HOMOGENEOUS CONTINUA WHICH ARE ALMOST CHAINABLE[1]

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The only known examples of nondegenerate homogeneous plane continua are the simple closed curve, the circle of pseudo-arcs **(6)**, and the pseudo-arc **(1; 13)**. Another example, called the pseudo-circle, has been suggested by Bing **(2)**, but it has not been proved to be homogeneous. (Definitions of some of these terms and a history of results on homogeneous plane continua can be found in **(6)**.) Of the three known examples, the pseudo-arc is both linearly chainable and circularly chainable, and the simple closed curve and the circle of pseudo-arcs are circularly chainable but not linearly chainable. It is not known whether every homogeneous plane continuum is either linearly chainable or circularly chainable. Bing has shown that a homogeneous continuum is a pseudo-arc provided it is linearly chainable **(4)**.

In this paper, a study is made of continua that are almost chainable, and the effect upon them by a homogeneity requirement is considered. It is hoped that these results might be of some help in a search for other examples of homogeneous plane continua or in an attempt to characterize such continua.

Bing has shown that a homogeneous plane continuum is a simple closed curve if it contains an arc **(5)**. Some of the theorems presented here give conditions under which a nondegenerate homogeneous plane continuum would contain a pseudo-arc. Perhaps this is a property of all such continua that do not contain an arc. Continua which are almost chainable and for which each point is an end point are characterized as continua for which every nondegenerate proper subcontinuum is a pseudo-arc. It is not known whether every such continuum is homogeneous. A more general question has been raised in **(8)**.

Throughout this paper, a *continuum* denotes a compact connected metric space. Where there is no reference to a space in which a continuum under discussion is imbedded, the continuum itself is considered as space. Where a plane continuum M is being discussed, M should be considered imbedded in a plane E and some of the coverings of M might be collections of open sets in E.

Definitions. Linear chains, circular chains, trees, and continua described with them are defined in **(10)**. Various types of homogeneity are defined in **(9)**.

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A continuum M is *almost chainable* if, for every positive number ϵ, there exist an ϵ-covering G of M and a linear chain C $(L_1, L_2, \ldots, L_n)$ of elements of G such that no L_i $(1 \leqslant i < n)$ intersects an element of $G - C$ and every point of M is within a distance ϵ of some element of C. The set L_1 is called an *end link* of G. A point p is called an *end point* of M if, for every positive number ϵ, there is an ϵ-covering G of M such that p is in an end link of G.

Definitions of end links of trees,[2] branches of trees, and k-branched continua are given in **(15)**. A *junction link* of a tree T is an element of T that intersects at least three other elements of T. A tree-like continuum is said to be *k-junctioned*, or to have k junctions, if k is the least integer such that, for every positive number ϵ, M can be covered by an ϵ-tree with k junction links.

THEOREM 1. *If the continuum M is nearly homogeneous and almost chainable, then M has a dense set of end points.*

Proof. That M has an end point can be shown by a method similar to the proof given by Bing **(4)** to show that a continuum has an end point if it is homogeneous and linearly chainable. Then Theorem 1 follows from the near-homogeneity of M and the fact that, under a homeomorphism of M onto itself, each end point of M goes into an end point of M.

THEOREM 2. *If the continuum M is almost chainable and K is a proper subcontinuum of M which contains an end point p of M, then K is linearly chainable with p as an end point.*

Proof. Let q be a point of $M - K$, and let ϵ be a positive number that is less than the distance from q to K. There exist an $\epsilon/2$-covering G of M and a linear chain $C(L_1, L_2, \ldots, L_n)$ in G such that: (1) no $L_i (1 \leqslant i < n)$ intersects an element of $G - C$; (2) every point of M is within a distance $\epsilon/2$ of some element of C; and (3) p is in L_1. There exists an element L_j of C such that the distance from q to L_j is less than $\epsilon/2$, and it follows that K does not intersect L_j. This implies that K is covered by the linear ϵ-chain $(L_1, L_2, \ldots, L_{j-1})$. Thus K is linearly chainable with p as an end point.

THEOREM 3. *In order that every nondegenerate proper subcontinuum of the continuum M should be a pseudo-arc, it is necessary and sufficient that M be almost chainable with each of its points as an end point.*

Proof of sufficiency. Let K be a nondegenerate proper subcontinuum of M and let p be a point of K. By Theorem 2, K is linearly chainable with p as an end point. It follows from Theorem 16 of **(3)** that K is a pseudo-arc.

