# RINGS SATISFYING $(x, y, z) = (y, z, x)$

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**1. Introduction.** Let $R$ be a ring satisfying the identity

$$(x, y, z) = (y, z, x) \tag{1}$$

for all $x, y, z \in R$, where $(x, y, z) = (xy)z - x(yz)$. If $R$ also satisfies the identity $(x, x, x) = 0$ for all $x \in R$, then $R$ is alternative. It is known that if $R$ satisfies (1), it need not be an alternative (see **6**). Thus, the class of rings satisfying (1) is a non-trivial extension of the class of alternative rings. P. Jordan remarked that $(x, x, x)^2 = 0$ is an identity in $R$ (see **9**). Outcalt strengthened this remark by proving that $(y, x, x)^2 = 0$ for all $x, y \in R$. He went on to show that if $R$ is a simple or primitive ring satisfying (1), with characteristic prime to 2 and 3, then $R$ is alternative (see **7**), and hence associative or a Cayley-Dickson algebra (see **2–5**).

In this paper we prove that a ring $R$ satisfying (1) with the same restrictions on its characteristic and containing no non-zero ideal whose square is zero must be alternative.

A prime ring has no non-zero ideal whose square is zero. Since arbitrary simple rings and primitive rings are prime, the latter result proved by San Soucie (**8**), this paper extends Outcalt's results.

**2. Preliminary results and identities.** Let $R$ be a ring satisfying (1) with characteristic prime to 2 and 3. We first introduce the $v$-function of Outcalt. For $x \in R$, define $v^n(a)$, where $a \in R$, as follows:

$$v(a) = (a, x, x), \qquad v^k(a) = v(v^{k-1}(a)).$$

It is clear that $v^n(v^m(a)) = v^m(v^n(a)) = v^{m+n}(a)$ and $v^n(a + b) = v^n(a) + v^n(b)$ for all $a, b \in R$. The $v$-function depends on $x$ as well as on $a$ but it will be clear from context when we use this fact. It was mentioned in the introduction that $(a, x, x)^2 = 0$ for all $a, x \in R$. In terms of the $v$-function this identity becomes

$$(v(a))^2 = 0 \tag{2}$$

for all $a \in R$. Linearizing (2) yields

$$E(a, b) = v(a)v(b) + v(b)v(a) = 0$$

for all $a, b \in R$. Next, using (1) and (2), we have

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$$3(v(a), v(a), v(b)) = -v(a)[v(a)v(b)] + [v(a)v(b)]v(a) - v(a)[v(b)v(a)] + [v(b)v(a)]v(a) = E(a, b)v(a) - v(a)E(a, b) = 0$$

and therefore

$$(v(a), v(a), v(b)) = 0.$$

When linearized, this equation becomes

$$(v(a), v(c), v(b)) + (v(c), v(a), v(b)) = 0$$

for all $a, b, c \in R$. Then

$$0 = v(b)E(a, c) - E(a, b)v(c) + (v(a), v(b), v(c)) + (v(b), v(a), v(c))$$

which, when expanded, gives us

$$v(a)[v(b)v(c)] = v(b)[v(c)v(a)]. \tag{3}$$

This identity appears in (**7**, p. 135). Outcalt derived the following identities for arbitrary $a, b, x \in R$:

$$F(a, b) = 3v(ab) - av(b) - 2v(b)a - v(a)b - 2bv(a) = 0,$$

$$G(a, b) = 9v^2(ab) - 5av^2(b) - 4v^2(b)a - 5v^2(a)b - 4bv^2(a) - 2v(a)v(b) = 0,$$

$$I(a, b) = 5v^2(ab) - 4v^2(ba) - av^2(b) - v^2(a)b - 2v(a)v(b) = 0,$$

$$J(a, b) = 27v^3(ab) - 13av^3(b) - 14v^3(b)a - 13v^3(a)b - 14bv^3(a) - 3v^2(b)v(a) - 3v(b)v^2(a) = 0,$$

$$9v^3(a) = (v(x), v(x), a). \tag{4}$$

We shall also make use of the following identity which is satisfied by all $x, y, z$ in an arbitrary ring, where $(x, y) = xy - yx$:

$$C(x, y, z) = (xy, z) - x(y, z) - (x, z)y - (x, y, z) + (x, z, y) - (z, x, y) = 0.$$

The $f$-function, which follows, was introduced in (**1**):

$$f(w, x, y, z) = (wx, y, z) - x(w, y, z) - (x, y, z)w,$$

where $w, x, y, z, \in R$. We shall need a property of the $f$-function which is derived from the Teichmüller identity:

$$h(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0$$

for all $w, x, y, z$ in an arbitrary ring $R$. Expanding $f$ and $h$ and collecting terms we obtain

$$f(w, x, y, z) - h(w, x, y, z) - h(x, y, z, w) = -(zw, x, y) + w(z, x, y) + (w, x, y)z.$$

Since $f(z, w, x, y) = (zw, x, y) - w(z, x, y) - (w, x, y)z$, we conclude that

$$f(w, x, y, z) = -f(z, w, x, y) \tag{5}$$

for all $w, x, y, z \in R$. Finally, let $x \in R$. Then

$$U(x) = \{u \in R \mid u(R, x, x) = (R, x, x)u = (u, x, x) = 0\}$$

is an ideal of $R$ (**7**, p. 136).

