J. Aust. Math. Soc. **116** (2024), 96–127
doi:10.1017/S1446788723000083

GHOSTS AND CONGRUENCES FOR p^s-APPROXIMATIONS OF HYPERGEOMETRIC PERIODS

ALEXANDER VARCHENKO and WADIM ZUDILIN

(Received 28 June 2022; accepted 2 July 2023; first published online 2 August 2023)

Communicated by Michael Coons

Abstract

We prove general Dwork-type congruences for constant terms attached to tuples of Laurent polynomials. We apply this result to establishing arithmetic and p-adic analytic properties of functions originating from polynomial solutions modulo p^s of hypergeometric and Knizhnik–Zamolodchikov (KZ) equations, solutions which come as coefficients of master polynomials and whose coefficients are integers. As an application, we show that the simplest example of a p-adic KZ connection has an invariant line subbundle while its complex analog has no nontrivial subbundles due to the irreducibility of its monodromy representation.

2020 *Mathematics subject classification*: primary 11D79; secondary 12H25, 32G34, 33C05, 33E30.

Keywords and phrases: hypergeometric equation, KZ equations, Dwork congruences, master polynomials, p^s-approximation polynomials.

1. Introduction

In the seminal work [Dw], Dwork laid the foundation of the theory of p-adic hypergeometric differential equations. His principal working example was the differential equation

$$x(1-x)I'' + (1-2x)I' - \tfrac{1}{4}I = 0, \tag{1-1}$$

whose analytic solution at the origin

$${}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; x\Big) = \frac{1}{\pi}\int_1^\infty t^{-1/2}(t-1)^{-1/2}(t-x)^{-1/2}\,dt = \sum_{k=0}^\infty \binom{-1/2}{k}^2 x^k \tag{1-2}$$

encodes periods of the Legendre family $y^2 = t(t-1)(t-x)$. Dwork used the approximations

© The Author(s), 2023. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

$$F_s(x) = \sum_{k=0}^{p^s-1} \binom{-1/2}{k}^2 x^k \quad \text{for } s = 1, 2, \dots, \tag{1-3}$$

which are nothing but truncations of the infinite sum in Equation (1-2) and clearly converge to it in the disk $D_{0,1} = \{x \mid |x|_p < 1\}$, to show that the uniform limit $F_{s+1}(x)/F_s(x^p)$ as $s \to \infty$ exists in a larger domain $\mathfrak{D}^{\mathrm{Dw}}$ and this limit, named the 'unit root', corresponds to a root of the local zeta function of the x-fiber in the family. Dwork's work boosted the whole body of research in the area; we limit ourselves to mentioning some recent contributions on the theme [AS, BV, LTYZ].

Dwork indicates in [Dw] that in the p-adic case, he adopts a point of view similar to that of Igusa in [Ig] on the modulo p solutions of Equation (1-1). Namely, the cycles of the elliptic curve $y^2 = t(t-1)(t-x)$ for a given x can be thought of as the local at x analytic solutions of the differential equation in Equation (1-1). At the same time, Igusa's modulo p solution

$$g(x) = \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k}^2 x^k \tag{1-4}$$

of Equation (1-1), though indeed coinciding with Dwork's $F_1(x)$ modulo p, hints at a somewhat different way for approximating the function in Equation (1-2) p-adically through different truncations of the infinite sum in Equation (1-2). Notice the difference of the limits of summations in Equations (1-3) and (1-4).

This recipe seemed to escape its own development until recently, when Schechtman and Varchenko constructed in [SV2] polynomial solutions modulo p to general Knizhnik–Zamolodchikov (KZ) equations, recovering Igusa's polynomial as a particular case.

The principal goal of this paper is to show that the p-approximation approach in [SV2] goes in parallel with that of Dwork in [Dw] and leads to Dwork-type congruences for solutions of the KZ equations, at least for the cases in which the technicality of proofs does not overshadow the beauty of outcomes.

In this paper, we study certain p^s-approximation polynomials of hypergeometric periods. We consider an integral of hypergeometric type like in Equation (1-1) without specifying the cycle of integration. For any positive integer s, we replace the integrand by a polynomial $\Phi_s(t, x)$ with integer coefficients called the master polynomial and define the p^s-approximation polynomial as the coefficient of t^{p^s-1} in the master polynomial. This is our p^s-analog of the initial integral. In the example of Equation (1-1), the master polynomial is $\Phi_s(t, x) = t^{(p^s-1)/2}(t-1)^{(p^s-1)/2}(t-x)^{(p^s-1)/2}$ and the p^s-approximation polynomial is

$$P_s(x) = (-1)^{(p^s-1)/2} \sum_{k=0}^{(p^s-1)/2} \binom{(p^s-1)/2}{k}^2 x^k. \tag{1-5}$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

We prove the Dwork-type congruence,

$$P_{s+1}(x)P_{s-1}(x^p) \equiv P_s(x)P_s(x^p) \pmod{p^s}, \tag{1-6}$$

in Theorem 4.2.

In Section 6, we consider the simplest example of the KZ connection. In our example, the KZ connection is identified with the Gauss–Manin connection of the family of elliptic curves $y^2 = (t - z_1)(t - z_2)(t - z_3)$. We study the p^s-approximation polynomials to the elliptic period $\int dt/y$ and show that the p-adic KZ connection of our example has an invariant line subbundle. This is a new p-adic feature since the corresponding complex KZ connection has no proper invariant subbundle due to the irreducibility of its monodromy representation.

Notice that usually, the invariant subbundles of the KZ connection over $\mathbb{C}$ are related to some additional conformal block constructions; for example see [FSV, SV2, V3]. Apparently, our subbundle is of a different p-adic nature.

The results above require proving p-adic convergence, which in turn rests upon establishing certain special congruences. Considering a general hypergeometric series $F(x) = \sum_{n=0}^{\infty} A(n)x^n$ and its p^s-truncations $F_s(x) = \sum_{n=0}^{p^s-1} A(n)x^n$, Dwork showed that

$$F_{s+1}(x)F_{s-1}(x^p) \equiv F_s(z)F_s(x^p) \pmod{p^s} \quad \text{for } s = 1, 2, \ldots, \tag{1-7}$$

in [Dw, Theorem 2]. This allowed him to conclude the existence of the p-adic limit $F_{s+1}(x)/F_s(x^p)$ as $s \to \infty$ in [Dw, Theorem 3]. As an auxiliary component of Dwork's argument, another set of congruences, à la Lucas, was established for the sequence of coefficients $A(n)$:

$$\frac{A(n + mp^s)}{A([n/p] + mp^{s-1})} \equiv \frac{A(n)}{A([n/p])} \pmod{p^s} \quad \text{for } m, n \in \mathbb{Z}_{\geqslant 0} \text{ and } s = 1, 2, \ldots \tag{1-8}$$

(see [Dw, Corollary 1(ii), page 36]). These two different-looking families of congruences in Equations (1-7) and (1-8) are both known as Dwork congruences, and to distinguish between the two, we name them type I and type II, respectively.

Our Igusa-inspired p^s-approximations of solutions of a hypergeometric equation, like the p^s-approximation in Equation (1-5), are of dual nature. Although congruences in Equation (1-6) look like congruences in Equation (1-7) of type I, we may view the sequence $(P_s(x))_{s\geqslant 1}$ as a subsequence of a suitable polynomial sequence $A(n; x)$ depending on the extra parameter x and satisfying the congruences

$$\frac{A(n + mp^s; x)}{A([n/p] + mp^{s-1}; x^p)} \equiv \frac{A(n; x)}{A([n/p]; x^p)} \pmod{p^s} \quad \text{for } m, n \in \mathbb{Z}_{\geqslant 0} \text{ and } s = 1, 2, \ldots. \tag{1-9}$$

Then the restriction of these type-II Dwork congruences to the subsequence $(P_s(x))_{s\geqslant 1}$ reads as type-I congruences for the p^s-approximation polynomials in parameter x. (We make this explicit in the remark after Theorem 4.4.)

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

In summary, our principal tools for establishing the existence of p-adic convergence are the Dwork-type congruences in Equation (1-9), of which the required type-I congruences are particular instances.

Initially, general theorems toward Dwork's congruences were given by Mellit [Me] and independently by Samol and van Straten [SvS]. They were generalized in [MV, Vl]. As these results are insufficiently general for us, we extend Dwork-type congruences further using the elegant method from Mellit's unpublished preprint [Me]. Our Dwork-type congruences are displayed in Section 2, and their power is illustrated by the congruences in Equation (1-6) from Theorem 4.2 and by several other quite different applications in later sections. In particular, see Section 6 for applications to the KZ equations.

In Section 7, we conjecture some stronger congruences for the polynomials $P_s(x)$.

We finish the introduction with a remark on the differential KZ equations. The differential KZ equations were discovered by physicists Knizhnik and Zamolodchikov [KZ] as differential equations satisfied by conformal blocks in conformal field theory. It was realized later that versions of KZ equations appear in different situations, for example, as quantum differential equations for Nakajima quiver varieties [MO]. The KZ equations are closely related to quantum integrable systems and the Bethe ansatz method.

The KZ equations were identified with suitable Gauss–Manin connections in [SV1], and integral representations for solutions of the KZ equations were constructed in the form of multidimensional hypergeometric integrals depending on parameters. Integral representations have connected the KZ equations with the theory of special functions and the hypergeometric functions.

In [SV2], a new side of the KZ equations was observed, namely, polynomial solutions modulo p of the KZ equations were constructed. Those are certain vectors of polynomials with integer coefficients which solve the KZ differential equations modulo p. Those polynomial solutions are, in some sense, p-approximations of the multidimensional hypergeometric solutions of the KZ equations. This brings us a general problem of studying arithmetic properties of solutions of the KZ equations with possible applications to enumerative geometry of Nakajima varieties. The present paper is a step in this direction.

2. On ghosts

In this paper, p is an odd prime.

2.1. Mellit's theorem. Let $\Lambda(t)$ be a Laurent polynomial in variables $t = (t_1, \dots, t_r)$ with coefficients in $\mathbb{Z}_p$ and constant term $\mathrm{CT}_t(\Lambda)$. Assume that the Newton polytope of $\Lambda(t)$ contains only one interior point $\{0\}$.

For a tuple $a = (a_0, a_1, \dots, a_{l-1})$, denote by $l(a) = l$ its length. For two tuples a and b, the concatenation product $a * b$ is the tuple of length $l(a) + l(b)$ obtained by gluing a and b together. For $a = (a_0, \dots, a_{l-1})$ of length l, denote by $a' = (a_1, \dots, a_{l-1})$ the 'derivative' tuple of length $l - 1$. If a is a tuple of numbers, denote $|a| = \sum_{i=0}^{l-1} a_i$.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

For a tuple $m = (m_0, \dots, m_{l-1})$ of integers from $\{1, \dots, p-1\}$, denote by $\mathrm{CT}_t(\Lambda^m)$ the constant term of the Laurent polynomial $\Lambda(t)^{m_0+m_1p+m_2p^2+\cdots+m_{l-1}p^{l-1}}$.

THEOREM 2.1 [Me]. *Let a, b, c be tuples of integers from $\{1, \dots, p-1\}$, where b, c, a' can be empty, that is, of length 0. Then,*

$$\mathrm{CT}_t(\Lambda^{a*b})\,\mathrm{CT}_t(\Lambda^{a'*c}) \equiv \mathrm{CT}_t(\Lambda^{a'*b})\,\mathrm{CT}_t(\Lambda^{a*c})\,(\mathrm{mod}\,p^{l(a)}).$$

We modify the statement and three-page Mellit's proof of Theorem 2.1 and prove Theorem 2.9.

2.2. Convex polytopes. Given a positive integer r, we consider convex polytopes, which are convex hulls of finite subsets of $\mathbb{Z}^r \subset \mathbb{R}^r$.

DEFINITION 2.2. A tuple $(N_0, N_1, \dots, N_{l-1})$ of convex polytopes is called admissible if for any $0 \leqslant i \leqslant j \leqslant l-1$,

$$(N_i + pN_{i+1} + \cdots + p^{j-i}N_j) \cap p^{j-i+1}\mathbb{Z}^r = \{0\}.$$

2.3. Definition of ghosts. Let $\Lambda(t, z)$ be a Laurent polynomial in some variables $t = (t_1, \dots, t_r)$, $z = (z_1, \dots, z_{r'})$ with coefficients in $\mathbb{Z}_p$. We define the ghost terms $R_m(\Lambda)$, $m \geqslant 0$, as the unique sequence of Laurent polynomials in t, z satisfying the following two properties.

(i) For each m,

$$\Lambda(t, z)^{p^m} = R_0(\Lambda)(t^{p^m}, z^{p^m}) + R_1(\Lambda)(t^{p^{m-1}}, z^{p^{m-1}}) + \cdots + R_m(\Lambda)(t, z).$$

(ii) For each m, the coefficients of $R_m(\Lambda)(t, z)$ are divisible by p^m in $\mathbb{Z}_p$.