The following lemma will be used in proving that the condition is necessary.

LEMMA 3.1. *If every nondegenerate proper subcontinuum of the continuum M is a pseudo-arc, K is a pseudo-arc in M, $C(L_1, L_2, \ldots, L_n)$ is a linear chain*

[2]A collection that is called a tree in **(10)** is called a tree-chain in **(15)**.

which is an essential covering of K, and p is a point of $K - (L_1 + L_n)$, then there is a linear chain $C'(L_1', L_2', \ldots, L_n')$ such that: (1) *for each $i(1 \leqslant i < n)$, L_i' is a subset of L_i;* (2) *for each $i(1 < i < n)$, the boundary of L_i' does not contain a point of M that is not covered by C'; and* (3) *p is in an element of C'.*[3]

Proof of Lemma 3.1. Let K' be a component of $K - (L_1 + L_n)$ that intersects both $\mathrm{cl}(L_1)$ and $\mathrm{cl}(L_n)$, and let K'' be the component of $K - (L_1 + L_n)$ that contains p. Let A denote the closed set $M - (L_1 + L_n)$ and let B denote the closed set $M - (L_1 + L_2 + \ldots + L_n)$. Now suppose that some continuum H in A intersects both $K' + K''$ and B. This leads to the contradiction that $H + K$ is decomposable. Hence it follows from **(14**, Theorem 35, p. 21**)** that A is the sum of two mutually separated closed sets A_1 and A_2 containing $K' + K''$ and B, respectively. Let L_1' and L_n' denote L_1 and L_n, respectively, and for each $i(1 < i < n)$, let $L_i' = A_1 \cdot L_i$. The chain $C'(L_1', L_2', \ldots, L_n')$ satisfies the conclusion of Lemma 3.1.

Proof of necessity. Since every proper subcontinuum of M is indecomposable **(1**; **12)**, it follows that M is indecomposable. Let ϵ be a positive number and let p be a point of M. There exists a pseudo-arc K in M such that p is in K and every point of M is within a distance $\epsilon/2$ of K. Let $D(R_1, R_2, \ldots, R_t)$ be a linear $\epsilon/2$ chain which is an essential covering of K such that p is in R_1. It follows from the proof of Theorem 13 of **(1)** that there exists a linear chain $C(L_1, L_2, \ldots, L_n)$ which is a refinement of D such that: (1) C is an essential covering of K; and (2) L_1 and L_n are subsets of R_t. By Lemma 3.1, there exists a linear chain $C'(L_1', L_2', \ldots, L_n')$ such that: (1) for each $i(1 \leqslant i \leqslant n)$, L_i' is a subset of L_i; (2) for each $i(1 < i < n)$, the boundary of L_i' does not contain a point of M that is not covered by C'; and (3) p is in an element of C'. Now for each $i(1 \leqslant i \leqslant t)$, let R_i' denote the sum of the elements of C' that lie in R_i. Let D' denote the linear chain $(R_1', R_2', \ldots, R_t')$. There exists an ϵ-covering G of M such that: (1) each link of D' is an element of G; (2) each point of M is within a distance ϵ of some link of D'; (3) no element of $G - D'$ intersects a link of D' different from R_t'; and (4) p is in R_1'. Hence M is almost chainable and each point of M is an end point of M.

THEOREM 4. *If the continuum M is circularly chainable and hereditarily indecomposable, then M is almost chainable and each point of M is an end point of M.*

Proof. Since every proper subcontinuum of M is linearly chainable and hereditarily indecomposable, it follows that every nondegenerate proper subcontinuum of M is a pseudo-arc **(2)**. Thus the conclusion of Theorem 4 follows from Theorem 3.

THEOREM 5. *If the continuum M is homogeneous and almost chainable, then every nondegenerate proper subcontinuum of M is a pseudo-arc.*

[3]This lemma and its proof might be compared with Property 17 and its proof in **(5)**.

Proof. Using the homogeneity of M, it can be shown by a method similar to the proof of Theorem 1 that every point of M is an end point of M. Hence it follows from Theorem 3 that every nondegenerate proper subcontinuum of M is a pseudo-arc.

