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WHEN IS THE LOWER RADICAL DETERMINED BY A SET OF RINGS STRONG?

M. FILIPOWICZ

Institute of Mathematics and Physics, Technical University of Białystok, 15-950 Białystok, Wiejska 45 A, Poland
e-mail: mfilipowicz@kki.net.pl

and E. R. PUCZYŁOWSKI*

Institute of Mathematics, University of Warsaw, Warsaw, Banacha 2, Poland
e-mail: edmundp@mimuw.edu.pl

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Abstract. We prove that the lower radical, determined by a set of rings, is strong if and only if it is the lower radical determined by a ring with zero multiplication.

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1. Introduction and preliminaries. Strong radicals were introduced in [**3**] and since then they have been extensively studied by many authors. Recall that a radical α is said to be *left* (respectively, *right*) *strong* if all left (respectively, right) α-ideals of every ring R are contained in $\alpha(R)$. It is well known that the prime, locally nilpotent, generalized nil and Jacobson radicals are all strong whereas the Brown-McCoy and strongly prime radicals are not. The question whether the nil radical is strong was posed by Koethe in 1930 and is still one of the most challenging open problems in ring theory. For some other concrete radicals, e.g. the uniformly strongly prime radical, it is also unknown whether they are strong or not. Many studies concerned general strong radicals, e.g. the left-right symmetry [**8**], lattices of such radicals [**2**] or strong radicals satisfying some additional properties [**5**, **7**].

In this paper we characterize strong radicals which are lower radicals determined by sets of rings. The lower radical determined by a set of rings is in fact determined by one ring. In the main result of this paper we prove that the lower radical determined by a ring R is strong if and only if it is equal to the lower radical determined by the ring R^0 with zero multiplication on the underlying additive group of R. The class of such rings R is strictly related to a class of rings studied in a quite different context by Sands in [**9**, **10**]. In the last section we study these classes. In particular we answer two questions raised by Sands in [**10**].

All rings in this paper are associative but it is not assumed that each ring has an identity element. For a given ring R we shall denote by $R^\star$ the usual extension of R to a ring with an identity. The ring of integers will be denoted by $\mathbf{Z}$.

For any abelian group A we denote by A^0 the ring with zero multiplication whose additive group is equal to A. For a ring R, R^0 will denote the ring with zero multiplication defined on the underlying additive group of R.

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To denote that I is an ideal (left ideal, right ideal) of a ring R we write $I \lhd R$ ($I <_l R, I <_r R$).

The fundamental results used on radicals of rings may be found in [**12**, **13**]. With some abuse of terminology we use the term "radical" to mean both "radical class" and "radical property".

We start with some general observations which will be used in the paper. Most of them are known but we present them in a more uniform setting and include proofs for completeness.

Proposition 1. *Suppose that R, Q are rings and N is an R-Q−bimodule. Then for every radical α, $\alpha(N^0) = M^0$ for some R-Q−subbimodule M of N.*

Proof. Denote by $\left(\begin{smallmatrix} R & N \\ 0 & Q \end{smallmatrix}\right)$ the set $\{\left(\begin{smallmatrix} r & n \\ 0 & q \end{smallmatrix}\right) \mid r \in R, q \in Q, n \in N\}$. It is clear that $\left(\begin{smallmatrix} R & N \\ 0 & Q \end{smallmatrix}\right)$ is a ring with respect to the usual matrix addition and multiplication. Clearly N^0 can be identified with $\left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} R & N \\ 0 & Q \end{smallmatrix}\right)$. Now $\alpha(N^0) = M^0$ for some subgroup M of the additive group of N, and so $\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right) = \alpha(\left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right))$. Since $\left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} R & N \\ 0 & Q \end{smallmatrix}\right)$, $\alpha(\left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)) \lhd \left(\begin{smallmatrix} R & N \\ 0 & Q \end{smallmatrix}\right)$. This easily implies that M is an R-Q−subbimodule of N. □

As a particular case of Proposition 1 one immediately obtains the following well known result.