**3. Main section.** The structure theorems involving simple and primitive rings satisfying (1) were derived using the ideal $U(x)$ and a property P (**7**, p. 137), which is shared by both simple and primitive rings. It does not seem obvious that this property P holds for a ring having no non-zero ideal whose square is zero but we can circumvent this difficulty with the aid of the following three lemmas. The main result will then be developed following Outcalt's argument with appropriate changes due to our lemmas.

LEMMA 1. *Let $R$ be a ring satisfying* (1). *If $A$ is an ideal of $R$, then*

$$B = \{x \in R \mid xA = Ax = 0\}$$

*is an ideal of $R$.*

*Proof.* It is immediate that

$$(x, a, y) = (y, a, x) = 0$$

for all $x \in B$, $a \in A$, and $y \in R$. It follows, using (1) and expanding the appropriate associators, that

$$a(yx) = a(xy) = (yx)a = (xy)a = 0.$$

Thus, $B$ is an ideal of $R$.

We shall assume that $R$ is a ring satisfying (1) with no non-zero ideal whose square is zero and with characteristic prime to 2 and 3 in all that follows.

LEMMA 2. *Let $y \in R$ such that $y^2 = 0$. Then $y \in U(y)$.*

*Proof.* Evidently, $(y, y, y) = 0$. Next, it follows from the fact that $y^2 = 0$ and $(y, x, y) = (y, y, x)$, that $f(y, y, x, y) = f(y, y, y, x)$.

Since $f(y, y, x, y) = -f(y, y, y, x)$ from (5), we obtain $0 = f(y, y, x, y)$. Hence

$$y(y, x, y) + (y, x, y)y = 0. \tag{6}$$

Next, using (1),

$$0 = C(y, yx, y) = (y(yx), y) - y(yx, y) - (y, y, yx)$$

which, when expanded and simplified, yields

$$0 = [y(yx)]y - y[(yx)y] + y[y(yx)]. \tag{7}$$

But, $[y(yx)]y = -(y, y, x)y$ and $y[y(yx)] = -y(y, y, x)$.

Adding these two equations, using (6), and comparing the result with (7), we obtain

$$(8) \qquad y[(yx)y] = 0.$$

Starting with $0 = C(xy, y, y)$ and following the same sequence of steps results in

$$(9) \qquad [y(xy)]y = 0.$$

Next, expanding (6) and using (8) and (9), we obtain

$$(10) \qquad [(yx)y]y = y[y(xy)].$$

Finally, expanding $0 = C(yx, y, y)$ and using (8), we obtain

$$(11) \qquad [(yx)y]y = [y(yx)]y.$$

However,

$$(y, x, y)y = (y, y, x)y.$$

Expanding these associators, and using (9), we get $[(yx)y]y = -[y(yx)]y$ which, when compared with (11), yields

$$(12) \qquad [y(yx)]y = [(yx)y]y = 0.$$

Then, from (10), $y[y(xy)] = 0$. This equation, together with (8) and (12), yields the equations

$$y(x, y, y) = (x, y, y)y = 0.$$

We conclude that $y \in U(y)$.

LEMMA 3. *Let* $a \in R$. *Then* $(R, v(a), v(a)) = 0$.

*Proof.* From (2), $[v(a)]^2 = 0$. Hence, from Lemma 2, $v(a) \in U(v(a))$. Let $A$ be the ideal generated by $(R, v(a), v(a))$. Then $A$ is contained in $U(v(a))$ and $A(R, v(a), v(a)) = (R, v(a), v(a))A = 0$. From Lemma 1, the two-sided annihilators of $A$ form an ideal of $R$, which must contain $A$. Hence $A^2 = 0$ and therefore $A = 0$.

