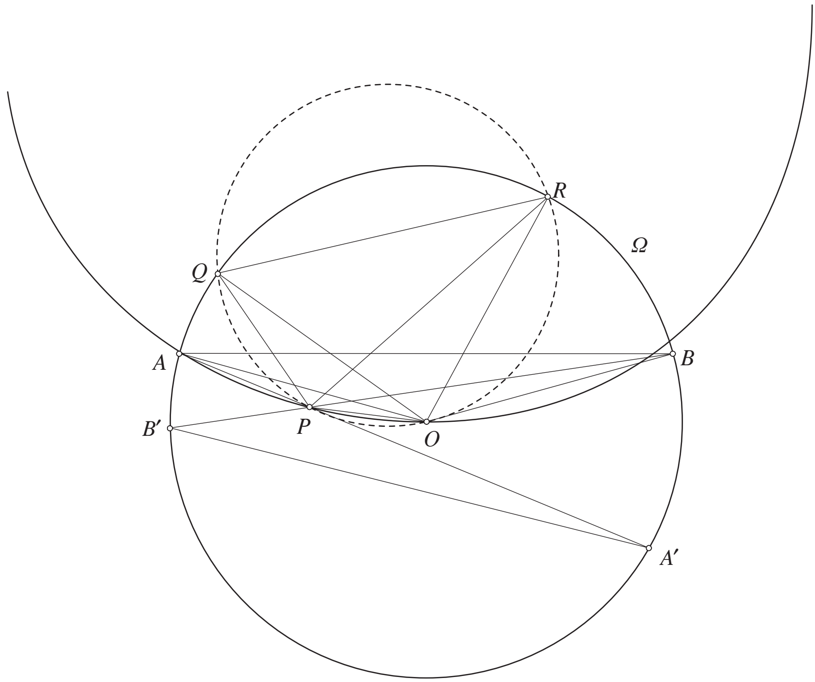

FIGURE 5: Proof of Theorem 4

Theorem 5 (An extension of Theorem 2): Let AB be a diameter of a circle Ω. Point P lies on the segment AB; points Q and R lie on the semicircle such that $\angle APQ = \angle RPB < 90°$. Then the Euler lines (see [3]) of triangles PAQ and PBR intersect on the circumcircle of triangle PQR.

Lemma 2 (Thébault's problem [4]): Let $A'B'C'$ be the orthic triangle of ABC. Then the Euler lines of the triangles $AB'C'$, $BC'A'$ and $CA'B'$ are concurrent at a point lying on the nine-point circle of triangle ABC.

For proof, see [5]. The concurrency point is known as the centre of the Jerabek hyperbola $X(125)$, see [3].

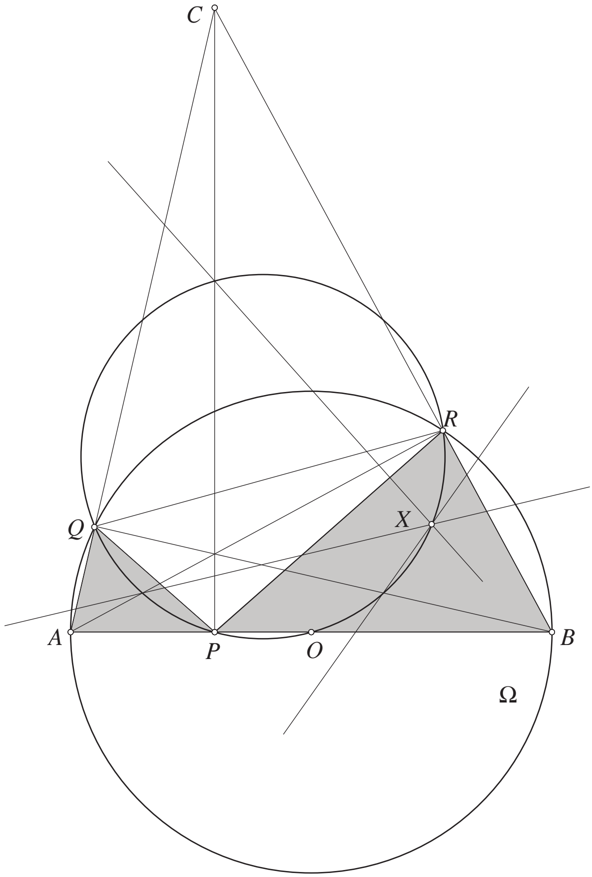

FIGURE 6: Proof of Theorem 5

Proof of Theorem 5: (See Figure 6.) Let O be the centre of Ω. Let C be the intersection of the two lines AQ and BR. Since AB is the diameter of circle Ω, $\angle AQB = \angle ARB = 90°$. This implies that AR and BQ are the altitudes of triangle CAB. Because O is the midpoint of AB, the circumcircle of triangle OQR must be the nine-point circle of triangle CAB. But from the proof of Theorem 2, the four points O, P, Q and R are concyclic. This means P is the second intersection of the circumcircle of triangle OQR with the line AB, in other words, P is the foot of altitude from C to the line AB. Thus triangle PQR is the orthic triangle of the triangle CBA. It follows from Lemma 2, that the Euler lines of triangles PQA and PRB meet on the nine-point circle of triangle CAB which is also the circumcircle of triangle PQR. This completes our proof.

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