THEOREM 6. *If the continuum M is almost chainable, then M is not a triod.*

Proof. Suppose that M is a triod. Let K be a subcontinuum of M such that $M - K$ is the sum of three mutually separated sets K_1, K_2, and K_3. For each i ($i \leqslant 3$), let D_i be an open set such that $\mathrm{cl}(D_i)$ is a subset of K_i. Let ϵ be a positive number such that, for each i, ϵ is less than the distance from $\mathrm{cl}(D_i)$ to $M - K_i$ and less than the distance from some point of D_i to the boundary of D_i. There exist an ϵ-covering G of M and a linear chain $C(L_1, L_2, \ldots, L_n)$ in G such that: (1) no $L_j (1 \leqslant j < n)$ intersects an element of $G - C$; and (2) every point of M is within a distance ϵ of some link of C. Hence for each $i (i \leqslant 3)$, some link L_{r_i} of C contains a point p_i of D_i and does not intersect $M - K_i$. Consider the case in which $r_1 < r_2 < r_3$. Then each of the links L_{r_1} and L_{r_3} of the linear chain C intersects the continuum $K + K_1 + K_3$, but L_{r_2} does not intersect this continuum. It follows that for some integer j less than n, the continuum $K + K_1 + K_3$ contains a point of the boundary of L_j that is not in a link of C. This involves the contradiction that L_j intersects an element of $G - C$. Hence M is not a triod.

Remark. While there does not exist a triod in a continuum that is linearly chainable **(10)**, there does exist a continuum which contains a triod and is almost chainable. A continuum which is the sum of a simple triod T and a ray spiralling around T is such an example.

THEOREM 7. *If the continuum M is almost chainable, then M is unicoherent.*

Proof. Suppose that M is the sum of two continua M_1 and M_2 and that p and q are two points of $M_1 \cdot M_2$. Consider the case in which, for every positive number ϵ, there exists an ϵ-covering G of M and a linear chain $C(L_1, L_2, \ldots, L_n)$ in G such that: (1) no $L_i (1 \leqslant i < n)$ intersects an element of $G - C$; (2) every point of M is within a distance ϵ of some link of C; and (3) L_1 intersects M_1. For a choice of ϵ that is sufficiently small, M_1 would be covered by C, and p and q would lie in two links L_i and L_j, respectively, of C. Hence every link of C between L_i and L_j would intersect both M_1 and M_2. That p and q lie in the same component of $M_1 \cdot M_2$, and hence that M is unicoherent, can be shown by a proof similar to the one given for Theorem 1 of **(6)**.

Remark. While a continuum is hereditarily unicoherent if it is linearly chainable **(6)**, this is not the case for continua that are almost chainable. A continuum which is the sum of a circle K and a ray spiralling around K is almost chainable but fails to be hereditarily unicoherent.

THEOREM 8. *If the continuum M is almost chainable, then M is irreducible between some two points.*

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Proof. By Theorems 6 and 7, M is unicoherent and is not a triod. Sorgenfrey **(16)** has shown that such a continuum is irreducible between some two points.

Remark. Theorem 8 is a generalization of Rosen's result that a continuum is irreducible between some two points if it is linearly chainable **(15)**.

THEOREM 9. *If the continuum M is nearly homogeneous and almost chainable, then M is indecomposable.*

Proof. By Theorem 8, M is irreducible between some two points, and such a continuum is indecomposable if it is nearly homogeneous **(7)**.

THEOREM 10. *If M is an indecomposable plane continuum and, for each positive number ϵ, there exists a circular ϵ-chain of open disks covering M, then M is almost chainable.*

The following definition and lemma will be used in the proof of this theorem.

Definition. A circular chain $C(L_1, L_2, \ldots, L_m)$ is said to *fold back one revolution* in a circular chain $D(K_1, K_2, \ldots, K_n)$ if C is a refinement of D and there exist two links K_i and K_j of D and three links L_r, L_s, and L_t of C such that: (1) K_i intersects K_j; (2) L_s is a subset of K_i; (3) L_r and L_t are subsets of K_j; and (4) there is a linear chain in C that contains L_s, has L_r and L_t as end links, and has no link that intersects both K_i and K_j.

