A CHARACTERIZATION OF IMPLICATIVE BOOLEAN RINGS

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In the theory of probability, the conditional can be treated by an operation analogous to division. Many properties of the conditional can best be studied by means of the corresponding multiplication (called the cross-product). An implicative Boolean ring is defined [**2**] in terms of a cross-product and the usual Boolean operations. The cross-product is the only device yet known in which the events corresponding to conditional probabilities are themselves elements of the Boolean ring. The fact that such advice was not introduced by Boole is probably the reason why Boolean algebra has been very little used in the theory of probability, although probability was one of the principal applications which Boole had in mind.

When one introduces a cross-product into the two element Boolean algebra, no additional elements are obtained. However the closure under cross-product of any other finite Boolean algebra is infinite. The same is, of course, true for the usual number systems, namely, the set $\{0, 1\}$ is closed under multiplication but the closure of the set obtained by including any additional positive element is infinite. The introduction of probabilities into a Boolean algebra maps the algebra into the reals between 0 and 1. In the case of an implicative Boolean ring, this mapping is a homomorphism in which cross-product corresponds to multiplication of the reals.

It is shown in this paper that implicative Boolean rings can be characterized in terms of familiar ring concepts only. More specifically, a Boolean ring B can contain a cross-product if and only if it is isomorphic to its quotient rings modulo the non-unit principal ideals. The isomorphisms enable one to set up a semigroup of transformations (not necessarily unique) of B into B. These are one-parameter transformations where the parameter is a non-zero element of the Boolean ring. The product of the transformations defines the cross-product of the parameters. The inverse of one of these transformations, when defined, is an implication which is neither strict nor material. The implication can be extended to elements for which the inverse does not exist and can be given a logical interpretation.

By definition a Boolean algebra $(B, \vee, \cdot, \sim)$, or equivalently a Boolean ring with unit $(B, +, \cdot)$ where $+$ denotes symmetric difference, is called *implicative* if there exists a binary operation, $\times$, such that the following postulates hold:

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P0 B is closed under $\times$.
P1 $a \times (b \times c) = (a \times b) \times c$.
P2 $a \times (b + c) = a \times b + a \times c$.
P3 $a \times (b \cdot c) = (a \times b) \cdot (a \times c)$.
P4 $x \neq 0$ and $x \times y = x \times z$ imply $y = z$.
P5 $x \times 1 = x$.
P6 If $x, y \in B$ and $y \neq 0$, then there is an element z such that $x \cdot y = y \times z$.
We use the notation $z = x \subset y$ for this element.

Let $x \leqslant y$ denote the condition $x \cdot y = x$; let (a) denote the principal ideal generated by a. Then (a) is the set of all $x \in B$ such that $x \leqslant a$, and $x \equiv y$ mod (a) if and only if $x \cdot \sim a = y \cdot \sim a$.

THEOREM. *A necessary and sufficient condition for a Boolean ring B to be implicative is that for all $a \in B$, $a \neq 1$, the Boolean rings B and $B/(a)$ are isomorphic.*

Equivalently we may consider $B \cong B/(\sim a)$ for all $a \neq 0$.

Proof of necessity. Let $C = \{x \subset a, x \in B\}$. Then C is a Boolean algebra and $B/(\sim a) \cong C$. For $x \subset a = y \subset a$ is equivalent to each of the following statements: $(x + y) \subset a = 0$; $(x + y) \cdot a = 0$; and $x \equiv y$ mod $(\sim a)$ and thus $x \subset a$ corresponds to that coset of $B/(\sim a)$ of which x is a representative. But for any $y \in B$, $y = (a \times y) \subset a \in C$. Hence $C = B \cong B/(\sim a)$.

