Further, x must be odd, but then all of $x - 1, x + 1$ and y are even meaning that solutions of the equation $a^2 + (a + 2)^2 = c^2$ are obtained from solutions of $a^2 + (a + 1)^2 = c^2$ by multiplication by 2. There are no other solutions.

References

1. K. H. Rosen, *Elementary Number Theory and its Applications*, Addison-Wesley (1992).
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press (1938).

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108.31 Generalised Thales intercept theorem

According to T. Heath [1, p. 124], [2, p. 128] and C. R. Fletcher [3, p. 268], Thales (about 624-547 B.C.) is a central figure in the evolution of geometry, as he was the first scientist to introduce proofs alongside empirical methods. One of the main results attributed to Thales is the so-called "intercept theorem", which the Greek scientist used to measure the heights of pyramids and distances of ships at sea [1, p. 124]. In [4, p. 9], John Stillwell underlines the importance of this theorem saying that it "is the key to using algebra in geometry". E. Moise, in [5, pp. 136-141], provides the following simple statement of Thales intercept theorem:

Parallel projections are one-to-one correspondences that preserve betweenness, congruence and ratio;

moreover, the author shows that the result on ratios and his converse can be deduced from the fact that parallel projections preserve the midpoint of a segment. Therefore, we focus our attention on the following statement:

Take two parallel segments A_1A_1' and A_2A_2' and find the midpoints M and M' of the segments A_1A_2 and $A_1'A_2'$. Then, the segment MM' is parallel to A_1A_1' and A_2A_2'.

Using the technique shown by N. Lord in [6], we can generalise the previous statement as follows:

Take n parallel segments A_1A_1', A_2A_2', … , A_nA_n' and find the centres of gravity M and M' of the sets $\{A_1, A_2, \dots, A_n\}$ and $\{A_1', A_2', \dots, A_n'\}$. Then, the segment MM' is parallel to the segments A_1A_1', A_2A_2', … , A_nA_n'.

Indeed, let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and $\mathbf{a}_1', \mathbf{a}_2', \dots, \mathbf{a}_n'$ be the position vectors of consecutive vertices $A_1, A_2, \dots, A_n$ and $A_1', A_2', \dots, A_n'$. Then, the position vectors of the centres of gravity M and M' are given by $\mathbf{m} = \frac{1}{n}(\mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_n)$ and $\mathbf{m}' = \frac{1}{n}(\mathbf{a}_1' + \mathbf{a}_2' + \dots + \mathbf{a}_n')$. It follows that

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$\mathbf{m}' - \mathbf{m} = \frac{1}{n}(\mathbf{a}_1' - \mathbf{a}_1) + \frac{1}{n}(\mathbf{a}_2' - \mathbf{a}_2) + \ldots + \frac{1}{n}(\mathbf{a}_n' - \mathbf{a}_n)$, which is the sum of n vectors that are parallel to the segments A_jA_j'. Thus, MM' is parallel to each A_jA_j'; moreover, it equals their average.

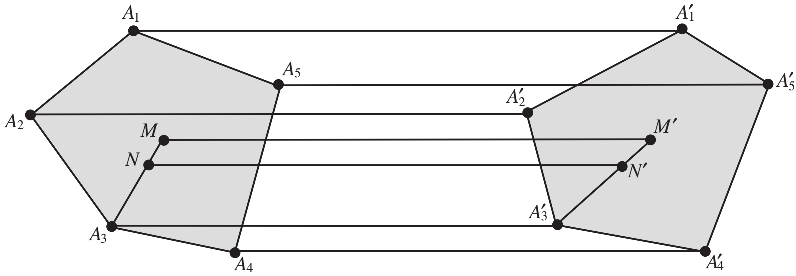

FIGURE 1

Figure 1 shows the situation outlined by the previous statement; moreover, it suggests that parallel projections also preserve the ratio in which they divide the corresponding medians. Indeed, let $\mathbf{n}$ and $\mathbf{n}'$ be the position vectors of points N and N' and assume that $A_iN : A_iM = A_i'N' : A_i'M' = k$. It follows that

$$\begin{aligned}\mathbf{n}' - \mathbf{n} &= \mathbf{a}_i - \mathbf{n} + \mathbf{a}_i' - \mathbf{a}_i + \mathbf{n}' - \mathbf{a}_i' \\ &= -k(\mathbf{m} - \mathbf{a}_i) + \mathbf{a}_i' - \mathbf{a}_i + k(\mathbf{m}' - \mathbf{a}_i') \\ &= k(\mathbf{m}' - \mathbf{m}) - k(\mathbf{a}_i' - \mathbf{a}_i) + \mathbf{a}_i' - \mathbf{a}_i.\end{aligned}$$