LEMMA 10.1. *If for each positive number ϵ, the continuum M can be covered by two circular ϵ-chains $C(L_1, L_2, \ldots, L_m)$ and $D(K_1, K_2, \ldots, K_n)$ such that C folds back one revolution in D, then M is almost chainable.*

Proof of Lemma 10.1. Let K_i and K_j be links of D and Let L_r, L_s, and L_t be links of C such that the requirements of the definition above are satisfied. For convenience, suppose that $i = 1$ and $j = n$. Let C' denote the linear chain in C that contains L_s, has L_r and L_t as end links, and has no link that intersects both K_1 and K_n. For each $q (1 \leqslant q \leqslant n)$, let H_q denote the sum of the elements of C' that lie in K_q. Let G denote the collection consisting of the sets $H_1, H_2, \ldots, H_n$ and the elements of $C - C'$. The collection G is an ϵ-covering of M such that: (1) no $H_q (1 \leqslant q < n)$ intersects an element of $G - C'$; and (2) each point of M is within a distance ϵ of one of the sets $H_1, H_2, \ldots, H_n$. Hence M is almost chainable.

Proof of Theorem 10. Let ϵ be a positive number. There exists a circular ϵ-chain $D(K_1, K_2, \ldots, K_n)$ of open disks covering M such that: (1) for each i $(1 \leqslant i \leqslant n)$, $\mathrm{cl}(K_i) \cdot \mathrm{cl}(K_{i+1, \bmod n})$ is a closed disk; and (2) the sum of the elements of D is an open annular ring. Let H and J be the two simple closed curves on the boundary of this annular ring. It follows from the indecomposability of M that there exist two disjoint subcontinua M_1 and M_2 of M and two consecutive links, say K_1 and K_n, of D such that M_1 and M_2 are covered by the linear chain $(K_1, K_2, \ldots, K_{n-1})$ and are irreducible from $K_1 \cdot \mathrm{cl}(K_n)$ to

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$K_{n-1} \cdot \text{cl}(K_n)$. Let δ be a positive number that is less than the distance from M_1 to M_2, and let $C(L_1, L_2, \ldots, L_m)$ be a circular δ-chain of open disks covering M such that each $\text{cl}(L_i)$ is a subset of an element of D and the links of C satisfy conditions similar to those required for D in (1) and (2) above. Let J_n denote the boundary of K_n. Then $J_n \cdot \text{cl}(K_1)$ and $J_n \cdot \text{cl}(K_{n-1})$ are arcs ab and cd, respectively, where $a + c$ and $b + d$ are subsets of H and J, respectively. Let W denote the collection of all linear chains in C that are refinements of the linear chain $(K_1, K_2, \ldots, K_{n-1})$ and are irreducible[4] from ab to cd. Let $C_1, C_2, \ldots, C_r$ denote the chains of W, and for each i $(1 \leqslant i \leqslant r)$, let L_{p_i} and L_{q_i} be the end links of C_i that intersect ab and cd, respectively. It follows from the choice of δ that $r > 1$. For convenience, suppose that $p_1 = 1$ and that the chain C_1 consists of the elements $L_1, L_2, \ldots, L_{q_1}$ of C. There are two cases to consider.

Case 1. There exist two integers i and j $(1 \leqslant i < j \leqslant r)$ such that either no p_u is between q_i and q_j or no q_u is between p_i and p_j. This implies that C folds back one revolution in D, and hence it follows from Lemma 10.1 that M is almost chainable.

Case 2. The requirements of Case 1 are not satisfied. It will be shown that this case is impossible. For convenience, suppose that the sets $L_{p_1}, L_{p_2}, \ldots, L_{p_r}$ intersect the arc ab in the order named from ab. It follows from (**14**, Theorem 17, p. 167) that the sets $L_{q_1}, L_{q_2}, \ldots, L_{q_r}$ intersect the arc cd in the order named from c to d. It follows from (**14**, Theorem 17, p. 189) that there exist two disjoint arcs ef and gh that are irreducible from H to J such that: (1) $e + g$ and $f + h$ are subsets of H and J, respectively; (2) ef and gh do not intersect $\text{cl}(K_n)$; (3) for each i $(1 \leqslant i \leqslant r)$, each of the arcs ef and gh intersects the closure of one and only one link of C_i and this intersection is a connected set; and (4) neither ef nor gh intersects the closure of a link of C unless that link is in one of the chains $C_1, C_2, \ldots, C_r$. Let Y be the simple closed curve formed by the arcs ef and gh and two arcs eg and gh of H and J, respectively, that do not intersect $\text{cl}(K_n)$. By considering the order on Y of the intersections of the arcs ef and gh with links of the chains $C_1, C_2, \ldots, C_r$, it follows from (**14**, Theorem 17, p. 167) that if the links of $C(L_1, L_2, \ldots, L_m)$ are followed in their natural order in C, then the end links of the chains $C_1, C_2, \ldots, C_r$ would occur as follows. First, $L_{p_1} = L_1$ would occur, next L_{q_1} would occur, then some L_{p_i} $(i > 1)$ would occur, then L_{q_i} would occur, then some L_{p_j} $(j > i)$ would occur, etc. By continuing this way until L_{q_r} occurs, then some L_{p_s} $(s < r)$ would occur next, and this would involve a contradiction to (**14**, Theorem 17, p. 167).