Proof of sufficiency. Since $B \cong B/(\sim a)$ for any $a \neq 0$, there exists by hypothesis a one-to-one correspondence between the elements x of B and the cosets which are the elements of $B/(\sim a)$. Let $C_x^{(a)}$ denote the coset corresponding to x. For any element $w \in C_x^{(a)}$, the element $y = w \cdot a$ of $C_x^{(a)}$ is uniquely determined since $v \cdot a = w \cdot a$ if $v, w \in C_x^{(a)}$. This procedure establishes a one-to-one correspondence between the elements x of B and those elements y of B such that $y \cdot a = y$. For, any two different elements y_1, y_2 contained in a must lie in different cosets mod $(\sim a)$ and thus must correspond to different elements of B. The following scheme indicates this one-to-one correspondence, which we denote by T_a, so that $T_a(x) = y$:

$$\begin{array}{l} B \leftrightarrow B/(\sim a) \\ x \leftrightarrow C_x^{(a)} \\ \updownarrow \\ y = w \cdot a \text{ where } w \in C_x^{(a)}. \end{array}$$

The following conditions C0 to C6 will be shown later to correspond to the postulates for implicative Boolean algebra. These conditions are restricted for the present to the single element $a \neq 0$.

C0 $T_a(x) \in B$ for all $x \in B$.

It is more convenient to discuss C1 after the other conditions have been considered.

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It follows at once from the definitions of $+$ and $\cdot$ on the elements of $B/(\sim a)$ that

C2 $T_a(x_1 + x_2) = T_a(x_1) + T_a(x_2)$,

C3 $T_a(x_1 \cdot x_2) = T_a(x_1) \cdot T_a(x_2)$.

Since it is shown that T_a is one-to-one, we have

C4 If $a \neq 0$, then $T_a(x_1) = T_a(x_2)$ implies $x_1 = x_2$.

Clearly $T_a(x) = a$ if and only if $x = 1$. For if $x = 1$, the coset $C_1^{(a)}$ in $B/(\sim a)$ corresponding to x must be the unit element of this quotient ring and therefore must consist of all $z \in B$ such that $z \cdot a = a$. The converse follows since T_a is a one-to-one transformation. Thus we have

C5 $T_a(1) = a$.

C6 The transformation $T_a(z) = w$ has an inverse $z = T_a^{-1}(w)$ if $w \leqslant a$.

This is assured by the fact that each element of B contained in a is the image under T_a of a unique element z.

We now consider the formulation of condition C1 corresponding to postulate P1. By way of preparation, we shall show how to select a semigroup of the above transformations in which multiplication is consistent.

If $a, b \neq 0$, let $T_a(b) = c$. Then $c \neq 0$ and $c \leqslant a$. The transformation T_c which is defined by the given system of isomorphisms is not necessarily the same transformation as $T_a T_b$, where T_a and T_b are defined by the given isomorphisms. In order that the set of all transformations T_a, $a \neq 0$, be consistent, we shall define a transformation T_c havingthe property that $T_c(x) = T_a T_b(x)$.

Since $T_a T_b(1) = T_a(b) = c$, it follows that $T_a T_b(x) \leqslant c$, $x \in B$. For the isomorphisms T_a and T_b preserve order, and $x \leqslant 1$. If $y \leqslant c$ is given, we wish to solve $T_a T_b(x) = y$ for x. To do this we first solve $T_a(u) = y$ for u. This may be done by C6 because $y \leqslant a$. We then solve $T_a(x) = u$ for x. In order to be able to do this by C6, it must be shown that $u \cdot b = u$. But this follows from $T_a(u \cdot b) = T_a(u) \cdot T_a(b) = y \cdot c = y = T_a(u)$ and the fact that T_a is one-to-one. Hence the transformation $T_a T_b$ is one-to-one on the set of all elements $x \leqslant c$.

Finally, if $T_a T_b(x) = y$ we let $C_x^{(c)} = \{w \in B, \ w \cdot c = y\}$. Then $C_x^{(c)}$ is one of the elements of the quotient algebra $B/(\sim c)$. It is easy to verify that the set of all *these* cosets forms a Boolean algebra isomorphic to B, i.e., that the map $x \rightarrow C_x^{(c)}$ is an isomorphism. If we define T_c using *this* system of cosets, then the transformation T_c is consistent with T_a and T_b, whereas the transformation defined by the given isomorphism might be inconsistent with T_a and T_b.