As we saw previously, the segment MM' is parallel to A_iA_i', thus the vector $\mathbf{n}' - \mathbf{n}$ is the sum of three vectors that are parallel to A_iA_i'. Therefore, NN' is parallel to the segments A_iA_i'. In other words, the following statement holds:

Take n parallel segments $A_1A_1', A_2A_2', \ldots, A_nA_n'$ and find the centres of gravity M and M' of sets $\{A_1, A_2, \ldots, A_n\}$ and $\{A_1', A_2', \ldots, A_n'\}$. Let N be a point on the line A_iM and *N' a point on the line $A_i'M'$ such that $A_iN : A_iM = A_i'N' : A_i'M'$. Then, the segment NN' is parallel to the segments $A_1A_1', A_2A_2', \ldots, A_nA_n'$.*

As is well known, we can describe the interior of any convex n-sided polygon $A_1A_2\ldots A_n$ as the set of points A whose position vectors are given by $\mathbf{a} = \sum_{i=1}^n \lambda_i\mathbf{a}_i$, where the λ_i are non-negative and sum to 1. The values λ_i are called the barycentric coordinates of A, with respect to the polygon $A_1A_2\ldots A_n$ [7, p. 216], [8]. If the barycentric coordinates of a point A' with respect to $A_1'A_2'\ldots A_n'$ coincide with the barycentric coordinates of the point A with respect to $A_1A_2\ldots A_n$, that is, if $\mathbf{a}' = \sum_{i=1}^n \lambda_i\mathbf{a}_i'$ is the position vector of the point A', it follows that $\mathbf{a}' - \mathbf{a} = \sum_{i=1}^n \lambda_i(\mathbf{a}_i' - \mathbf{a}_i)$. Then, AA' is parallel to the segments A_iA_i'. Therefore, although such coordinates for $n > 3$ are not unique [8], we can say that

Parallel projections preserve barycentric coordinates.

More precisely, the following statement holds:

Take n parallel segments $A_1A_1', A_2A_2', \ldots, A_nA_n'$. Let $\mathbf{a} = \sum_{i=1}^n \lambda_i\mathbf{a}_i$

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and $\mathbf{a}' = \sum_{i=1}^{n} \lambda_i \mathbf{a}'_i$ *be the position vectors of points* A *and* A', *where the* λ_i *are non-negative and sum to* 1. *Then, the segment* AA' *is parallel to the segments* $A_1A'_1, A_2A'_2, \ldots, A_nA'_n$.

We can note that the barycentric coordinates of the points A_i and A'_i (to the respective polygons) coincide since they are all zero except for the i-th, which is equal to 1. The barycentric coordinates of the centres of gravity M and M' also coincide since they are all equal to $\frac{1}{n}$. Moreover, the position vectors of the points N and N' in Figure 1 can be obtained from the formulas $\mathbf{n} = \mathbf{a}_i + k(\mathbf{m} - \mathbf{a}_i)$ and $\mathbf{n}' = \mathbf{a}'_i + k(\mathbf{m}' - \mathbf{a}'_i)$, which act equally on the points A_i, M and A'_i, M'. Hence, N and N' have equal barycentric coordinates (to their respective polygons). Therefore, the result for the points N and N' is a special case of the last statement.

At this point, we can further extend Thales intercept theorem. Indeed, as shown in [9], given an n-sided polygon $A_1A_2\ldots A_n$ putting $A_{n+j} = A_j$ for $1 \leqslant j \leqslant n - 1$, we can consider the n-sided polygons $B_{h,1}B_{h,2}\ldots B_{h,n}$ whose vertex $B_{h,i}$, for $i = 1, 2, \ldots, n - 1$, is the centre of gravity of the set $\{A_i, A_{i+1}, \ldots, A_{i+h-1}\}$. For example, the derived polygon $B_{2,1}B_{2,2}B_{2,3}$, $B_{2,4}$ of a given quadrilateral $A_1A_2A_3A_4$, is the well-known *Varignon parallelogram* that joins the midpoints of the sides of $A_1A_2A_3A_4$. In [10] the author calls such polygons $B_{h,1}B_{h,2}\ldots B_{h,n}$ the "h-Varignon polygons of $A_1A_2\ldots A_n$" and notes that the centre of gravity of $A_1A_2\ldots A_n$ can be seen as the n-Varignon polygon $B_{n,1}B_{n,2}\ldots B_{n,n}$ of $A_1A_2\ldots A_n$. Thus, in a certain sense, the n-Varignon polygons generalise the concept of centre of gravity. Then, it might be interesting to note that

Parallel projections preserve the h-Varignon polygons.