Properties (i) and (ii) recursively determine the polynomials $R_m(\Lambda)(t, z)$. Namely,

$$R_m(\Lambda)(t, z) = \Lambda(t, z)^{p^m} - \Lambda(t^p, z^p)^{p^{m-1}}, \quad R_0(\Lambda)(t, z) = \Lambda(t, z).$$

Let $F(t, z)$ be a Laurent polynomial in t, z. Let $N(F)$ be the Newton polytope of $F(t, z)$ with respect to the t *variables only*. Clearly,

$$N(R_m(\Lambda)) \subset p^m N(\Lambda).$$

2.4. Composed ghosts. Let $\lambda = (\Lambda_0(t, z), \dots, \Lambda_{l-1}(t, z))$ be a tuple of Laurent polynomials with coefficients in $\mathbb{Z}_p$. We decompose the product

$$\tilde{\lambda}(t, z) := \Lambda_0(t, z)(\Lambda_1(t, z))^p \cdots (\Lambda_{l-1}(t, z))^{p^{l-1}}$$

into the sum of ghost terms of $\Lambda_0, \dots, \Lambda_{l-1}$. As the result, we obtain that $\tilde{\lambda}$ is the sum of the products

$$R_{m,\lambda}(t, z) := R_{m_0}(\Lambda_0)(t, z) \cdot R_{m_1}(\Lambda_1)(t^{p^{1-m_1}}, z^{p^{1-m_1}}) \cdot R_{m_2}(\Lambda_2)(t^{p^{2-m_2}}, z^{p^{2-m_2}}) \cdots$$
$$R_{m_{l-1}}(\Lambda_{l-1})(t^{p^{l-1-m_{l-1}}}, z^{p^{l-1-m_{l-1}}}),$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

where $m = (m_0, \dots, m_{l-1})$ runs over the set of all l-tuples of integers satisfying $0 \leqslant m_i \leqslant i$. Clearly,

$$R_{m,\lambda}(t, z) \equiv 0 \,(\mathrm{mod}\, p^{|m|})$$

and

$$N(R_{m,\lambda}(t, z)) \subset N(\Lambda_0(t, z)) + pN(\Lambda_1(t, z)) + \cdots + p^{l-1}N(\Lambda_{l-1}(t, z)).$$

2.5. Indecomposable tuples. Denote by S_k the set of all k-tuples $m = (m_0, \dots, m_{k-1})$ of integers such that $0 \leqslant m_i \leqslant i$. Put $S = \bigcup_{k=1}^{\infty} S_k$. A tuple $m \in S$ is called *indecomposable* if it cannot be presented as $m' * m''$ for $m', m'' \in S$. Denote by S_k^{ind} the set of all indecomposable k-tuples and put $S^{\mathrm{ind}} = \bigcup_{k=1}^{\infty} S_k^{\mathrm{ind}}$.

LEMMA 2.3. *If $m \in S_k^{\mathrm{ind}}$, then $|m| \geqslant k - 1$.*

PROOF. If m is indecomposable, then for each $i \in \{1, \dots, k-1\}$, there exists $j \geqslant i$ such that $m_j > j - i$, that is, $j \geqslant i > j - m_j$. The number of such i for a given j is m_j. The total number of such i is $k - 1$; therefore, the sum of m_j is at least $k - 1$. □

LEMMA 2.4. *For each $m \in S$, there exist unique indecomposable $m^1, \dots, m^r$ such that $m = m^1 * \cdots * m^r$.*

PROOF. The proof is by induction on $l(m)$. If $l(m) = 1$, then $m = (m_0) = (0)$ and m is indecomposable. Let us prove the induction step. Let

$$m = m^1 * \cdots * m^r = n^1 * \cdots * n^s$$

be two decompositions into indecomposable factors. We may assume that $l(n^s) \geqslant l(m^r)$. If $l(n^s) = l(m^r)$, then $n^s = m^r$. In this case, we can conclude that $m^1 * \cdots * m^{r-1} = n^1 * \cdots * n^{s-1}$, and the statement follows from the induction hypothesis. If $l(n^s) > l(m^r)$, then the sequence n^s contains the sequence m^r as its last $l(m^r)$-part. This contradicts to the indecomposability of n^s. The lemma is proved. □

2.6. Polynomials I_λ. For an l-tuple $\lambda = (\Lambda_0(t, z), \Lambda_1(t, z), \dots, \Lambda_{l-1}(t, z))$ of Laurent polynomials with coefficients in $\mathbb{Z}_p$, define

$$I_\lambda(t, z) = \sum_{m \in S_l^{\mathrm{ind}}} R_{m,\lambda}(t, z).$$

We have

$$I_\lambda(x, z) \equiv 0 \,(\mathrm{mod}\, p^{l-1})$$

by Lemma 2.3 and

$$N(I_\lambda(t, z)) \subset N(\Lambda_0(t, z)) + pN(\Lambda_1(t, z)) + \cdots + p^{l-1}N(\Lambda_{l-1}(t, z)).$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

LEMMA 2.5. *We have*

$$\tilde{\lambda}(t,z) = \sum_{\lambda=\lambda^1*\cdots*\lambda^s} I_{\lambda^1}(t,z) I_{\lambda^2}(t^{p^{l(\lambda^1)}}, z^{p^{l(\lambda^1)}}) \cdots I_{\lambda^s}(t^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}, z^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}),$$

where the sum is over the set of all possible decompositions of the tuple λ into a product of tuples.

PROOF. We have

$$\tilde{\lambda}(t,z) = \sum_{m\in S_l} R_{m,\lambda}(t,z).$$

For any $m \in S_l$, let $m = m^1 * \cdots * m^s$ be its unique indecomposition into indecomposable factors. Let $\lambda = \lambda^1 * \cdots * \lambda^s$ be the corresponding factorization of the sequence λ. Then,

$$R_{m,\lambda}(t,z) = R_{m^1,\lambda^1}(t,z) R_{m^2,\lambda^2}(t^{p^{l(\lambda^1)}}, z^{p^{l(\lambda^1)}}) \cdots R_{m^s,\lambda^s}(t^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}, z^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}). \quad (2\text{-}1)$$

This product contributes to the expansion of the product

$$I_{\lambda^1}(t,z) I_{\lambda^2}(t^{p^{l(\lambda^1)}}, z^{p^{l(\lambda^1)}}) \cdots I_{\lambda^s}(t^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}, z^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}) \quad (2\text{-}2)$$

into the sum, and conversely each summand in the expansion of Equation (2-2) comes from Equation (2-1) for a unique indecomposable factorization $m = m^1 * \cdots * m^s$. □

2.7. Admissible tuples of Laurent polynomials.

DEFINITION 2.6. A tuple $\lambda = (\Lambda_0(t,z), \Lambda_1(t,z), \ldots, \Lambda_{l-1}(t,z))$ of Laurent polynomials is called admissible if the tuple $(N(\Lambda_0(t,z)), N(\Lambda_1(t,z)), \ldots, N(\Lambda_{l-1}(t,z)))$ of its Newton polytopes with respect to variables t is admissible.

Denote by $\mathrm{CT}_t(\Lambda)(z)$ the constant term of the Laurent polynomial $\Lambda(t,z)$ with respect to the variables t. The constant term $\mathrm{CT}_t(\Lambda)(z)$ is a Laurent polynomial in z.

LEMMA 2.7. *Let $\lambda = (\Lambda_0(t,z), \Lambda_1(t,z), \ldots, \Lambda_{l-1}(t,z))$ be an admissible tuple of Laurent polynomials with coefficients in $\mathbb{Z}_p$ and $\lambda = \lambda^1 * \cdots * \lambda^s$. Then,*

$$\mathrm{CT}_t\Big(\prod_{i=1}^{s} I_{\lambda^i}(t^{p^{l(\lambda^1)+\cdots+l(\lambda^{i-1})}}, z^{p^{l(\lambda^1)+\cdots+l(\lambda^{i-1})}})\Big)(z) = \prod_{i=1}^{s} \mathrm{CT}_t(I_{\lambda^i}(t,z))(z^{p^{l(\lambda^1)+\cdots+l(\lambda^{i-1})}}).$$

PROOF. We have

$$N(I_{\lambda^1}(t,z)) \subset N(\Lambda_0(t,z)) + pN(\Lambda_1(t,z)) + \cdots + p^{l(\lambda^1)-1} N(\Lambda_{l(\lambda^1)-1}(t,z)).$$

Hence,

$$N(I_{\lambda^1}(t,z)) \cap p^{l(\lambda^1)}\mathbb{Z}^r = \{0\}$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

and

$$\mathrm{CT}_t\Big(\prod_{i=1}^{s} I_{\lambda^i}(t^{p^{l(\lambda^1)+\cdots+l(\lambda^{i-1})}}, z^{p^{l(\lambda^1)+\cdots+l(\lambda^{i-1})}})\Big)(z)$$
$$= \mathrm{CT}_t(I_{\lambda^1}(t,z))(z)\,\mathrm{CT}_t\Big(\prod_{i=2}^{s} I_{\lambda^i}(t^{p^{l(\lambda^2)+\cdots+l(\lambda^{i-1})}}, z^{p^{l(\lambda^2)+\cdots+l(\lambda^{i-1})}})\Big)(z^{p^{l(\lambda^1)}}).$$

Thus, by induction on s, we prove the statement. □

COROLLARY 2.8. *We have*

$$\mathrm{CT}_t(\tilde{\lambda})(z) = \sum_{\lambda=\lambda^1*\cdots*\lambda^s} \mathrm{CT}_t(I_{\lambda^1})(z)\cdot \mathrm{CT}_t(I_{\lambda^2})(z^{p^{l(\lambda^1)}})\cdots \mathrm{CT}_t(I_{\lambda^s})(z^{p^{l(\lambda^1)+\cdots+l(\lambda^{s-1})}}),$$

where the sum is over the set of all decompositions of λ into a product of tuples.

2.8. Dwork congruence for tuples of Laurent polynomials.

THEOREM 2.9. *Let a, b, c be tuples of Laurent polynomials in t, z with coefficients in $\mathbb{Z}_p$, where b, c, a' can be empty. Assume that the tuples $a * b, a * c, a' * b, a' * c$ of Laurent polynomials are admissible. Then,*

$$\mathrm{CT}_t(\widetilde{a * b})(z)\ \mathrm{CT}_t(\widetilde{a' * c})(z^p) \equiv \mathrm{CT}_t(\widetilde{a' * b})(z^p)\ \mathrm{CT}_t(\widetilde{a * c})(z) \pmod{p^{l(a)}}. \tag{2-3}$$

PROOF. The left-hand side and right-hand side of Equation (2-3) are

$$\sum_{\substack{a*b=x^1*\cdots*x^q\\ a'*c=y^1*\cdots*y^s}} \prod_{i=1}^{q} \mathrm{CT}_t(I_{x^i})(z^{p^{l(x^1)+\cdots+l(x^{i-1})}}) \prod_{i=1}^{s} \mathrm{CT}_t(I_{y^j})(z^{p^{1+l(y^1)+\cdots+l(y^{j-1})}})$$

and

$$\sum_{\substack{a'*b=x^1*\cdots*x^q\\ a*c=y^1*\cdots*y^s}} \prod_{i=1}^{q} \mathrm{CT}_t(I_{x^i})(z^{p^{1+l(x^1)+\cdots+l(x^{i-1})}}) \prod_{i=1}^{s} \mathrm{CT}_t(I_{y^j})(z^{p^{l(y^1)+\cdots+l(y^{j-1})}}), \tag{2-4}$$

respectively. Since we work modulo $p^{l(a)}$, all the terms with

$$\sum_{i=1}^{q} l(x^i) + \sum_{j=1}^{s} l(y^j) - q - s \geqslant l(a)$$

may be dropped off from consideration. That inequality can be reformulated as $l(a) + l(b) + l(a) + l(c) - 1 - q - s \geqslant l(a)$, equivalently, as

$$l(a) + l(b) + l(c) \geqslant q + s + 1. \tag{2-5}$$

Let us prove that the remaining terms in both expressions are in a bijective correspondence such that the corresponding terms are *equal*.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Namely, take one of the remaining summands on the left-hand side:

$$\prod_{i=1}^{q} \mathrm{CT}_t(I_{x^i})(z^{p^{l(x^1)+\cdots+l(x^{i-1})}}) \prod_{i=1}^{s} \mathrm{CT}_t(I_{y^i})(z^{p^{1+l(y^1)+\cdots+l(y^{i-1})}}), \tag{2-6}$$

the summand corresponding to the presentation $a*b=x^1*\cdots*x^q$, $a'*c=y^1*\cdots*y^s$.

LEMMA 2.10. *There exist indices $i \geqslant 1$ and $j \geqslant 0$ such that*

$$l(x^1)+\cdots+l(x^i) = l(y^1)+\cdots+l(y^j)+1 \leqslant l(a). \tag{2-7}$$

PROOF. If $l(x^1)=1$, then $i=1$ and $j=0$ are the required indices.

Assume that $l(x^1)>1$ and the required i, j do not exist. Then each number in $\{2,\dots,l(a)\}$ cannot be represented simultaneously as $l(x^1)+\cdots+l(x^i)$ and $l(y^1)+\cdots+l(y^j)+1$. Therefore, the sum of the total number of $i\geqslant 1$, such that $l(x^1)+\cdots+l(x^i)\leqslant l(a)$, and the total number of $j\geqslant 1$, such that $l(y^1)+\cdots+l(y^j)+1\leqslant l(a)$ is at most $l(a)-1$. The number of remaining i is at most $l(b)$ and the number of remaining j is at most $l(c)$. Therefore, $q+s\leqslant l(a)-1+l(b)+l(c)$, which is the same as Equation (2-5). Hence, the corresponding summand must have been dropped off. This establishes the existence of indices i and j required.

Now we return to the remaining summand in Equation (2-6). Choose the minimal indices $i\geqslant 1$ and $j\geqslant 0$ such that Equation (2-7) holds. Then it is easy to see that

$$a'*b = y^1*\cdots*y^j*x^{i+1}*\cdots*x^q, \quad a*c = x^1*\cdots*x^i*y^{j+1}*\cdots*x^s, \tag{2-8}$$

and the summand in Equation (2-4) corresponding to the presentations in Equation (2-8) equals the product in Equation (2-6). This clearly gives the desired bijection. □

3. p^s-Approximation of a hypergeometric integral

Let α,β,γ be rational numbers with $|\alpha|_p=|\beta|_p=|\gamma|_p=1$. Consider a hypergeometric integral

$$I^{(C)}(x) = \int_C t^\alpha(t-1)^\beta(t-x)^\gamma\,dt, \tag{3-1}$$

where $C\subset\mathbb{C}-\{0,1,x\}$ is a contour on which the integrand takes its initial value when t encircles C. The function $I^{(C)}(x)$ satisfies the hypergeometric differential equation

$$x(1-x)I''+((\alpha+\beta+2\gamma)x-(\alpha+\gamma))I'-\gamma(\alpha+\beta+\gamma+1)I=0. \tag{3-2}$$

This follows from Stokes' theorem and the following identity of differential forms. Denote $\Phi(t,x)=t^\alpha(t-1)^\beta(t-x)^\gamma$,

$$\mathcal{D} = x(1-x)\frac{d^2}{dx^2}+((\alpha+\beta+2\gamma)x-(\alpha+\gamma))\frac{d}{dx}-\gamma(\alpha+\beta+\gamma+1).$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Then,

$$d_t\Big(\gamma \frac{t(t-1)}{t-x}\Phi(t,x)\Big) = \mathcal{D}\,\Phi(t,x)\,dt. \tag{3-3}$$

The differential equation in Equation (3-2) turns into the standard hypergeometric differential equation

$$x(1-x)I'' + (c-(a+b+1)x)I' - abI = 0$$

if $\alpha = a-c$, $\beta = c-b-1$, $\gamma = -a$. If $c \notin \mathbb{Z}_{\leqslant 0}$, then for a suitable choice of C and multiplication of the integral by a constant, the integral in Equation (3-1) can be expanded as a power series

$${}_2F_1(a,b;c;x) = {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!\,(c)_k} x^k.$$

Here, $(a)_n = \Gamma(a+n)/\Gamma(a) = \prod_{k=0}^{n-1}(a+k)$ stands for Pochhammer's symbol.