Remark. It would be interesting to know whether every plane continuum M that is circularly chainable can be imbedded in the plane so that, for

[4]A linear chain C is *irreducible between two sets* X and Y if one end link of C intersects X and the other intersects Y but no proper subchain of C has this property.

every positive number ϵ, M can be covered by a circular ϵ-chain of open disks.[5] Every continuum M that is linearly chainable can be imbedded in the plane so that, for every positive number ϵ, M can be covered by a linear ϵ-chain of open disks **(3)**. However, there do exist continua, for example solenoids **(5)**, which are circularly chainable and cannot be imbedded in the plane.

THEOREM 11. *If M is a homogeneous indecomposable plane continuum such that, for each positive number ϵ, M can be covered by a circular ϵ-chain of open disks, then every nondegenerate proper subcontinuum of M is a pseudo-arc.*

Proof. By Theorem 10, M is almost chainable. Hence, it follows from Theorem 5 that every nondegenerate proper subcontinuum of M is a pseudo-arc.

Remark. The pseudo-arc **(1; 13)** is the only known example of a continuum which satisfies the hypothesis of Theorem 11. While the pseudo-circle **(2)** is not known to be homogeneous, it is described with circular chains of open disks and each of its nondegenerate proper subcontinua is a pseudo-arc. It would be interesting to know whether a plane continuum is a pseudo-circle if it is circularly chainable, hereditarily indecomposable, and different from a pseudo-arc. This is suggested by Bing's result that a continuum is a pseudo-arc if it is linearly chainable and hereditarily indecomposable **(2)**.

THEOREM 12. *If the tree-like continuum M is k-branched and nearly homogeneous, then M is indecomposable.*

Proof. Rosen has shown that every k-branched continuum is irreducible about some k points **(15)**, and such an irreducible continuum is indecomposable if it is nearly homogeneous **(7)**.

Remark. Since every tree-like continuum is hereditarily unicoherent **(6)**, it follows from a result by F. B. Jones that every homogeneous tree-like continuum is indecomposable **(11)**. However, it is necessary in Theorem 12 to require that M be k-branched, or at least that it be k-junctioned, as there exists a dendron which is nearly homogeneous **(9)**.

THEOREM 13. *If the indecomposable tree-like continuum M is k-junctioned and nearly homogeneous, then M is almost chainable.*

The following definition and lemma will be used in the proof of Theorem 13.

Definition. A junction link L of a tree T is said to be a *free junction link* of T if there does not exist a linear chain in T which contains L and has two junction links of T different from L as end links.

[5]A forthcoming paper by R. H. Bing will include an affirmative answer to this question. Hence the hypotheses of Theorems 10 and 11 can be weakened accordingly.

LEMMA 13.1. *If the tree-like continuum M is k-junctioned and nearly homogeneous, U is an open subset of M, and ϵ is a positive number, then there exists an ϵ-tree T which covers M and contains only k junction links such that some free junction link of T is a subset of U.*

Proof of Lemma 13.1. It is easy to see that each tree different from a linear chain has a free junction link. For each positive integer i, let T_i be a $1/i$-tree covering M such that T_i has exactly k junction links, and let K_i be a free junction link in T_i. Some subsequence of the sequence $K_1, K_2, K_3, \ldots$ converges to a point p. For convenience, suppose that $K_1, K_2, K_3, \ldots$ converges to p. There is a homeomorphism f of M onto itself that carries p into a point of U. Hence for infinitely many integers i, $f(K_i)$ is a subset of U. From this and the uniform continuity of f, it follows that, for some integer n, $f(K_n)$ is a subset of U and each link of T_n has an image, under f, with a diameter less than ϵ. The collection consisting of all images, under f, of links of T_n is a tree T satisfying the requirements of the conclusion of Lemma 13.1.