In particular, $(R, v(x), v(x)) = 0$. Thus, from (4), we have

$$(13) \qquad v^3(a) = 0$$

for all $a \in R$. This result, coupled with $J(a, b) = 0$, yields

$$(14) \qquad 0 = v^2(b)v(a) + v(b)v^2(a).$$

Then, replacing $a$ by $v(a)$ in (14) and using (13) along with the fact that $v^3(a) = v^2(v(a))$, we get

$$(15) \qquad 0 = v^2(a)v^2(b).$$

From Lemma 3, $(v^2(a), v^2(a), r) = 0$ for all $a, r \in R$.

Expanding and using (2), we have

$$(16) \qquad v^2(a)[v^2(a)r] = 0.$$

Furthermore, linearizing the above associator yields

$$(17) \qquad 0 = (v^2(a), v^2(b), r) + (v^2(b), v^2(a), r).$$

It follows from (17) that

$$0 = I(a, a)v^2(r) - v^2(a)I(a, r) + (v^2(a), a, v^2(r)) + (a, v^2(a), v^2(r)).$$

Expanding the above equation, applying (15) and (16), and using the fact that $E(x, y) = 0$ for all $x, y \in R$, we get $0 = 2v^2(a)[v(a)v(r)]$.

Applying (3) with $a$, $b$, and $c$ replaced by $r$, $v(a)$, and $a$, respectively, we obtain

$$(18) \qquad 0 = v(r)[v^2(a)v(a)].$$

Passing to the anti-isomorphic copy of $R$ yields $0 = [v(a)v^2(a)]v(r)$.

Using the fact that $0 = E(v(a), v^2(a))$ we conclude that

$$(19) \qquad [v^2(a)v(a)]v(r) = 0.$$

Finally, applying $E(x, y) = 0$ and (13) to $0 = F(v^2(a), v(a))$, we obtain

$$(20) \qquad v(v^2(a)v(a)) = 0.$$

Recall that $v(a) = (a, x, x)$. Then equations (18), (19), and (20) imply that, given $x \in R$, $v^2(a)v(a) \in U(x)$. Let $B$ be the ideal generated by $v^2(a)v(a)$. Then $B$ is contained in $U(x)$. Hence, in particular, $B(v(a)) = (v(a))B = 0$. From Lemma 1, the ideal generated by $v(a)$ annihilates $B$ from both sides. Since $B$ is contained in this ideal we conclude that $B^2$, and hence $B$, is zero. In particular,

$$(21) \qquad v^2(a)v(a) = 0$$

for all $a, x \in R$.

Linearizing (21) we obtain $v^2(a)v(b) + v^2(b)v(a) = 0$. Subtracting (14) from this equation we obtain

$$0 = v^2(a)v(b) - v(b)v^2(a) + E(b, v(a)) = 2v^2(a)v(b)$$

whence

$$(22) \qquad v^2(a)v(b) = v(b)v^2(a) = 0.$$

Equations (13) and (22) imply that for each $x \in R$, $v^2(a) \in U(x)$.

The ideal, $C$, generated by $v^2(a)$, is contained in $U(x)$ and therefore $C(v(a)) = (v(a))C = 0$. We conclude, as before, that $C^2 = 0$ and, hence, $C = 0$. In particular,

$$(24) \qquad v^2(a) = 0$$

for all $a \in R$. But then computing $0 = G(a, b)$ yields $v(a)v(b) = v(b)v(a) = 0$ for all $a, b \in R$. Therefore, $v(a) \in U(x)$ for each $x \in R$. Reasoning as above, we conclude that $v(a) = 0$ for all $a \in R$, and $R$ must be alternative.

THEOREM. *If $R$ is a ring satisfying* (1) *with characteristic prime to* 2 *and* 3, *then, if $R$ contains no non-zero ideal whose square is zero, $R$ is alternative.*

REFERENCES


1. R. H. Bruck and E. Kleinfeld, *The structure of alternative division rings*, Proc. Amer. Math. Soc. *2* (1951), 878–890.
2. E. Kleinfeld, *Extension of a theorem on alternative division rings*, Proc. Amer. Math. Soc. *3* (1952), 348–351.
3. ——— *Simple alternative rings*, Ann. of Math. (2), *58* (1953), 544–547.
4. ——— *On simple alternative rings*, Amer. J. Math. *75* (1953), 98–104.
5. ——— *Alternative nil rings*, Ann. of Math. (2) *66* (1957), 395–399.
6. ——— *Assosymmetric rings*, Proc. Amer. Math. Soc. *8* (1957), 983–986.
7. D. L. Outcalt, *An extension of the class of alternative rings*, Can. J. Math. *17* (1965), 130–141.
8. R. L. San Soucie, *Weakly standard rings*, Amer. J. Math. *79* (1957), 80–86.
9. M. Zorn, *Alternativkörper und quadratische Systeme*, Abh. Math. Sem. Hamburg Univ. *9* (1933), 395–402.


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