We consider the following p^s-approximation of the integral in Equation (3-1). Given a positive integer s, let $1 \leqslant \alpha_s, \beta_s, \gamma_s \leqslant p^s$ be the unique positive integers such that

$$\alpha_s \equiv \alpha, \quad \beta_s \equiv \beta, \quad \gamma_s \equiv \gamma \pmod{p^s}.$$

Define the *master polynomial*

$$\Phi_s(t,x) = t^{\alpha_s}(t-1)^{\beta_s}(t-x)^{\gamma_s}$$

and the *p^s-approximation polynomial* $I_s(x)$ as the coefficient of t^{p^s-1} in the master polynomial $\Phi_s(t,x)$. Then,

$$I_s(x) = (-1)^{\alpha_s+\beta_s+\gamma_s-p^s+1} \sum_{k_1+k_2=\alpha_s+\beta_s+\gamma_s-p^s+1} \binom{\beta_s}{k_1}\binom{\gamma_s}{k_2} x^{k_2}.$$

The polynomial $I_s(x)$ has integer coefficients.

THEOREM 3.1. *The polynomial $I_s(x)$ is a solution of the hypergeometric differential equation in Equation (3-2) modulo p^s,*

$$\mathcal{D}I_s(x) \in p^s\mathbb{Z}_p[x].$$

PROOF. The theorem follows from Equation (3-3). □

In this paper, we prove Dwork-type congruences for the p^s-approximation polynomials $I_s(x)$ in several basic examples and leave general considerations for another occasion.

For more general versions of the p^s-approximation construction, see [SV2].

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

4. Function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, x)$

4.1. Polynomials $P_s(x)$. The function

$$ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; x\Big) = \frac{1}{\pi} \int_1^\infty t^{-1/2}(t-1)^{-1/2}(t-x)^{-1/2}\,dt = \sum_{k=0}^\infty \binom{-1/2}{k}^2 x^k \tag{4-1}$$

satisfies the hypergeometric differential equation

$$ x(1-x)I'' + (1-2x)I' - \tfrac{1}{4}I = 0. \tag{4-2}$$

Define the master polynomial

$$ \Phi_{p^s}(t,x) = t^{(p^s-1)/2}(t-1)^{(p^s-1)/2}(t-x)^{(p^s-1)/2}. $$

The number $M = (p^s-1)/2 = (p-1)/2 + (p-1)/2p + \cdots + (p-1)/2p^{s-1}$ is the unique positive integer such that $1 \leqslant M \leqslant p^s$ and $M \equiv -1/2 \pmod{p^s}$. Define the p^s-approximation polynomial $P_s(x)$ as the coefficient of t^{p^s-1} in the master polynomial $\Phi_{p^s}(t,x)$. Then,

$$ P_s(x) = (-1)^{(p^s-1)/2} \sum_{k=0}^{(p^s-1)/2} \binom{(p^s-1)/2}{k}^2 x^k. \tag{4-3}$$

Define $P_0(x) = 1$.

Recall the hypergeometric function ${}_2F_1(a,b;c;x)$. Then,

$$ P_s(x) = (-1)^{(p^s-1)/2} {}_2F_1\Big(\frac{1-p^s}{2}, \frac{1-p^s}{2}; 1; x\Big). \tag{4-4}$$

The polynomial $P_s(x)$ is a solution of the hypergeometric equation in Equation (4-2) modulo p^s. This follows from Theorem 3.1 or from Equation (4-4).

4.2. Baby congruences. Let $\varphi_s(x) = (x+1)^{(p^s-1)/2}$. Then,

$$ \varphi_{s+1}(x)\varphi_{s-1}(x^p) \equiv \varphi_s(x)\varphi_s(x^p) \pmod{p^s}. \tag{4-5}$$

This follows from $(x+1)^{p^s} \equiv (x^p+1)^{p^{s-1}} \pmod{p^s}$.

LEMMA 4.1. *The master polynomials $\Phi_s(t,x)$ satisfy the baby congruence*

$$ \Phi_{s+1}(t,x)\Phi_{s-1}(t^p,x^p) \equiv \Phi_s(t,x)\Phi_s(t^p,x^p) \pmod{p^s}. \tag{4-6}$$

PROOF. The lemma follows from Equation (4-5). □

4.3. Congruences for $P_s(x)$.

THEOREM 4.2. *The approximation polynomials $P_s(x)$ satisfy the congruence*

$$ P_{s+1}(x)P_{s-1}(x^p) \equiv P_s(x)P_s(x^p) \pmod{p^s}. \tag{4-7}$$

This theorem follows from a more general Theorem 4.4.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Using Equation (4-4), we may rewrite Equation (4-7) as the congruence

$$
\begin{aligned}
&{}_2F_1\Big(\frac{1-p^{s+1}}{2}, \frac{1-p^{s+1}}{2}; 1; x\Big){}_2F_1\Big(\frac{1-p^{s-1}}{2}, \frac{1-p^{s-1}}{2}; 1; x^p\Big) \\
&\quad \equiv {}_2F_1\Big(\frac{1-p^{s}}{2}, \frac{1-p^{s}}{2}; 1; x\Big){}_2F_1\Big(\frac{1-p^{s}}{2}, \frac{1-p^{s}}{2}; 1; x^p\Big) \pmod{p^s}. \qquad (4\text{-}8)
\end{aligned}
$$

Let α be a rational number which is a p-adic unit, $\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots$. Denote by $[\alpha]_s$ the sum of the first s summands. Then the congruence in Equation (4-8) takes the form:

$$
\begin{aligned}
&{}_2F_1([-\tfrac{1}{2}]_{s+1}, [-\tfrac{1}{2}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{1}{2}]_{s-1}, [-\tfrac{1}{2}]_{s-1}; 1; x^p) \\
&\quad \equiv {}_2F_1([-\tfrac{1}{2}]_{s}, [-\tfrac{1}{2}]_{s}; 1; x)\, {}_2F_1([-\tfrac{1}{2}]_{s}, [-\tfrac{1}{2}]_{s}; 1; x^p) \pmod{p^s}.
\end{aligned}
$$

4.4. Coefficients of master polynomials. Consider

$$
\begin{aligned}
\hat{\Phi}_s(t,x) &:= t^{-(p^s-1)}\Phi_s(t,x) = t^{-(p^s-1)/2}((t-1)(t-x))^{(p^s-1)/2} \\
&= ((t-1)(1-x/t))^{(p^s-1)/2} = \sum_{j=-(p^s-1)/2}^{(p^s-1)/2} C_{s,j}(x) t^j,
\end{aligned}
$$

where

$$
C_{s,j}(x) = (-1)^{((p^s-1)/2)-j} \sum_m \binom{\frac{p^s-1}{2}}{m+j}\binom{\frac{p^s-1}{2}}{m} x^m.
$$

In particular, $C_{s,0}(x) = P_s(x)$. Every coefficient $C_{s,j}(x)$ is a hypergeometric function:

$$
C_{s,j}(x) = (-1)^{((p^s-1)/2)-j}\binom{\frac{p^s-1}{2}}{j}{}_2F_1\Big(\frac{1-p^s}{2}, \frac{1-p^s}{2}+j; j+1; x\Big) \quad \text{for } j \geqslant 0,
$$

while a hypergeometric expression in the case $j < 0$ comes out from the following simple fact.

LEMMA 4.3. *We have $\hat{\Phi}_s(x/t,x) = \hat{\Phi}_s(t,x)$ and hence*

$$
C_{s,-j}(x) = x^j C_{s,j}(x). \qquad (4\text{-}9)
$$

We expand the congruence in Equation (4-6) into a congruence of polynomials $C_{s,j}(x)$. The constant term in t gives us

$$
\sum_k C_{s+1,kp}(x) C_{s-1,-k}(x^p) \equiv \sum_k C_{s,kp}(x) C_{s,-k}(x^p) \pmod{p^s}. \qquad (4\text{-}10)
$$

The following Theorem 4.4 establishes the congruences of individual pairs of terms in Equation (4-10).

THEOREM 4.4. *For any k appearing in Equation (4-10),*

$$
C_{s+1,kp}(x) C_{s-1,-k}(x^p) \equiv C_{s,kp}(x) C_{s,-k}(x^p) \pmod{p^s}. \qquad (4\text{-}11)
$$

In particular, for $k = 0$, we have the congruence in Equation (4-7).

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

PROOF. Every index k appearing in Equation (4-10) can be written uniquely as

$$k = k_0 + k_1 p + \cdots + k_{s-2} p^{s-2}, \quad -(p-1)/2 \leqslant k_i \leqslant (p-1)/2.$$

Using Equation (4-9), we reformulate Equation (4-11) as

$$C_{s+1,kp}(x) C_{s-1,k}(x^p) \equiv C_{s,kp}(x) C_{s,k}(x^p) \pmod{p^3} \tag{4-12}$$

and prove it. We have

$$C_{s+1,kp}(x) = \mathrm{CT}_t \Big[\hat{\Phi}_1(t,x) \Big(\prod_{i=0}^{s-2} (t^{-k_i} \hat{\Phi}_1(t,x))^{p^{i+1}} \Big) \hat{\Phi}_1(t,x)^{p^s} \Big],$$

$$C_{s-1,k}(x) = \mathrm{CT}_t \Big[\prod_{i=0}^{s-2} (t^{-k_i} \hat{\Phi}_1(t,x))^{p^i} \Big],$$

$$C_{s,kp}(x) = \mathrm{CT}_t \Big[\hat{\Phi}_1(t,x) \prod_{i=0}^{s-2} (t^{-k_i} \hat{\Phi}_1(t,x))^{p^{i+1}} \Big],$$

$$C_{s,k}(x) = \mathrm{CT}_t \Big[\Big(\prod_{i=0}^{s-2} (t^{-k_i} \hat{\Phi}_1(t,x))^{p^i} \Big) \hat{\Phi}_1(t,x)^{p^{s-1}} \Big].$$

It is easy to see that the $(s+1)$-tuple of Laurent polynomials

$$\hat{\Phi}_1(t,x),\ t^{-k_0} \hat{\Phi}_1(t,x),\ \ldots,\ t^{-k_{s-2}} \hat{\Phi}_1(t,x),\ \hat{\Phi}_1(t,x)$$

is admissible in the sense of Definition 2.6. Now the application of Theorem 2.9 gives the congruence in Equation (4-12) and hence the congruence in Equation (4-11). □

REMARK 4.5. Denote $A(n,x) := {}_2F_1(-n,-n;1;x) = \sum_k \binom{n}{k}^2 x^k$. Let

$$n = n_0 + n_1 p + \cdots + n_{s-1} p^{s-1}, \quad [n/p] = n_1 + \cdots + n_{s-1} p^{s-2},$$

where $0 \leqslant n_i < p$. Then for any $m \in \mathbb{Z}_{\geqslant 0}$,

$$A(n + mp^s, x) A([n/p], x^p) \equiv A(n,x) A([n/p] + mp^{s-1}, x^p) \pmod{p^s}.$$

The proof follows from Theorem 2.9 and the identity

$$A(n,x) = \mathrm{CT}_t[((t+1)(1+x/t))^n].$$

4.5. Limits of $P_s(x)$. For $\alpha \in \mathbb{Z}_p$, there exists a unique solution $\omega(\alpha) \in \mathbb{Z}_p$ of the equation $\omega(\alpha)^p = \omega(\alpha)$ that is congruent to x modulo p. The element $\omega(\alpha)$ is called the Teichmüller representative of α. For $\alpha \in \mathbb{F}_p$, $r > 0$, define the disc

$$D_{\alpha,r} = \{x \in \mathbb{Z}_p \mid |x - \omega(\alpha)|_p < r\}.$$

Denote

$$\bar{P}_s(x) := (-1)^{(p^s-1)/2} P_s(x) = {}_2F_1([-\tfrac{1}{2}]_s, [-\tfrac{1}{2}]_s; 1; x);$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

see Equation (4-4). Denote

$$\mathfrak{D} = \{x \in \mathbb{Z}_p \mid |\bar{P}_1(x)|_p = 1\}.$$

THEOREM 4.6. *For $s \geqslant 1$, the rational function $\bar{P}_{s+1}(x)/\bar{P}_s(x^p)$ is regular on $\mathfrak{D}$. The sequence $(\bar{P}_{s+1}(x)/\bar{P}_s(x^p))_{s\geqslant 1}$ uniformly converges on $\mathfrak{D}$. The limiting analytic function $f(x)$ equals the ratio $F(x)/F(x^p)$ on the disc $D_{0,1}$, where $F(x) := {}_2F_1(1/2, 1/2; 1; x)$ is defined by the convergent power series in Equation (4-1). We also have $|f(x)|_p = 1$ for $x \in \mathfrak{D}$.*

PROOF. We have $\mathbb{Z}_p = \bigcup_{\alpha\in\mathbb{F}_p} D_{\alpha,1}$ and also $\mathfrak{D} = \bigcup_{\alpha\in\mathbb{F}_p, |\bar{P}_1(\omega(\alpha))|_p=1} D_{\alpha,1}$ since $\bar{P}_1(x)$ has coefficients in $\mathbb{Z}_p$. In particular, $D_{0,1} \subset \mathfrak{D}$. We also have

$$\{x \in \mathbb{Z}_p \mid |\bar{P}_1(x^p)|_p = 1\} = \bigcup_{\alpha\in\mathbb{F}_p, |\bar{P}_1(\omega(\alpha))|_p=1} D_{\alpha,1} = \mathfrak{D}$$

for the same reason.