Assuming for the moment that all of the above mentioned inconsistencies have been removed, then to each non-zero element $a \in B$ there corresponds a transformation T_a. The set of all such transformations is a one-parameter semigroup in which the parameter is an element of B. The product of the transformations defines the cross-product of the parameters. More specifically, let $a \times b$ be that element c of B such that $T_a T_b(x) = T_c(x)$. Then

$$T_a(b) = T_a T_b(1) = T_c(1) = c = a \times b.$$

If the non-zero elements of B are well ordered, then the set of corresponding transformations may be made consistent by the following procedure. If a is the first element of B in the well ordering, let T_a be defined as above by means of the given isomorphism, $B \cong B/(\sim a)$. Let $a^1 = a$, and a^n be defined recursively by $a^n = a^{n-1} \times a$, $n \geqslant 2$. Then by the argument of the preceding paragraph, the transformations $T_{a^2}, \ldots, T_{a^n}, \ldots$ may all be consistently defined.

Now suppose that the transformations T_x have been defined for all x preceding a given element b in the well-ordered series. Then all finite cross-products of such elements have been defined. If b is one of these cross-products, then T_b is already defined. Otherwise let T_b be defined by the given system of isomorphisms. We have thus obtained a consistent family of transformations T_x for all non-zero $x \in B$. Finally $T_0(z) = 0$ defines T_x for $x = 0$.

Since the consistent family of transformations is a semigroup, we have the condition

C1 $T_a(T_b T_c) = (T_a T_b) T_c$.

Therefore

$$\begin{aligned} [T_a(T_b T_c)]\,(1) &= [(T_a T_b)\, T_c]\,(1),\\ T_a T_{b \times c}(1) &= T_{a \times b} T_c(1),\\ T_a\,(b \times c) &= T_{a \times b}\,(c),\\ a \times (b \times c) &= (a \times b) \times c. \end{aligned}$$

Similarly the remaining conditions C0, C2, C3, C4, C5, C6 are immediately seen from the relation $T_a(b) = a \times b$ to be verifications of the postulates for implicative Boolean algebra. This completes the proof of the characterization theorem.

In C6 we discussed the inverse transformation $z = T_a^{-1}(w)$ defined for $w \leqslant a$. We use the notation $z = w \subset a = T_a^{-1}(w)$. Thus z is a function of w and a. This function can be extended so that w can range over the entire ring B. Namely, $z = w \subset a$ is the solution of the equation $w \cdot a = T_a(z)$ for any $w \in B$.

The element $x \subset a$ of an implicative Boolean ring can be interpreted as the sentence "x if a" or the sentence "a implies x." This "if" operation is the conditional in probability theory. The conventional treatment of the conditional is based on ordered pairs of propositions, whereas in our system such an ordered pair is itself a proposition, i.e., an element of the ring. Koopman [**4**] uses this particular implication as a model for conditional probability, but still treats the conditional as an ordered pair of propositions. Material implication "a implies x" is the proposition $\sim a \vee x$. Strict implication can be defined equivalently by either of the equations $\sim a \vee x = 1$ or $x \subset a = 1$. These three implications are all distinct. Our implication $x \subset a$ is the only implication which is appropriate to the theory of probability.

In addition, the implication $x \subset a$ has an interpretation in formal logic. Consider a set of postulates $P_1, P_2, \ldots, P_n$. Let B be the set of all propositions which are statable on the base of these postulates. We extend B to an implicative Boolean ring B^* by the method of [3]. Let $P = P_1 \cdot P_2 \cdot \ldots \cdot P_n$. These elements of B which are in the unit coset of $B^*/(\sim P)$ are those which are strictly implied by the postulates. For $x \subset P = 1$ if and only if x is the unit coset of $B^*/(\sim P)$.

It is meaningful in this extended language B^* to consider propositions of the form $x \subset P$ where x is not necessarily an element of the unit coset of $B^*/(\sim P)$. That is, we consider such implications as valid sentences even though they may not be true in the sense of being strict implications. This last property is also true of material implication. The language B^* is thus seen to be a metalanguage containing the original language B. In the probabilistic interpretation, this metalanguage also contains all conditional sentences.

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