In other words, Thales intercept Theorem can be extended to the h-Varignon polygons as follows:

Take n parallel segments $A_1A'_1, A_2A'_2, \ldots, A_nA'_n$ *and find the h-Varignon polygons* $B_{h,1}B_{h,2}\ldots B_{h,n}$ *and* $B'_{h,1}B'_{h,2}\ldots B'_{h,n}$ *of the n-sided polygons* $A_1A_2\ldots A_n$ *and* $A'_1A'_2\ldots A'_n$. *Then, each segment* $B_{h,i}B'_{h,i}$ *is parallel to the segments* $A_1A'_1, A_2A'_2, \ldots, A_nA'_n$.

Indeed, for each $h, k = 1, 2, \ldots, n$, the position vectors of the vertices $B_{h,k}$ and $B'_{h,k}$ are respectively given by $\mathbf{b}_{h,k} = \frac{1}{h}\left(\mathbf{a}_k + \mathbf{a}_{k+1} + \ldots + \mathbf{a}_{k+h-1}\right)$ and $\mathbf{b}'_{h,k} = \frac{1}{h}\left(\mathbf{a}'_k + \mathbf{a}'_{k+1} + \ldots + \mathbf{a}'_{k+h-1}\right)$. It follows that

$$\mathbf{b}'_{h,k} - \mathbf{b}_{h,k} = \tfrac{1}{h}\left(\mathbf{a}'_k - \mathbf{a}_k\right) + \tfrac{1}{h}\left(\mathbf{a}'_{k+1} - \mathbf{a}_{k+1}\right) + \ldots + \tfrac{1}{h}\left(\mathbf{a}'_{k+h+1} - \mathbf{a}_{k+h+1}\right).$$

Thus, the segment $B_{h,k}B'_{h,k}$ is parallel to each segment $A_jA'_j$. Moreover, $B_{h,k}B'_{h.k}$ equals the average of segments $A_kA'_k, A_{k+1}A'_{k+1}, \ldots, A_{k+h-1}A'_{k+h-1}$. In Figure 2 we show two 7-sided polygons whose vertices are in a parallel projection. As we can see, their 4-Varignon polygons are in the same parallel projection.

We can, finally, observe that all the previous results were obtained by proving parallelism between segments through the proportionality of the position vectors that join their extremes. Since this condition of parallelism

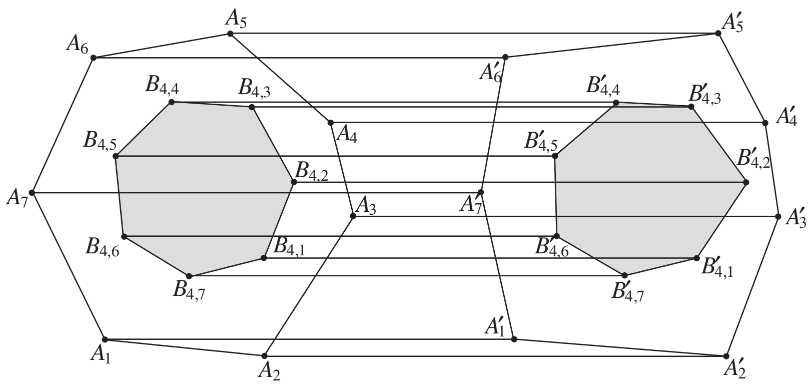

FIGURE 2

continues to hold in the Euclidean space $\mathbb{R}^3$ and, more generally, in $\mathbb{R}^m$, the previous results can be generalised to any polytope in every space dimension.

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References

1. T. Heath, Greek mathematics and science, *Math. Gaz.* **32** (July 1948) pp. 120-133.
2. T. Heath, *A history of greek mathematics*, *Volume* 1. Oxford University Press (1921).
3. C. R. Fletcher, Thales–our founder?, *Math. Gaz.* **66** (December 1982) pp. 266-272.
4. J. Stillwell, *The four pillars of geometry*, Springer (2005).
5. E. Moise, *E.G. from an advanced standpoint* (3rd edn.) Addison-Wesley (1990).
6. N. Lord, Maths bite: averaging polygons, *Math. Gaz.* **92** (March 2008) p. 134.
7. H. S. M. Coxeter, *Introduction to geometry* (2nd edn.) Wiley (1969).
8. M. S. Floater, Generalized barycentric coordinates and applications, *Acta Numer.* **29** (2015) pp. 161-214.
9. F. Laudano, Generalised averages of polygons, *Math. Gaz.* **107** (November 2023) pp. 528-530.
10. F. Laudano, Generalized Varignon's and medial triangle theorems, Commun. Korean Math. Soc. **38** (2023) (2) pp. 561-573.

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