We have $\bar{P}_s(x) \equiv \bar{P}_1(x)\bar{P}_1(x^p)\cdots\bar{P}_1(x^{p^{s-1}}) \pmod p$. Indeed, the polynomial $P_s(x)$ is the coefficient of x^{p^s-1} in the master polynomials $\Phi_{p^s}(t,x)$ and

$$\Phi_{p^s}(t,x) \equiv \Phi_p(t,x)\Phi_p(t^p,x^p)\cdots\Phi_p(t^{p^{s-1}},x^{p^{s-1}}) \pmod p.$$

Hence, $|\bar{P}_s(x)|_p = |\bar{P}_s(x^p)|_p = 1$ for $s \geqslant 1$, $x \in \mathfrak{D}$. Hence, the rational functions $\bar{P}_{s+1}(x)/\bar{P}_s(x^p)$ are regular on $\mathfrak{D}$.

The congruence in Equation (4-7) implies that

$$\left|\frac{\bar{P}_{s+1}(x)}{\bar{P}_s(x^p)} - \frac{\bar{P}_s(x)}{\bar{P}_{s-1}(x^p)}\right|_p \leqslant p^{-s} \quad \text{for } x \in \mathfrak{D}.$$

This shows the uniform convergence of our sequence of rational functions on the domain $\mathfrak{D}$. For the limiting function $f(x)$, we have $|f(x)|_p = 1$ for $x \in \mathfrak{D}$.

Clearly, for any fixed index k, the coefficient $\binom{(p^s-1)/2}{k}^2$ of x^k in $\bar{P}_s(x)$ converges p-adically to the coefficient $\binom{-1/2}{k}^2$ of x^k in $F(x)$. Hence, the sequence $(\bar{P}_s(x))_{s\geqslant 1}$ converges to $F(x)$ on $D_{0,1}$, so that $f(x) = F(x)/F(x^p)$ on $D_{0,1}$. The theorem is proved. □

Dwork gives in [Dw] a different construction of analytic continuation of the ratio $F(x)/F(x^p)$ from $D_{0,1}$ to a larger domain. He considers the sequence of polynomials

$$F_s(x) = \sum_{k=0}^{p^s-1} \binom{-1/2}{k}^2 x^k,$$

which are truncations of the hypergeometric series $F(x)$, and shows that the sequence of rational functions $(F_{s+1}(x)/F_s(x^p))_{s\geqslant 1}$ uniformly converges on the domain $\mathfrak{D}^{\mathrm{Dw}} = \{x \in \mathbb{Z}_p \mid |g(x)|_p = 1\}$, where the polynomial

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

$$g(x) = \sum_{k=0}^{(p-1)/2} \binom{-1/2}{k}^2 x^k \tag{4-13}$$

is attributed by Dwork to Igusa [Ig]. Clearly, his limiting function $f^{\mathrm{Dw}}(x)$ equals the ratio $F(x)/F(x^p)$ on $D_{0,1}$.

It is easy to see that the two sequences of rational functions $(\bar{P}_{s+1}(x)/\bar{P}_s(x^p))_{s\geqslant 1}$ and $(F_{s+1}(x)/F_s(x^p))_{s\geqslant 1}$ have the same limiting functions on the same domain. Indeed, $\bar{P}_1(x) \equiv g(x) \pmod p$ and hence $\mathfrak{D} = \mathfrak{D}^{\mathrm{Dw}}$. Also, $f(x) = f^{\mathrm{Dw}}(x)$ on $D_{0,1}$ and hence on $\mathfrak{D}$.

Dwork shows in [Dw] interesting properties of the function $f(x)$. For example, let $\alpha \in \mathbb{F}_p^\times - \{1\}$ be such that $\omega(\alpha) \in \mathfrak{D}$. Dwork shows that the zeta function of the elliptic curve defined over $\mathbb{F}_p$ by the equation $y^2 = x(x-1)(x-\alpha)$ has two zeros, which are $1/((-1)^{(p-1)/2} f(\omega(\alpha)))$ and $(-1)^{(p-1)/2} f(\omega(\alpha))/p$. Clearly, $|f(\omega(\alpha))|_p = 1$. The number $(-1)^{(p-1)/2} f(\omega(\alpha))$ is called the unit root of that elliptic curve.

According to our discussion, this unit root can be calculated as the value at $x = \omega(\alpha)$ of the limit as $s \to \infty$ of the ratio $\bar{P}_{s+1}(x)/\bar{P}_s(x^p)$ of approximation polynomials multiplied by $(-1)^{(p-1)/2}$.

5. Function ${}_2F_1(2/3, \frac{1}{3}; 1; x)$

5.1. Two hypergeometric integrals. The integral

$$I^{(C)}(x) = \int_C t^{-1/3}(t-1)^{-1/3}(t-x)^{-2/3}\,dt,$$

where $C \subset \mathbb{C} - \{0, 1, x\}$ is a contour on which the integrand takes its initial value when t encircles C, satisfies the hypergeometric differential equation

$$x(1-x)I'' + (1-2x)I' - \tfrac{2}{9}I = 0. \tag{5-1}$$

For a suitable choice of C, the integral $I^{(C)}(x)$ presents the hypergeometric function

$${}_2F_1\Big(\frac{2}{3}, \frac{1}{3}; 1, x\Big) = \sum_{k=0}^{\infty} \binom{-1/3}{k}\binom{-2/3}{k} x^k.$$

The integral

$$J^{(D)}(x) = \int_D t^{-2/3}(t-1)^{-2/3}(t-x)^{-1/3}\,dt,$$

where $D \subset \mathbb{C} - \{0, 1, x\}$ is a contour on which the integrand takes its initial value when t encircles D, satisfies the same hypergeometric differential equation. For a suitable choice of D, the integral $J^{(D)}(x)$ presents the same hypergeometric function ${}_2F_1(2/3, \frac{1}{3}; 1, x)$.

The differential form $t^{-1/3}(t-1)^{-1/3}(t-z)^{-2/3}\,dt$ is transformed to the differential form $-t^{-2/3}(t-1)^{-2/3}(t-z)^{-1/3}\,dt$ by the change of variable $t \mapsto (t-z)/(t-1)$.

In this section, we discuss the p^s-approximations of the integrals $I^{(C)}(x)$ and $J^{(D)}(x)$.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

5.2. The case $p = 3\ell + 1$. The master polynomial for $I^{(C)}(x)$ is given by the formula

$$\Phi_s(t,x) = t^{(p^s-1)/3}(t-1)^{(p^s-1)/3}(t-x)^{2(p^s-1)/3}.$$

The p^s-approximation polynomial $Q_s(x)$ is defined as the coefficient of t^{p^s-1} in $\Phi_s(t,x)$,

$$Q_s(x) = (-1)^{(p^s-1)/3}\sum_k \binom{2(p^s-1)/3}{k}\binom{(p^s-1)/3}{k}x^k.$$

Define $Q_0(x) = 1$. We have

$$Q_s(x) = {}_2F_1\Big(\frac{2-2p^s}{3}, \frac{1-p^s}{3}; 1; x\Big), \tag{5-2}$$

since $(-1)^{(p^s-1)/3} = 1$.

The polynomial $Q_s(x)$ is a solution of the hypergeometric equation in Equation (5-1) modulo p^s. This follows from Theorem 3.1 or from Equation (5-2).

The master polynomial for $J^{(D)}(x)$ is given by the formula

$$\Psi_s(t,x) = t^{2(p^s-1)/3}(t-1)^{2(p^s-1)/3}(t-x)^{(p^s-1)/3}.$$

The p^s-approximation polynomial $R_s(x)$ is defined as the coefficient of t^{p^s-1} in $\Psi_s(t,x)$,

$$R_s(x) = \sum_k \binom{2(p^s-1)/3}{k}\binom{(p^s-1)/3}{k}x^k.$$

Define $R_0(x) = 1$. We have

$$Q_s(x) = {}_2F_1\Big(2\frac{1-p^s}{3}, \frac{1-p^s}{3}; 1; x\Big). \tag{5-3}$$

The master polynomials satisfy the baby congruences,

$$\begin{aligned}\Phi_{s+1}(t,x)\Phi_{s-1}(t^p,x^p) &\equiv \Phi_s(t,x)\Phi_s(t^p,x^p) \pmod{p^s},\\ \Psi_{s+1}(t,x)\Psi_{s-1}(t^p,x^p) &\equiv \Psi_s(t,x)\Psi_s(t^p,x^p) \pmod{p^s},\end{aligned} \tag{5-4}$$

by Equation (4-5).

THEOREM 5.1. *For $p = 3\ell + 1$, the approximation polynomials $R_s(x)$ and $Q_s(x)$ satisfy the congruences*

$$Q_{s+1}(x)Q_{s-1}(x^p) \equiv Q_s(x)Q_s(x^p) \pmod{p^s}, \tag{5-5}$$

$$R_{s+1}(x)R_{s-1}(x^p) \equiv R_s(x)R_s(x^p) \pmod{p^s}. \tag{5-6}$$

PROOF. Denote $\hat\Phi_1(t,x) = (t-1)^\ell(1-x/t)^{2\ell}$. Then, $Q_{s+1}(x) = \mathrm{CT}_t[\hat\Phi_1(t,x)^{1+p+\cdots+p^s}]$. It is easy to see that the $(s+1)$-tuple $(\hat\Phi_1(t,x), \hat\Phi_1(t,x), \ldots, \hat\Phi_1(t,x))$ of Laurent polynomials is admissible in the sense of Definition 2.6. Now the application of Theorem 2.9 gives the congruence in Equation (5-5). The congruence in Equation (5-6) is proved in the same way applied to the formula $R_{s+1}(x) = \mathrm{CT}_t[\hat\Psi_1(t,x)^{1+p+\cdots+p^s}]$,

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

where $\hat{\Psi}_1(t,x) = (t-1)^{2\ell}(1-x/t)^{\ell}$. The congruence in Equation (5-6) also follows from Equation (5-5) since $R_s(x) = Q_s(x)$. □

Equations (5-2) and (5-3) imply that for $p = 3\ell+1$, $s \geqslant 1$,

$$\begin{aligned}&{}_2F_1\Big(\frac{2-2p^{s+1}}{3}, \frac{1-p^{s+1}}{3}; 1; x\Big) {}_2F_1\Big(\frac{2-2p^{s-1}}{3}, \frac{1-p^{s-1}}{3}; 1; x^p\Big)\\ &\quad\equiv {}_2F_1\Big(\frac{2-2p^{s}}{3}, \frac{1-p^{s}}{3}; 1; x\Big) {}_2F_1\Big(\frac{2-2p^{s}}{3}, \frac{1-p^{s}}{3}; 1; x^p\Big) \pmod{p^s}.\end{aligned}$$

Using the expansions

$$\begin{aligned}-1/3 &= \ell + \ell p + \ell p^2 + \cdots, & -2/3 &= 2\ell + 2\ell p + 2\ell p^2 + \cdots,\\ (p^s-1)/3 &= \ell + \ell p + \ell p^2 + \cdots + \ell p^{s-1}, & (p^s-2)/3 &= 2\ell + 2\ell p + 2\ell p^2 + \cdots + 2\ell p^{s-1},\end{aligned}$$

we conclude that for $p = 3\ell+1$ and $s \geqslant 1$,

$$\begin{aligned}&{}_2F_1([-\tfrac{2}{3}]_{s+1}, [-\tfrac{1}{3}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{2}{3}]_{s-1}, [-\tfrac{1}{3}]_{s-1}; 1; x^p)\\ &\quad\equiv {}_2F_1([-\tfrac{2}{3}]_{s}, [-\tfrac{1}{3}]_{s}; 1; x)\, {}_2F_1([-\tfrac{2}{3}]_{s}, [-\tfrac{1}{3}]_{s}; 1; x^p) \pmod{p^s}.\end{aligned} \tag{5-7}$$

5.3. The case $p = 3\ell + 2 > 2$. The master polynomial for $I^{(C)}(x)$ is given by the formulas

$$\begin{aligned}\Phi_s(t,x) &= t^{(2p^s-1)/3}(t-1)^{(2p^s-1)/3}(t-x)^{(p^s-2)/3}, \quad \text{odd } s,\\ \Phi_s(t,x) &= t^{(p^s-1)/3}(t-1)^{(p^s-1)/3}(t-x)^{2(p^s-1)/3}, \quad \text{even } s.\end{aligned}$$

The p^s-approximation polynomial $Q_s(x)$ is defined as the coefficient of t^{p^s-1} in $\Phi_s(t,x)$,

$$\begin{aligned}Q_s(x) &= (-1)^{(2p^s-1)/3} \sum_k \binom{(2p^s-1)/3}{k}\binom{(p^s-2)/3}{k} x^k, \quad \text{odd } s,\\ Q_s(x) &= (-1)^{(p^s-1)/3} \sum_k \binom{2(p^s-1)/3}{k}\binom{(p^s-1)/3}{k} x^k, \quad \text{even } s.\end{aligned}$$

Define $Q_0(x) = 1$. We have

$$\begin{aligned}Q_s(x) &= -\,{}_2F_1\Big(\frac{2-p^s}{3}, \frac{1-2p^s}{3}; 1; x\Big), \quad \text{odd } s,\\ Q_s(x) &= {}_2F_1\Big(\frac{2-2p^s}{3}, \frac{1-p^s}{3}; 1; x\Big), \quad \text{even } s.\end{aligned} \tag{5-8}$$

Here we use the fact that for $p = 3\ell+2 > 2$, we have $(-1)^{(2p^s-1)/3} = -1$ for odd s and $(-1)^{(p^s-1)/3} = 1$ for even s.