Proof of Theorem 13. Suppose that M fails to be almost chainable. There exists a positive number ϵ such that every ϵ-tree covering M has at least k junction links and such that no ϵ-covering of M satisfies the requirements for M to be almost chainable. It follows from the indecomposability of M that there exists a collection W consisting of $2k$ disjoint subcontinua of M such that, for each element X of W, each point of M is within a distance $\epsilon/2$ of X. Let δ be a positive number less than $\epsilon/2$ such that no two continua of W are within a distance δ of each other. Let G be a δ-tree which covers M and has only k junction links. No two continua of W intersect the same link of G, so there exist at least k continua of W that do not intersect a junction link of G. From the supposition that M fails to be almost chainable, it follows that no branch of G covers a continuum of W. Now by induction on k, it can be shown that, for any k linear chains in G each of which has two junction links of G as end links, one such chain must contain at least three junction links of G. Hence there exist two continua H and K of W and a linear chain $C(L_1, L_2, \ldots, L_j, \ldots, L_n)$ in G such that: (1) no link of C is a junction link of G; (2) H is covered by the linear chain $(L_1, L_2, \ldots, L_j)$; and (3) K is covered by the linear chain $(L_{j+1}, L_{j+2}, \ldots, L_n)$. By Lemma 13.1, there exists a tree G', covering M such that: (1) G' is a refinement of G; (2) G' has exactly k junction links; and (3) some free junction link R of G' is a subset of L_j and is not a subset of any other element of C. Let A denote the collection of all elements X of G' such that some linear chain in G' has both R and X as links and has no more than one link that intersects $L_1 + L_n$. There are two cases to consider.

Case 1. One of the sets L_1 and L_n, say L_1, does not intersect an element of A. Let r be the least positive integer such that L_r contains an element of A that is not in L_{r+1}. For each $i (r \leqslant i < n)$, let K_i denote the sum of all elements

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of A that lie in L_i. Now, since K is covered by the linear $\epsilon/2$-chain $(L_{j+1}, L_{j+2}, \ldots, L_n)$ and every point of M is within a distance $\epsilon/2$ of K, it follows that every point of M is within a distance ϵ of some link of the linear ϵ-chain $(K_r, K_{r+1}, \ldots, K_{n-1})$. However, since no element of $G' - A$ intersects one of the sets $K_r, K_{r+1}, \ldots, K_{n-2}$, this is contrary to the supposition that M fails to be almost chainable.

Case 2. Each of the sets L_1 and L_n intersects an element of A. There exist linear chains C_1 and C_2 in G' such that C_1 is irreducible from R to L_1 and C_2 is irreducible from R to L_n. Let B denote the collection of all links of G' that lie in a branch of G' that starts at R. From the supposition that M fails to be almost chainable, it follows that neither L_1 nor L_n intersects an element of B. For each i $(1 < i < n)$, let L_i' denote the sum of the elements of the collection $B + C_1 + C_2$ that lie in L_i but not in L_{i+1}. Let G'' denote the collection consisting of $L_2', L_3', \ldots, L_{n-1}'$ and the elements of $G' - (B + C_1 + C_2)$. Then G'' is an ϵ-tree covering M, and each junction link of G'' contains a junction link of G'. However, unless L_j' contains a junction link of G' different from R, L_j' is not a junction link of G''. Hence G'' has no more than k–1 junction links. This involves a contradiction as ϵ was chosen so that every ϵ-tree covering M would have at least k junction links.

THEOREM 14. *If the tree-like continuum M is k-branched and nearly homogeneous, then M is almost chainable.*

Proof. A k-branched continuum is at most (k–2)-junctioned. Hence Theorem 14 follows from Theorems 12 and 13.

THEOREM 15. *If the tree-like continuum M is k-junctioned and homogeneous, then every nondegenerate proper subcontinuum of M is a pseudo-arc.*

Proof. As observed in the remark following Theorem 12, M is indecomposable. Hence it follows from Theorems 5 and 13 that every nondegenerate proper subcontinuum of M is a pseudo-arc.

COROLLARY. *If the tree-like continuum M is k-branched and homogeneous, then every nondegenerate proper subcontinuum of M is a pseudo-arc.*

Remark. By slight modifications of the arguments, it can be shown that Theorems 5 and 11 and the above corollary hold for a weaker type of homogeneity where, for each point p in the continuum M and each nondegenerate subcontinuum K of M, there is a homeomorphism of M onto itself that carries p into a point of K.

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