The polynomial $Q_s(x)$ is a solution of the hypergeometric equation in Equation (5-1) modulo p^s. This follows from Theorem 3.1 or from Equation (5-8).

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

The master polynomial for $J^{(D)}(x)$ is given by the formulas

$$\Psi_s(t,x) = t^{(p^s-2)/3}(t-1)^{(p^s-2)/3}(t-x)^{(2p^s-1)/3}, \quad \text{odd } s,$$
$$\Psi_s(t,x) = t^{2(p^s-1)/3}(t-1)^{2(p^s-1)/3}(t-x)^{(p^s-1)/3}, \quad \text{even } s.$$

The p^s-approximation polynomial $R_s(x)$ is defined as the coefficient of t^{p^s-1} in $\Phi_s(t,x)$,

$$R_s(x) = (-1)^{(p^s-2)/3} \sum_k \binom{(2p^s-1)/3}{k}\binom{(p^s-2)/3}{k} x^k, \quad \text{odd } s,$$
$$R_s(x) = (-1)^{2(p^s-1)/3} \sum_k \binom{2(p^s-1)/3}{k}\binom{(p^s-1)/3}{k} x^k, \quad \text{even } s.$$

Define $R_0(x) = 1$. We have

$$\begin{aligned} R_s(x) &= -\,{}_2F_1\Big(\frac{2-p^s}{3}, \frac{1-2p^s}{3}; 1; x\Big), \quad \text{odd } s,\\ R_s(x) &= {}_2F_1\Big(\frac{2-2p^s}{3}, \frac{1-p^s}{3}; 1; x\Big), \quad \text{even } s. \end{aligned} \tag{5-9}$$

Here we use the fact that for $p = 3\ell + 2 > 2$, we have $(-1)^{(p^s-2)/3} = -1$ for odd s and $(-1)^{(2p^s-1)/3} = 1$ for even s.

The polynomial $R_s(x)$ is a solution of the hypergeometric equation in Equation (5-1) modulo p^s. This follows from Theorem 3.1 or from Equation (5-9).

LEMMA 5.2. *The master polynomials satisfy the baby congruences,*

$$\Phi_{s+1}(t,x)\Psi_{s-1}(t^p,x^p) \equiv \Phi_s(t,x)\Psi_s(t^p,x^p) \pmod{p^s}, \tag{5-10}$$

$$\Psi_{s+1}(t,x)\Phi_{s-1}(t^p,x^p) \equiv \Psi_s(t,x)\Phi_s(t^p,x^p) \pmod{p^s}. \tag{5-11}$$

PROOF. We prove Equation (5-10) for an odd s. The case of an even s and the congruence in Equation (5-11) are proved similarly. The left-hand side of Equation (5-10) for an odd $s = 2k+1$ equals

$$t^{(p^{s+1}-1)/3}(t-1)^{(p^{s+1}-1)/3}(t-x)^{2(p^{s+1}-1)/3} t^{2p(p^{s-1}-1)/3}(t^p-1)^{2(p^{s-1}-1)/3}(t^p-x^p)^{(p^{s-1}-1)/3},$$

while the right-hand side equals

$$t^{(2p^s-1)/3}(t-1)^{(2p^s-1)/3}(t-x)^{(p^s-2)/3} t^{p(p^s-2)/3}(t^p-1)^{(p^s-2)/3}(t^p-x^p)^{(2p^s-1)/3}.$$

Now the congruence in Equation (5-10) for an odd s follows from Equation (4-5). □

THEOREM 5.3. *For $p = 3\ell + 2 > 2$, the approximation polynomials $R_s(x)$ and $Q_s(x)$ satisfy the congruences*

$$Q_{s+1}(x)R_{s-1}(x^p) \equiv Q_s(x)R_s(x^p) \pmod{p^s}, \tag{5-12}$$

$$R_{s+1}(x)Q_{s-1}(x^p) \equiv R_s(x)Q_s(x^p) \pmod{p^s}. \tag{5-13}$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

PROOF. We prove Equation (5-12) for an odd s. The case of an even s and the congruence in Equation (5-13) are proved similarly. Denote

$$f(t,x) = (t-1)^{(2p-1)/3}(1-x/t)^{(p-2)/3} = (t-1)^{2\ell+1}(1-x/t)^{\ell},$$
$$g(t,x) = (t-1)^{(p-2)/3}(1-x/t)^{(2p-1)/3} = (t-1)^{\ell}(1-x/t)^{2\ell+1}.$$

It is easy to see that for an odd s,

$$\begin{aligned} Q_{s+1}(x) &= \mathrm{CT}_t[(t-1)^{(p^{s+1}-1)/3}(1-x/t)^{2(p^{s+1}-1)/3}] \\ &= \mathrm{CT}_t[f(t,x)g(t,x)^p \cdots f(t,x)^{p^{s-1}}g(t,x)^{p^s}], \\ R_{s-1}(x) &= \mathrm{CT}_t[(t-1)^{2(p^{s-1}-1)/3}(1-x/t)^{(p^{s-1}-1)/3}] \\ &= \mathrm{CT}_t[g(t,x)f(t,x)^p \cdots g(t,x)^{p^{s-3}}f(t,x)^{p^{s-2}}], \end{aligned}$$

$$\begin{aligned} Q_s(x) &= \mathrm{CT}_t[(t-1)^{(2p^s-1)/3}(1-x/t)^{(p^s-2)/3}] \\ &= \mathrm{CT}_t[f(t,x)g(t,x)^p \cdots g(t,x)^{p^{s-2}}f(t,x)^{p^{s-1}}], \\ R_s(x) &= \mathrm{CT}_t[(t-1)^{(p^s-2)/3}(1-x/t)^{(2p^s-1)/3}] \\ &= \mathrm{CT}_t[g(t,x)f(t,x)^p \cdots f(t,x)^{p^{s-2}}g(t,x)^{p^{s-1}}]. \end{aligned}$$

Observe that the $(s+1)$-tuple of Laurent polynomials $(f(t,x), g(t,x), \dots, f(t,x), g(t,x))$ is admissible in the sense of Definition 2.6. Now the application of Theorem 2.9 gives the congruence in Equation (5-12) for an odd s. □

REMARK 5.4. In general, we may take any admissible tuple of Laurent polynomials and obtain the corresponding Dwork congruences. For example, the tuples $(f, g, f, f, f, g, g, f, \dots)$ and $(f, f, \dots)$ are admissible.

Using Equations (5-8) and (5-9), we may reformulate the congruences in Equations (5-12) and (5-13) as

$$\begin{aligned} &{}_2F_1\Big(\frac{2-p^{s+1}}{3}, \frac{1-2p^{s+1}}{3}; 1; x\Big){}_2F_1\Big(\frac{2-p^{s-1}}{3}, \frac{1-2p^{s-1}}{3}; 1; x^p\Big) \\ &\quad\equiv {}_2F_1\Big(\frac{2-2p^s}{3}, \frac{1-p^s}{3}; 1; x\Big){}_2F_1\Big(\frac{2-2p^s}{3}, \frac{1-p^s}{3}; 1; x^p\Big) \pmod{p^s}, \quad \text{odd } s, \end{aligned} \tag{5-14}$$

$$\begin{aligned} &{}_2F_1\Big(\frac{2-2p^{s+1}}{3}, \frac{1-p^{s+1}}{3}; 1; x\Big){}_2F_1\Big(\frac{2-2p^{s-1}}{3}, \frac{1-p^{s-1}}{3}; 1; x^p\Big) \\ &\quad\equiv {}_2F_1\Big(\frac{2-p^s}{3}, \frac{1-2p^s}{3}; 1; x\Big){}_2F_1\Big(\frac{2-p^s}{3}, \frac{1-2p^s}{3}; 1; x^p\Big) \pmod{p^s}, \quad \text{even } s. \end{aligned} \tag{5-15}$$

Recall that in these congruences, we have $p = 3\ell + 2$.

Consider the p-adic presentations

$$-1/3 = 2\ell + 1 + \ell p + (2\ell+1)p^2 + \ell p^3 + \cdots, \tag{5-16}$$

$$-2/3 = \ell + (2\ell+1)p + \ell p^2 + (2\ell+1)p^3 + \cdots. \tag{5-17}$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Recall that $[-1/3]_s$ (respectively, $[-2/3]_s$) is the sum of the first s summands in Equation (5-16) (respectively, (5-17)). Then the congruences in Equations (5-14) and (5-15) imply that for $p = 3\ell + 2$, $s \geqslant 1$,

$$\begin{aligned} &{}_2F_1([-\tfrac{2}{3}]_{s+1}, [-\tfrac{1}{3}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{2}{3}]_{s-1}, [-\tfrac{1}{3}]_{s-1}; 1; x^p) \\ &\quad \equiv {}_2F_1([-\tfrac{2}{3}]_s, [-\tfrac{1}{3}]_s; 1; x)\, {}_2F_1([-\tfrac{2}{3}]_s, [-\tfrac{1}{3}]_s; 1; x^p) \pmod{p^s}. \end{aligned} \tag{5-18}$$

5.4. Limits of $\bar{Q}_s(x)$. Define

$$\bar{Q}_s(x) = {}_2F_1([-\tfrac{2}{3}]_s, [-\tfrac{1}{3}]_s; 1; x).$$

Then for any prime $p > 3$,

$$\bar{Q}_{s+1}(x)\bar{Q}_{s-1}(x^p) \equiv \bar{Q}_s(x)\bar{Q}_s(x^p) \pmod{p^s}$$

by Equations (5-7) and (5-18).

THEOREM 5.5. *For any prime $p > 3$ and integer $s \geqslant 1$, the rational function $\bar{Q}_{s+1}(x)/\bar{Q}_s(x^p)$ is regular on the domain*

$$\mathfrak{D} = \{x \in \mathbb{Z}_p \mid |\bar{Q}_1(x)|_p = 1\}.$$

The sequence $(\bar{Q}_{s+1}(x)/\bar{Q}_s(x^p))_{s\geqslant 1}$ uniformly converges on $\mathfrak{D}$. The limiting analytic function $f(x)$ equals the ratio $F(x)/F(x^p)$ on the disc $D_{0,1}$, where $F(x) := {}_2F_1(2/3, 1/3; 1; x)$ is defined by the corresponding convergent power series.

PROOF. The proof is the same as the proof of Theorem 4.6. □

5.5. Remark. Although the congruences in Equations (5-7) and (5-18) look the same for $p = 3\ell + 1$ and $p = 3\ell + 2$, the proofs of them are different as already presented. The proof of Equation (5-7) for $p = 3\ell + 1$ uses just any one of the two master polynomials: $\Phi_s(t, x)$ or $\Psi_s(t, x)$, while the proof of Equation (5-18) for $p = 3\ell + 2$ uses the interaction of the two master polynomials $\Phi_s(t, x)$ and $\Psi_s(t, x)$. See the baby congruences in Equations (5-4), (5-10), (5-11).

5.6. Congruences related to $-\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}$. In this section, we formulate the congruences related to the above rational numbers. The proof of these congruences is similar to the corresponding proofs in Sections 4 and 5.

For $p = 5\ell \pm 2$ and any $s \geqslant 1$,

$$\begin{aligned} &{}_2F_1([-\tfrac{4}{5}]_{s+1}, [-\tfrac{1}{5}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{3}{5}]_{s-1}, [-\tfrac{2}{5}]_{s-1}; 1; x^p) \\ &\quad \equiv {}_2F_1([-\tfrac{4}{5}]_s, [-\tfrac{1}{5}]_s; 1; x)\, {}_2F_1([-\tfrac{3}{5}]_s, [-\tfrac{2}{5}]_s; 1; x^p) \pmod{p^s}, \end{aligned}$$

$$\begin{aligned} &{}_2F_1([-\tfrac{3}{5}]_{s+1}, [-\tfrac{2}{5}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{4}{5}]_{s-1}, [-\tfrac{1}{5}]_{s-1}; 1; x^p) \\ &\quad \equiv {}_2F_1([-\tfrac{3}{5}]_s, [-\tfrac{2}{5}]_s; 1; x)\, {}_2F_1([-\tfrac{4}{5}]_s, [-\tfrac{1}{5}]_s; 1; x^p) \pmod{p^s}. \end{aligned}$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

For $p = 5\ell \pm 1$ and any $s \geqslant 1$,

$$\begin{aligned}
&{}_2F_1([-\tfrac{4}{5}]_{s+1}, [-\tfrac{1}{5}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{4}{5}]_{s-1}, [-\tfrac{1}{5}]_{s-1}; 1; x^p) \\
&\quad \equiv {}_2F_1([-\tfrac{4}{5}]_{s}, [-\tfrac{1}{5}]_{s}; 1; x)\, {}_2F_1([-\tfrac{4}{5}]_{s}, [-\tfrac{1}{5}]_{s}; 1; x^p) \pmod{p^s},
\end{aligned}$$

$$\begin{aligned}
&{}_2F_1([-\tfrac{3}{5}]_{s+1}, [-\tfrac{2}{5}]_{s+1}; 1; x)\, {}_2F_1([-\tfrac{3}{5}]_{s-1}, [-\tfrac{2}{5}]_{s-1}; 1; x^p) \\
&\quad \equiv {}_2F_1([-\tfrac{3}{5}]_{s}, [-\tfrac{2}{5}]_{s}; 1; x)\, {}_2F_1([-\tfrac{3}{5}]_{s}, [-\tfrac{2}{5}]_{s}; 1; x^p) \pmod{p^s}.
\end{aligned}$$

Similar congruences hold for rational numbers of the form a/b, where b is a prime and $1 - b \leqslant a \leqslant -1$. These congruences are described somewhere else.

6. KZ equations

6.1. KZ equations. Let $\mathfrak{g}$ be a simple Lie algebra with an invariant scalar product. The *Casimir element* is

$$\Omega = \sum_i h_i \otimes h_i \quad \in \mathfrak{g} \otimes \mathfrak{g},$$

where $(h_i) \subset \mathfrak{g}$ is an orthonormal basis. Let $V = \otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{g}$-modules, $\kappa \in \mathbb{C}^\times$ a nonzero number. The *KZ equations* are the system of differential equations on a V-valued function $I(z_1, \ldots, z_n)$,

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} I, \quad i = 1, \ldots, n,$$

where $\Omega_{i,j} : V \to V$ is the Casimir operator acting in the ith and jth tensor factors; see [KZ, EFK].

This system is a system of Fuchsian first-order linear differential equations. The equations are defined on the complement in $\mathbb{C}^n$ to the union of all diagonal hyperplanes.

The object of our discussion is the following particular case. We consider the following system of differential and algebraic equations for a column 3-vector $I = (I_1, I_2, I_3)$ depending on variables $z = (z_1, z_2, z_3)$:

$$\begin{aligned}
&\frac{\partial I}{\partial z_1} = \frac{1}{2}\Big(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{z_1 - z_3}\Big) I, \quad \frac{\partial I}{\partial z_2} = \frac{1}{2}\Big(\frac{\Omega_{21}}{z_2 - z_1} + \frac{\Omega_{23}}{z_2 - z_3}\Big) I, \\
&\frac{\partial I}{\partial z_3} = \frac{1}{2}\Big(\frac{\Omega_{31}}{z_3 - z_1} + \frac{\Omega_{32}}{z_3 - z_2}\Big) I, \quad 0 = I_1 + I_2 + I_3,
\end{aligned} \tag{6-1}$$

where $\Omega_{ij} = \Omega_{ji}$ and

$$\Omega_{12} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega_{13} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \Omega_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Denote

$$H_1(z) = \frac{1}{2}\Big(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{z_1 - z_3}\Big), \quad H_2(z) = \frac{1}{2}\Big(\frac{\Omega_{21}}{z_2 - z_1} + \frac{\Omega_{23}}{z_2 - z_3}\Big),$$
$$H_3(z) = \frac{1}{2}\Big(\frac{\Omega_{31}}{z_3 - z_1} + \frac{\Omega_{32}}{z_3 - z_2}\Big), \quad \nabla_i^{\mathrm{KZ}} = \frac{\partial}{\partial z_i} - H_i(z), \quad i = 1,2,3.$$

Then the KZ equations can be written as the system of equations,

$$\nabla_i^{\mathrm{KZ}} I = 0, \quad i = 1,2,3, \quad I_1 + I_2 + I_3 = 0.$$

System in Equation (6-1) is the system of KZ equations with parameter $\kappa = 2$ associated with the Lie algebra $\mathfrak{sl}_2$ and the subspace of singular vectors of weight 1 of the tensor power $(\mathbb{C}^2)^{\otimes 3}$ of two-dimensional irreducible $\mathfrak{sl}_2$-modules, up to a gauge transformation; see this example in [V2, Section 1.1].

6.2. Solutions over $\mathbb{C}$. Define the *master function*

$$\Phi(t,z) = (t - z_1)^{-1/2}(t - z_2)^{-1/2}(t - z_3)^{-1/2}$$

and the column 3-vector

$$I^{(C)}(z) = (I_1(z), I_2(z), I_3(z)) := \int_C \Big(\frac{\Phi(t,z)}{t - z_1}, \frac{\Phi(t,z)}{t - z_2}, \frac{\Phi(t,z)}{t - z_3}\Big)\, dt, \tag{6-2}$$

where $C \subset \mathbb{C} - \{z_1, z_2, z_3\}$ is a contour on which the integrand takes its initial value when t encircles C.

THEOREM 6.1 (See [V4]). *The function $I^{(C)}(z)$ is a solution of the system in Equation (6-1).*

This theorem is a very particular case of the results in [SV1].

PROOF. The theorem follows from Stokes' theorem and the two identities:

$$-\frac{1}{2}\Big(\frac{\Phi(t,z)}{t - z_1} + \frac{\Phi(t,z)}{t - z_2} + \frac{\Phi(t,z)}{t - z_3}\Big) = \frac{\partial \Phi}{\partial t}(t,z), \tag{6-3}$$

$$\Big(\frac{\partial}{\partial z_i} - \frac{1}{2}\sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j}\Big)\Big(\frac{\Phi(t,z)}{t - z_1}, \frac{\Phi(t,z)}{t - z_2}, \frac{\Phi(t,z)}{t - z_3}\Big) = \frac{\partial \Psi^i}{\partial t}(t,z), \tag{6-4}$$

where $\Psi^i(t,z)$ is the column 3-vector $(0,\dots,0,-\Phi(t,z)/t - z_i, 0,\dots,0)$ with the nonzero element at the ith place. □

THEOREM 6.2 (See [V1, Equation (1.3)]). *All solutions of the system in Equation (6-1) have this form. Namely, the complex vector space of solutions of the form in Equation (6-2) is 2-dimensional.*

6.3. Solutions as vectors of first derivatives. Consider the elliptic integral

$$T(z) = T^{(C)}(z) = \int_C \Phi(t,z)\, dt.$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Then,

$$I^{(C)}(z) = 2\left(\frac{\partial T^{(C)}}{\partial z_1}, \frac{\partial T^{(C)}}{\partial z_2}, \frac{\partial T^{(C)}}{\partial z_3}\right).$$

Denote $\nabla T = (\partial T/\partial z_1, \partial T/\partial z_2, \partial T/\partial z_3)$. Then the column gradient vector of the function $T(z)$ satisfies the following system of (KZ) equations:

$$\nabla_i^{\mathrm{KZ}} \nabla T = 0, \quad i = 1, 2, 3, \quad \frac{\partial T}{\partial z_1} + \frac{\partial T}{\partial z_2} + \frac{\partial T}{\partial z_3} = 0.$$

This is a system of second-order linear differential equations on the function $T(z)$.

6.4. Solutions modulo p^s. For an integer $s \geqslant 1$, define the master polynomial

$$\Phi_s(t, z) = ((t - z_1)(t - z_2)(t - z_3))^{(p^s-1)/2}.$$

Define the column 3-vector

$$I_s(z) = (I_{s,1}(z), I_{s,2}(z), I_{s,3}(z))$$

as the coefficient of t^{p^s-1} in the polynomial

$$\left(\frac{\Phi_s(t, z)}{t - z_1}, \frac{\Phi_s(t, z)}{t - z_2}, \frac{\Phi_s(t, z)}{t - z_3}\right).$$

THEOREM 6.3 [V4]. *The polynomial $I_s(z)$ is a solution of the system in Equation (6-1) modulo p^s.*

PROOF. We have the following modifications of the identities in Equations (6-3), (6-4):

$$\begin{aligned} \frac{p^s-1}{2}\left(\frac{\Phi_s(t, z)}{t - z_1} + \frac{\Phi_s(t, z)}{t - z_2} + \frac{\Phi_s(t, z)}{t - z_3}\right) &= \frac{\partial \Phi_s}{\partial t}(t, z), \\ \left(\frac{\partial}{\partial z_i} + \frac{p^s-1}{2}\sum_{j\neq i}\frac{\Omega_{i,j}}{z_i - z_j}\right)\left(\frac{\Phi_s(t, z)}{t - z_1}, \frac{\Phi_s(t, z)}{t - z_2}, \frac{\Phi_s(t, z)}{t - z_3}\right) &= \frac{\partial \Psi_s^i}{\partial t}(t, z), \end{aligned} \tag{6-5}$$

where $\Psi_s^i(t, z)$ is the column 3-vector $(0, \ldots, 0, -\Phi_s(t, z)/(t - z_i), 0, \ldots, 0)$ with the nonzero element at the ith place. Theorem 6.3 follows from these identities. □

6.5. p^s-Approximation polynomials of $T(z)$. Define the p^s-approximation polynomial $T_s(z)$ of the elliptic integral $T(z)$ as the coefficient of t^{p^s-1} in the master polynomial $\Phi_s(t, z)$,

$$T_s(z) = (-1)^{(p^s-1)/2} \sum_{k_1+k_2+k_3=(p^s-1)/2} \binom{(p^s-1)/2}{k_1}\binom{(p^s-1)/2}{k_2}\binom{(p^s-1)/2}{k_3} z_1^{k_1} z_2^{k_2} z_3^{k_3}. \tag{6-6}$$

We put $T_0(x) = 1$.

The polynomial $T_s(z_1, z_2, z_3)$ is symmetric with respect to permutations of z_1, z_2, z_3 and

$$T_s(1, z_2, 0) = P_s(z_2),$$

where $P_s(x)$ is defined in Equation (4-3). The gradient vector

$$\nabla T_s := (\partial T_s/\partial z_1, \partial T_s/\partial z_2, \partial T_s/\partial z_3)$$

of the p^s-approximation polynomial $T_s(z)$ is a solution modulo p^s of the system in Equation (6-1) since

$$\nabla T_s = \frac{1-p^s}{2}\,(I_{s,1}(z), I_{s,2}(z), I_{s,3}(z)).$$

LEMMA 6.4. *For $s \geqslant 1$, the master polynomials satisfy the baby congruences,*

$$\Phi_{s+1}(t,z)\Phi_{s-1}(t^p, z_1^p, z_2^p, z_3^p) \equiv \Phi_s(t,z)\Phi_s(t^p, z_1^p, z_2^p, z_3^p) \pmod{p^s}.$$

THEOREM 6.5. *For $s \geqslant 1$,*

$$T_{s+1}(z_1, z_2, z_3)\, T_{s-1}(z_1^p, z_2^p, z_3^p) \;\equiv\; T_s(z_1, z_2, z_3)\, T_s(z_1^p, z_2^p, z_3^p) \pmod{p^s}. \tag{6-7}$$

PROOF. Let $h(t,z) = t^{1-p}((t-z_1)(t-z_2)(t-z_3))^{(p-1)/2}$. Then,

$$T_s(z) = \mathrm{CT}_t[h(t,z)h(t,z)^p \cdots h(t,z)^{p^{s-1}}].$$

The tuple of Laurent polynomials $(h(t,z), h(t,z), \ldots)$ is admissible in the sense of Definition 2.6. Now the application of Theorem 2.9 gives the congruence in Equation (6-7). □

6.6. Limits of $T_s(z)$. Denote $\bar{T}_s(z) := (-1)^{(p^s-1)/2} T_s(z)$; see Equation (6-6),

$$\mathfrak{D} = \{(z_1, z_2, z_3) \in \mathbb{Z}_p^3 \,\|\bar{T}_1(z_1, z_2, z_3)|_p = 1\}. \tag{6-8}$$

Notice that $\mathbb{Z}_p^3 = \bigcup_{\alpha,\beta,\gamma\in\mathbb{F}_p} D_{\alpha,1} \times D_{\beta,1} \times D_{\gamma,1}$. Since $\bar{T}_1(z)$ has coefficients in $\mathbb{Z}_p$,

$$\mathfrak{D} = \bigcup{}^{o} D_{\alpha,1} \times D_{\beta,1} \times D_{\gamma,1},$$

where the summation $\bigcup^o$ is over all $\alpha, \beta, \gamma \in \mathbb{F}_p$ such that $|T(\omega(\alpha), \omega(\beta), \omega(\gamma))|_p = 1$. For the same reason,

$$\mathfrak{D} = \{(z_1, z_2, z_3) \in \mathbb{Z}_p^3 \mid |\bar{T}_1(z_1^p, z_2^p, z_3^p)|_p = 1\}.$$

Denote

$$\mathfrak{E} = \{(1, z_2, 0) \in \mathbb{Z}_p^3 \mid |z_2|_p < 1\}.$$

LEMMA 6.6. *We have $\mathfrak{E} \subset D_{1,1} \times D_{0,1} \times D_{0,1} \subset \mathfrak{D}$.*

PROOF. The first inclusion is clear. The second inclusion follows from the equality $\bar{T}_s(1, 0, 0) = 1$. □

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

THEOREM 6.7. *For $s \geqslant 1$, the rational function $\bar{T}_{s+1}(z)/\bar{T}_s(z^p)$ is regular on $\mathfrak{D}$. The sequence $(\bar{T}_{s+1}(z)/\bar{T}_s(z^p))_{s\geqslant 1}$ uniformly converges on $\mathfrak{D}$. The limiting analytic function $f(z)$, restricted to $\mathfrak{E}$, equals the ratio $F(z_2)/F(z_2^p)$, where $F(z_2) := {}_2F_1(1/2, 1/2; 1; z_2)$ is defined by the convergent power series in Equation (4-1). We also have $|f(z)|_p = 1$ for every $z \in \mathfrak{D}$.*

PROOF. Similarly to the proof of Theorem 4.6, we have $\bar{T}_s(z) \equiv \bar{T}_1(z)\bar{T}_1(z^p)\cdots\bar{T}_1(x^{p^{s-1}})$ (mod p). Hence, $|\bar{T}_s(z)|_p = |\bar{T}_s(z^p)|_p = 1$ for $s \geqslant 1$, $z \in \mathfrak{D}$. Hence, the rational functions $\bar{T}_{s+1}(z)/\bar{T}_s(z^p)$ are regular on $\mathfrak{D}$.

The congruence in Equation (6-7) implies that

$$\left|\frac{\bar{T}_{s+1}(z)}{\bar{T}_s(z^p)} - \frac{\bar{T}_s(z)}{\bar{T}_{s-1}(z^p)}\right|_p \leqslant p^{-s} \quad \text{for } z \in \mathfrak{D}.$$

This shows the uniform convergence of $(\bar{T}_{s+1}(z)/\bar{T}_s(z^p))_{s\geqslant 1}$ on $\mathfrak{D}$. For the limiting function $f(z)$, we have $|f(z)|_p = 1$ for $z \in \mathfrak{D}$.

We have $\bar{T}_s(1, z_2, 0) = \bar{P}_s(z_2) = \sum_k \binom{(p^s-1)/2}{k}^2 z_2^k$. Clearly, for any fixed index k, the coefficient $\binom{(p^s-1)/2}{k}^2$ of z_2^k in $\bar{T}_s(1, z_2, 0)$ converges p-adically to the coefficient $\binom{-1/2}{k}^2$ of z_2^k in $F(z_2)$. Hence, the sequence $(\bar{T}_s(1, z_2, 0))_{s\geqslant 1}$ converges to $F(z_2)$ on $\mathfrak{E}$, so that $f(1, z_2, 0) = F(z_2)/F(z^p)$ on $\mathfrak{E}$. The theorem is proved. □

REMARK 6.8. The analytic function $f(z)$ of Theorem 6.10 exhibits behavior very different from the behavior of the corresponding ratio $T^{(C)}(z)/T^{(C)}(z^p)$ of complex elliptic integrals.

By Theorem 6.10, the function $f(z)$ restricted to the one-dimensional discs $\{(z_1, 0, 1) \in \mathbb{Z}_p^3 \mid |z_1|_p < 1\}$, $\{(1, z_2, 0) \in \mathbb{Z}_p^3 \mid |z_2|_p < 1\}$, $\{(0, 1, z_3) \in \mathbb{Z}_p^3 \mid |z_3|_p < 1\}$ equals $F(z_1)/F(z_1^p)$, $F(z_2)/F(z_2^p)$, $F(z_3)/F(z_3^p)$, respectively.

In the complex case, for the ratio $T^{(C_1)}(z)/T^{(C_1)}(z^p)$ to be equal to $F(z_1)/F(z_1^p)$ on $\{(z_1, 0, 1) \in \mathbb{C}^3 \mid |z_1| < 1\}$, the contour C_1 must be the cycle on the elliptic curve $y^2 = (t - z_1)t(t - 1)$ vanishing at $z_1 = 0$. Similarly, for $T^{(C_2)}(z)/T^{(C_2)}(z^p)$ to be equal to $F(z_2)/F(z_2^p)$ on $\{(1, z_2, 0) \in \mathbb{C}^3 \mid |z_2| < 1\}$, the contour C_2 must be the cycle on the elliptic curve $y^2 = (t - 1)(t - z_2)t$ vanishing at $z_2 = 0$, and for $T^{(C_3)}(z)/T^{(C_3)}(z^p)$ to be equal to $F(z_3)/F(z_3^p)$ on $\{(0, 1, z_3) \in \mathbb{C}^3 \mid |z_3| < 1\}$, the contour C_3 must be the cycle on the elliptic curve $y^2 = t(t - 1)(t - z_3)$ vanishing at $z_3 = 0$. However, these three local complex analytic functions are not restrictions of a single univalued complex analytic function due to the irreducibility of the monodromy representation of the Gauss–Manin connection associated with the family of elliptic curves $y^2 = (t - z_1)(t - z_2)(t - z_3)$.

For $i, j \in \{1, 2, 3\}$ and $s \geqslant 1$, denote

$$f_s(z) = T_s(z)/T_{s-1}(z^p), \quad \eta_s^{(i)}(z) = \frac{\partial T_s}{\partial z_i}(z)/T_s(z), \quad \eta_s^{(ij)}(z) = \frac{\partial^2 T_s}{\partial z_i \partial z_j}(z)/T_s(z).$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

THEOREM 6.9. *For $s \geqslant 1$, the rational functions $\eta_s^{(i)}(z)$ and $\eta_s^{(ij)}(z)$ are regular on $\mathfrak{D}$. The sequences of rational functions $(\eta_s^{(i)}(z))_{s\geqslant 1}$ and $(\eta_s^{(ij)}(z))_{\geqslant 1}$ converge uniformly on $\mathfrak{D}$ to analytic functions. If $\eta^{(j)}$ and $\eta^{(ij)}$ denote the corresponding limits, then*

$$\eta^{(1)} + \eta^{(2)} + \eta^{(3)} = 0, \tag{6-9}$$

$$\eta^{(j1)} + \eta^{(j2)} + \eta^{(j3)} = 0, \quad j = 1, 2, 3, \tag{6-10}$$

$$\frac{\partial}{\partial z_j}\eta^{(i)} = \eta^{(ji)} - \eta^{(i)}\eta^{(j)}. \tag{6-11}$$

PROOF. Denote $\delta_i = z_i(\partial/\partial z_i)$. By Theorem 6.7, the sequence (f_s) uniformly converges to the analytic function f on $\mathfrak{D}$. Therefore, the sequence of the derivatives $((\partial/\partial z_i)f_s)$ uniformly converges on $\mathfrak{D}$ to $(\partial/\partial z_i)f$. Hence, the sequence $((\delta_i f_s)/f_s)$ uniformly converges on $\mathfrak{D}$ to the function $(\delta_i f)/f$. At the same time,

$$\frac{\delta_i f_s}{f_s}(z) = \frac{\delta_i T_s}{T_s}(z) - p\,\frac{\delta_i T_{s-1}}{T_{s-1}}(z^p)$$

and, more generally,

$$\frac{\delta_i f_{s-k}}{f_{s-k}}(z^{p^k}) = \frac{\delta_i T_{s-k}}{T_{s-k}}(z^{p^k}) - p\,\frac{\delta_i T_{s-k-1}}{T_{s-k-1}}(z^{p^{k+1}}) \quad \text{for } k = 0, 1, \ldots, s.$$

Summing the relations up with suitable weights to get telescoping, we obtain, for any $r \leqslant s$,

$$\sum_{k=0}^{r-1} p^k\,\frac{\delta_i f_{s-k}}{f_{s-k}}(z^{p^k}) \;=\; \frac{\delta_i T_s}{T_s}(z) \;-\; p^r\,\frac{\delta_i T_{s-r}}{T_{s-r}}(z^{p^r}).$$

Choosing $r = [s/2]$ and taking the limit as $s \to \infty$ on both sides,

$$\sum_{k=0}^{\infty} p^k\,\frac{\delta_i f}{f}(z^{p^k}) = \lim_{s\to\infty}\frac{\delta_i T_s}{T_s}(z).$$

The series on the left uniformly converges on $\mathfrak{D}$. Hence, there exists the limit on the right-hand side. This means that

$$\eta^{(i)}(x) = \lim_{s\to\infty}\frac{\frac{\partial}{\partial z_i}T_s}{T_s}(z) = \frac{1}{z_i}\sum_{k=0}^{\infty} p^k\,\frac{\delta_i f}{f}(z^{p^k}).$$

One can further differentiate the resulting equality with respect to any of the variables z_1, z_2, z_3 to get, by induction, formulas for $\eta^{(ij)}$ and more generally for $\eta^{(ijk\ldots)}$. Note that Equation (6-11) comes out from differentiating logarithmic derivatives.

Equations (6-9) and (6-10) follow from Equation (6-5). The theorem is proved. □

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

THEOREM 6.10. *We have the following system of equations on* $\mathfrak{D}$:

$$\begin{pmatrix}\eta^{(11)}\\ \eta^{(12)}\\ \eta^{(13)}\end{pmatrix}=\frac{1}{2}\Big(\frac{\Omega_{12}}{z_1-z_2}+\frac{\Omega_{13}}{z_1-z_3}\Big)\begin{pmatrix}\eta^{(1)}\\ \eta^{(2)}\\ \eta^{(3)}\end{pmatrix},\quad \begin{pmatrix}\eta^{(21)}\\ \eta^{(22)}\\ \eta^{(23)}\end{pmatrix}=\frac{1}{2}\Big(\frac{\Omega_{21}}{z_2-z_1}+\frac{\Omega_{23}}{z_2-z_3}\Big)\begin{pmatrix}\eta^{(1)}\\ \eta^{(2)}\\ \eta^{(3)}\end{pmatrix},$$
$$\begin{pmatrix}\eta^{(31)}\\ \eta^{(32)}\\ \eta^{(33)}\end{pmatrix}=\frac{1}{2}\Big(\frac{\Omega_{31}}{z_3-z_1}+\frac{\Omega_{33}}{z_3-z_2}\Big)\begin{pmatrix}\eta^{(1)}\\ \eta^{(2)}\\ \eta^{(3)}\end{pmatrix},\quad \eta^{(1)}+\eta^{(2)}+\eta^{(3)}=0. \tag{6-12}$$

PROOF. The theorem follows from Theorems 6.3 and 6.10. □

THEOREM 6.11. *The column vector*

$$\vec{\eta}(z):=(\eta^{(1)}(z),\eta^{(2)}(z),\eta^{(3)}(z))$$

is nonzero at every point $z\in\mathfrak{D}$.

PROOF. On the one hand, if $\vec{\eta}(a)=0$ for some $a\in\mathfrak{D}$, then all derivatives of $\vec{\eta}(z)$ at a are equal to zero. This follows from the first three equations in Equation (6-12) written as

$$\frac{\partial}{\partial z_i}\vec{\eta}=(H_i-\eta^{(i)})\,\vec{\eta},\quad i=1,2,3. \tag{6-13}$$

Hence, $\vec{\eta}(z)$ equals zero identically on $\mathfrak{D}$. On the other hand, $\eta^{(2)}(1,0,0)=F'(0)/F(0)=1/4$ by Theorem 6.7. This contradiction implies the theorem. □

6.7. Subbundle $\mathcal{L}\to\mathfrak{D}$. Denote $W=\{(I_1,I_2,I_3)\in\mathbb{Q}_p^3\mid I_1+I_2+I_3=0\}$. The differential operators ∇_i^{KZ}, $i=1,2,3$, define a connection on the trivial bundle $W\times\mathfrak{D}\to\mathfrak{D}$, called the KZ connection. The KZ connection is flat,

$$[\nabla_i^{\mathrm{KZ}},\nabla_j^{\mathrm{KZ}}]=0\quad\text{for all } i,j.$$

The flat sections of the KZ connection are solutions of the system in Equation (6-1) of KZ equations.

For any $a\in\mathfrak{D}$, let $\mathcal{L}_a\subset W$ be the one-dimensional vector subspace generated by $\vec{\eta}(a)$. Then,

$$\mathcal{L}:=\bigcup_{a\in\mathfrak{D}}\mathcal{L}_a\to\mathfrak{D}$$

is an analytic line subbundle of the trivial bundle $W\times\mathfrak{D}\to\mathfrak{D}$.

THEOREM 6.12. *The subbundle $\mathcal{L}\to\mathfrak{D}$ is invariant with respect to the KZ connection. In other words, if $s(z)$ is any section of $\mathcal{L}\to\mathfrak{D}$, then the sections $\nabla_i s(z)$, $i=1,2,3$, also are sections of $\mathcal{L}\to\mathfrak{D}$.*

PROOF. The theorem follows from Equation (6-13). □

REMARK 6.13. For any $a\in\mathfrak{D}$, we may find locally a scalar analytic function $u(z)$ such that $u(z)\cdot\vec{\eta}(z)$ is a solution of the KZ equations in Equation (6-1). Such a function

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

is a solution of the system of equations $\partial u/\partial z_i = -\eta^{(i)}u$, $i = 1, 2, 3$. This system is compatible since $\partial\eta^{(j)}/\partial z_i = \eta^{(ij)} - \eta^{(i)}\eta^{(j)} = \partial\eta^{(i)}/\partial z_j$.

REMARK 6.14. The corresponding complex KZ connection does not have invariant line subbundles due to irreducibility of the monodromy of the KZ connection, which in our case is the Gauss–Manin connection of the family $y^2 = (t - z_1)(t - z_2)(t - z_3)$. Thus, the existence of the KZ invariant line subbundle $\mathcal{L} \to \mathfrak{D}$ is a pure p-adic feature.

REMARK 6.15. The invariant subbundles of the KZ connection over $\mathbb{C}$ usually are related to some additional conformal block constructions; see [FSV, SV2, V3]. Apparently, the subbundle $\mathcal{L} \to \mathfrak{D}$ is of a different p-adic nature; see [V4].

REMARK 6.16. Following Dwork, we may expect that locally at any point $a \in \mathfrak{D}$, the solutions of the KZ equations of the form $u(z) \cdot \vec{\eta}(z)$, where $u(z)$ is a scalar function, are given at a by power series in $z_i - a_i$, $i = 1, 2, 3$, bounded in their polydiscs of convergence, while any other local solution at a is given by a power series unbounded in its polydisc of convergence; see [Dw] and [V4, Theorem A.4].

6.8. Other definitions of subbundle $\mathcal{L} \to \mathfrak{D}$.

6.8.1. Line subbundle $\mathcal{M} \to \mathfrak{D}$. Define a polynomial $U_s(z)$ as the coefficient of t^{p^s-1} in the master polynomial $\Phi_s(t + z_3, z) = ((t - (z_1 - z_3))(t - (z_2 - z_3))t)^{(p^s-1)/2}$. It is easy to see that $U_1(z) = T_1(z) \pmod{p}$. Similarly to Theorem 6.5, we conclude that

$$U_{s+1}(z_1, z_2, z_3)\, U_{s-1}(z_1^p, z_2^p, z_3^p) \equiv U_s(z_1, z_2, z_3)\, U_s(z_1^p, z_2^p, z_3^p) \pmod{p^s}.$$

Hence, the sequence $(U_{s+1}(z)/U_s(z^p))_{s\geqslant 1}$ uniformly converges to an analytic function on the domain $\mathfrak{D}$ defined in Equation (6-8). The vector-valued polynomial $\nabla U_s(z) = (\partial U_s/\partial z_1, \partial U_s/\partial z_2, \partial U_s/\partial z_3)$ is a solution modulo p^s of the KZ equations in Equation (6-1); see [V4, Theorem 9.1], and see the proof of Theorem 6.3. Consider the function

$$\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) := \lim_{s\to\infty} \frac{\nabla U_s}{U_s}$$

defined on the same domain $\mathfrak{D}$. Similarly to the proofs of Theorems 6.9–6.12, we conclude that the function $\vec{\mu}(z)$ is nonzero on $\mathfrak{D}$ and its values span an analytic line subbundle

$$\mathcal{M} := \bigcup_{a\in\mathfrak{D}} \mathcal{M}_a \to \mathfrak{D}$$

of the trivial bundle $W \times \mathfrak{D} \to \mathfrak{D}$; here, $\mathcal{M}_a \subset W$ is the one-dimensional subspace generated by $\vec{\mu}(a)$. The line subbundle $\mathcal{M} \to \mathfrak{D}$ is invariant with respect to the KZ connection.

THEOREM 6.17. *The line bundles $\mathcal{M} \to \mathfrak{D}$ and $\mathcal{L} \to \mathfrak{D}$ coincide.*

PROOF. The proof rests on the following two lemmas.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

LEMMA 6.18. *The line bundles $\mathcal{M} \to \mathfrak{D}$ and $\mathcal{L} \to \mathfrak{D}$ coincide if there is $a \in \mathfrak{D}$ such that $\mathcal{M}_a = \mathcal{L}_a$.* □

PROOF. Let $\mathcal{M}_a = \mathcal{L}_a$ for some $a \in \mathfrak{D}$. Then, $\mathcal{M}_z = \mathcal{L}_s$ in some neighborhood of a, since locally the subbundles are generated by the values of the solutions with the same initial condition at $z = a$. Hence, $\mathcal{M}_z = \mathcal{L}_z$ on $\mathfrak{D}$. □

LEMMA 6.19. *For $i = 1, 2, 3$, the functions $\partial T_s/\partial z_i(z)/T_s(z)$ and $\partial U_s/\partial z_i(z)/U_s(z)$ are equal on the line $z_1 = 1$, $z_3 = 0$.*

Hence, $\mathcal{M} = \mathcal{L}$ over the points of that line and, therefore, $\mathcal{M}_z = \mathcal{L}_z$ for $z \in \mathfrak{D}$.

6.8.2. Line subbundle $\mathcal{N} \to \hat{\mathfrak{D}}$. Let $\omega(x) = F'(x)/F(x)$, where $F(x)$ is defined as ${}_2F_1(1/2, 1/2; 1; x)$. We have $\omega(x) = \lim_{s\to\infty}(P_s'(x)/P_s(x))$ on $D_{0,1}$. Introduce new variables

$$u_1 = z_1 - z_3, \quad u_2 = \frac{z_2 - z_3}{z_1 - z_3}, \quad u_3 = z_1 + z_2 + z_3,$$

and a vector-valued function

$$\vec{\omega}(u) = \frac{1}{u_1}(-1/2 - \omega(u_2)u_2,\ \omega(u_2),\ 1/2 + \omega(u_2)(u_2 - 1)).$$

Define

$$\hat{\mathfrak{D}}_0 = \{(z_1, z_2, z_3) \in \mathbb{Q}_p^3 \mid z_i \neq z_j \quad \text{for all } i \neq j\}.$$

For any $\sigma = (i, j, k) \in S_3$, define

$$\hat{\mathfrak{D}}_1^\sigma = \left\{(z_1, z_2, z_3) \in \hat{\mathfrak{D}}_0 \;\middle|\; \frac{z_j - z_k}{z_i - z_k} \in \mathbb{Z}_p,\ \left|g\left(\frac{z_j - z_k}{z_i - z_k}\right)\right|_p = 1\right\},$$

$$\hat{\mathfrak{D}}_2^\sigma = \left\{(z_1, z_2, z_3) \in \hat{\mathfrak{D}}_0 \;\middle|\; \frac{z_i - z_k}{z_j - z_k} \in \hat{\mathfrak{D}}_1^\sigma\right\}, \quad \hat{\mathfrak{D}}^\sigma = \hat{\mathfrak{D}}_1^\sigma \cup \hat{\mathfrak{D}}_2^\sigma, \quad \hat{\mathfrak{D}} = \sum_{\sigma\in S_3} \hat{\mathfrak{D}}^\sigma,$$

where $g(\lambda)$ is the Igusa polynomial in Equation (4-13).

Using Dwork's results in [Dw], it is shown in [V4, Appendix] that the values of the analytic continuation of the function $\vec{\omega}(u)$ generate a line bundle $\mathcal{N} \to \hat{\mathfrak{D}}$ invariant with respect to the KZ connection.

THEOREM 6.20. *The line bundles $\mathcal{M} \to \mathfrak{D}$ and $\mathcal{N} \to \hat{\mathfrak{D}}$ coincide on $\mathfrak{D} \cap \hat{\mathfrak{D}}$.*

Thus, we identify the line bundles $\mathcal{L} \to \mathfrak{D}$, $\mathcal{M} \to \mathfrak{D}$, and $\mathcal{N} \to \hat{\mathfrak{D}}$ over $\mathfrak{D} \cap \hat{\mathfrak{D}}$.

PROOF. We have

$$U_s(z) = (z_1 - z_3)^{(p^s-1)/2} P_s\left(\frac{z_2 - z_3}{z_1 - z_3}\right), \quad \frac{\partial U_s}{\partial z_2} = (z_1 - z_3)^{(p^s-1)/2-1} P_s'\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

$$\frac{\partial U_s}{\partial z_1} = \frac{p^s - 1}{2} \operatorname{const}(z_1 - z_3)^{(p^s-1)/2-1} P_s\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$
$$- \operatorname{const}(z_1 - z_3)^{(p^s-1)/2-1} \frac{z_2 - z_3}{z_1 - z_3} P_s'\left(\frac{z_2 - z_3}{z_1 - z_3}\right).$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

Hence,

$$\frac{1}{U_s(z)}\Big(\frac{\partial U_s}{\partial z_1},\frac{\partial U_s}{\partial z_2},\frac{\partial U_s}{\partial z_3}\Big)=\frac{1}{u_1}\Big(\frac{p^s-1}{2}-u_2\frac{P_s'(u_2)}{P_s(u_2)},\frac{P_s'(u_2)}{P_s(u_2)},-\frac{p^s-1}{2}+(u_2-1)\frac{P_s'(u_2)}{P_s(u_2)}\Big).$$

Clearly, the limit of this vector equals $\vec{\omega}(u)$ as $s\to\infty$. The theorem is proved. □

7. Concluding remarks

7.1. Conjectural stronger congruences for $\overline{P}_s(x)$. By Theorem 4.2, we have for polynomials $\bar{P}_s(x):=(-1)^{(p^s-1)/2}P_s(x)$,

$$\bar{P}_4(x)\bar{P}_2(x^p)-\bar{P}_3(x)\bar{P}_3(x^p)\equiv 0 \pmod{p^3}.$$

In particular, for the coefficient of $x^{N_0+N_1p+N_2p^2+N_3p^3}$ in $\bar{P}_4(x)\bar{P}_2(x^p)-\bar{P}_3(x)\bar{P}_3(x^p)$,

$$\sum_{\substack{k_1+l_1=N_1\\k_2+l_2=N_2}}\Bigg(\binom{(p^4-1)/2}{N_0+k_1p+k_2p^2+N_3p^3}^2\binom{(p^2-1)/2}{l_1+l_2p}^2 \\ -\binom{(p^3-1)/2}{N_0+k_1p+k_2p^2}^2\binom{(p^3-1)/2}{l_1+l_2p+N_3p^2}^2\Bigg)\equiv 0 \pmod{p^3}.$$

Computer experiments show that this sum can be split into subsums with at most four terms so that each subsum is divisible by p^3. More precisely, let $0\leqslant a,b,\ c,c',d,d'\leqslant p-1$ be integers. Define

$$A(a,b;c,c';d,d')=\binom{(p^4-1)/2}{a+cp+dp^2+bp^3}^2\binom{(p^2-1)/2}{c'+d'p}^2-\binom{(p^3-1)/2}{a+cp+dp^2}^2\binom{(p^3-1)/2}{c'+d'p+bp^2}^2$$

and

$$\begin{aligned}B(a,b;c,c';d,d')&=\operatorname{Sym}A(a,b;c,c';d,d')\\&:=A(a,b;c,c';d,d')+A(a,b;c',c;d,d')+A(a,b;c,c';d',d)+A(a,b;c',c;d',d).\end{aligned} \tag{7-1}$$

We expect that the integer $B(a,b;c,c';d,d')$ is divisible by p^3.

More generally, define

$$k=(k^{(1)},\dots,k^{(s)}),\qquad k^{(i)}=(k_1^{(i)},k_2^{(i)}),$$

$$A(a,b;k)=\binom{(p^{s+2}-1)/2}{a+\sum_{i=1}^s k_1^{(i)}p^i+bp^{s+1}}^2\binom{(p^s-1)/2}{\sum_{i=1}^s k_2^{(i)}p^{i-1}}^2-\binom{(p^{s+1}-1)/2}{a+\sum_{i=1}^s k_1^{(i)}p^i}^2\binom{(p^{s+1}-1)/2}{\sum_{i=1}^s k_2^{(i)}p^{i-1}+bp^s}^2.$$

Set

$$B(a,b;k)=\operatorname{Sym}A(a,b;k),$$

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

where Sym denotes the symmetrization with respect to the index j in $k_j^{(i)}$ in each group $k^{(i)} = (k_1^{(i)}, k_2^{(i)})$. Thus, the symmetrization has 2^s summands; the case $s = 2$ of this symmetrization is displayed in Equation (7-1).

CONJECTURE 7.1. The integer $B(a, b; k)$ is divisible by p^{s+1}.

This conjecture is supported by computer experiments and is checked for $s = 1$ using [Gr].

7.2. Papers [BV, Vl]. After reading this paper, Masha Vlasenko was able to invent a new proof of our congruences in Theorems 4.2 and 5.1 (private communication). Her proof was based on the results of [BV, Vl].

Acknowledgements

The authors thank Frits Beukers, Andrew Granville, Anton Mellit, Richárd Rimányi, Steven Sperber, and Masha Vlasenko for useful discussions.

Support in part by NSF grant DMS-1954266 is acknowledged.

References

[AS] A. Adolphson and S. Sperber, 'A-hypergeometric series and a p-adic refinement of the Hasse–Witt matrix', *Abh. Math. Semin. Univ. Hambg.* **91** (2021), 225–256.

[BV] F. Beukers and M. Vlasenko, 'Dwork crystals. I', *Int. Math. Res. Not. IMRN* **2021**(12) (2021), 8807–8844; II, Int. Math. Res. Not. IMRN **2021**(6) (2021), 4427–4444.

[Dw] B. Dwork, 'p-adic cycles', *Publ. Math. Inst. Hautes Études Sci.* **37** (1969), 27–115

[EFK] P. Etingof, I. Frenkel and A. Kirillov, *Lectures on Representation Theory and Knizhnik–Zamolodchikov Equations*, Mathematical Surveys and Monographs, 58 (American Mathematical Society, Providence, RI, 1998), xiv+198 pages.

[FSV] B. Feigin, V. Schechtman and A. Varchenko, 'On algebraic equations satisfied by hypergeometric correlators in WZW models. I', *Comm. Math. Phys.* **163** (1994), 173–184; II, *Comm. Math. Phys.* **70** (1995), 219–247.

[Gr] A. Granville, 'Binomial coefficients modulo prime powers', Preprint, 1–24; https://dms.umontreal.ca/~andrew/Binomial/.

[Ig] J. Igusa, 'Class number of a definite quaternion with prime discriminant', *Proc. Natl. Acad. Sci. USA* **44**(4) (1958), 312–314.

[KZ] V. Knizhnik and A. Zamolodchikov, 'Current algebra and the Wess–Zumino model in two dimensions', *Nucl. Phys.* **B247** (1984), 83–103.

[LTYZ] L. Long, F.-T. Tu, N. Yui and W. Zudilin, 'Supercongruences for rigid hypergeometric Calabi–Yau threefolds', *Adv. Math.* **393** (2021), 108058.

[MO] D. Maulik and A. Okounkov, 'Quantum groups and quantum cohomology', *Astérisque* **408** (2019), 1–277; doi:10.24033/ast.1074.

[Me] A. Mellit, 'A proof of Dwork's congruences', unpublished (October 20, 2009), 1–3.

[MV] A. Mellit and M. Vlasenko, 'Dwork's congruences for the constant terms of powers of a Laurent polynomial', *Int. J. Number Theory* **12**(2) (2016), 313–321.

[SvS] K. Samol and D. van Straten, 'Dwork congruences and reflexive polytopes', *Ann. Math. Qué.* **39**(2) (2015), 185–203.

[SV1] V. Schechtman and A. Varchenko, 'Arrangements of hyperplanes and Lie algebra homology', *Invent. Math.* **106** (1991), 139–194.

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press

[SV2] V. Schechtman and A. Varchenko, 'Solutions of KZ differential equations modulo p', *Ramanujan J.* **48**(3) (2019), 655–683; doi:10.1007/s11139-018-0068-x.

[V1] A. Varchenko, 'The Euler beta-function, the Vandermonde determinant, Legendre's equation and critical values of linear functions on a configuration of hyperplanes. I', *Izv. Akademii Nauk USSR, Seriya Mat.* **53**(6) (1989), 1206–1235; II, *Izv. Akademii Nauk USSR, Seriya Mat.* **54**(1) (1990), 146–158.

[V2] A. Varchenko, *Special Functions, KZ Type Equations, and Representation Theory*, CBMS Regional Conference Series in Mathematics, 98 (American Mathematical Society, Providence, RI, 2003), viii+118 pages.

[V3] A. Varchenko, 'An invariant subbundle of the KZ connection mod p and reducibility of $\widehat{\mathfrak{sl}_2}$ Verma modules mod p', *Math. Notes* **109**(3) (2021), 386–397.

[V4] A. Varchenko, 'Notes on solutions of KZ equations modulo p^s and p-adic limit $s \to \infty$', *Contemp. Math.* **780** (2022), 309–347, with Appendix written jointly with S. Sperber; doi:10.1090/conm/780/15695.

[Vl] M. Vlasenko, 'Higher Hasse–Witt matrices', *Indag. Math.* **29** (2018), 1411–1424.

ALEXANDER VARCHENKO, Department of Mathematics,
University of North Carolina at Chapel Hill,
Chapel Hill, NC 27599-3250, USA
e-mail: anv@email.unc.edu

WADIM ZUDILIN, Institute for Mathematics, Astrophysics and Particle Physics,
Radboud University, PO Box 9010, 6500 GL Nijmegen, The Netherlands
e-mail: w.zudilin@math.ru.nl

https://doi.org/10.1017/S1446788723000083 Published online by Cambridge University Press