Corollary 2. *For every radical α and arbitrary $I \lhd R$, $\alpha(I^0) = J^0$ for some ideal J of R.*

Corollary 3. (i) *Suppose that N is an R-Q−bimodule and α is a radical. If, for a subgroup M of the additive group of N, $M^0 \in \alpha$, then $(R^\star M Q^\star)^0 \in \alpha$.*

(ii) *If S is a subring of a ring R and $S^0 \in \alpha$, then $(R^\star S R^\star)^0 \in \alpha$. In particular if $L <_l R$, $L^2 = 0$ and $L \in \alpha$, then $LR^\star \in \alpha$.*

Proof. (i) Clearly $M^0 \lhd (R^\star M Q^\star)^0$. Hence $M^0 \subseteq \alpha(R^\star M Q^\star)^0$. Since $R^\star M Q^\star$ is the R-Q−bimodule of N generated by M, Proposition 1 gives $(R^\star M Q^\star)^0 \in \alpha$.

The statement (ii) is a clear consequence of (i). □

Proposition 4. (i) *Let M be a left (respectively, right) R-module. If for a radical α, $R^0 \in \alpha$, then $(RM)^0 \in \alpha$ (respectively, $(MR)^0 \in \alpha$).*

(ii) *Suppose that α is a radical, $I \lhd R$, $J \lhd Q$ with $(R/I)^0 \in \alpha$, $(Q/J)^0 \in \alpha$ and M is an R-Q−bimodule such that $(IMJ)^0 \subseteq \alpha(M^0)$. Then $(RMQ)^0 \subseteq \alpha(M^0)$.*

Proof. (i) If M is a left R−module, then the maps $f_m(r) : R \longrightarrow M$, where $m \in M$, defined by $f_m(r) = rm$, induce a canonical R-module epimorphism of $\oplus_{m \in M} R$ onto RM. Clearly it is also a ring epimorphism of $\oplus_{m \in M} R^0$ onto $(RM)^0$. Since $R^0 \in \alpha$, also $(RM)^0 \in \alpha$. Dual arguments can be applied when M is a right R-module.

(ii) By Proposition 1, $\alpha(M^0) = N^0$ for an R-Q−subbimodule N of M. Passing to the factor bimodule M/N, we can assume that $\alpha(M^0) = 0$, so that $IMJ = 0$ and we have to show that $RMQ = 0$. By (i) applied to the left R/I-module MJ we get that $(RMJ)^0 \subseteq \alpha(M^0) = 0$, and so $RMJ = 0$. Now applying (i) to the right Q/J-module RM we obtain $(RMQ)^0 \subseteq \alpha(M^0) = 0$, so that $RMQ = 0$. □

In what follows β denotes the prime radical.

We shall need the following known results, which easily follow from Proposition 4 (i).

COROLLARY 5. (i) (cf. [**11**]). *Let α be a radical and R a ring with $R/R^2 \in \alpha$. Then $R^n/R^{n+1} \in \alpha$, for every integer $n \geq 2$. If R is nilpotent then $R^0 \in \alpha$ and $R \in \alpha$.*

(ii) (cf. [**5**, Lemma 5]). *If α is a radical and R is a β-radical ring such that $R^0 \in \alpha$, then $R \in \alpha$.*

2. Main results. Recall that the *lower radical $l_{\mathcal{N}}$* determined by the class $\mathcal{N}$ of rings is the smallest radical containing $\mathcal{N}$. If $\mathcal{N} = \{R\}$, then $l_{\mathcal{N}}$ will be denoted by l_R.

Observe that if $\mathcal{N}$ is a set, then $l_{\mathcal{N}} = l_R$, where $R = \oplus_{A \in \mathcal{N}} A$.

A subring A of a ring R is called *accessible* (respectively, *left accessible*) if there are subrings A_i, $0 \leq i \leq n$, of R such that $A = A_0 \lhd A_1 \lhd \ldots \lhd A_n = R$ (respectively, $A_0 <_l A_1 <_l \ldots <_l A_n = R$). For a given ring R, $l_{\mathcal{N}}(R) \neq 0$ if and only if R contains a nonzero accessible subring which is a homomorphic image of a ring in $\mathcal{N}$ [**12**, **13**].

In what follows for a ring R and a set X we denote by $M_X(R)$ the ring of all matrices over R indexed by elements from X that have finitely many nonzero entries.

If R is a ring with identity, then each ideal of $M_X(R)$ is of the form $M_X(I)$, for some ideal I of R.

It is known [**6**] that for every ring A and every radical α, $\{R \mid R \otimes_{\mathbf{Z}} A \in \alpha\}$ is again a radical. For every ring R, $R^0 \simeq R \otimes_{\mathbf{Z}} \mathbf{Z}^0$ and for every set X, $M_X(R) \simeq R \otimes_{\mathbf{Z}} M_X(\mathbf{Z})$.

Consequently for every radical α and for every set X, $\alpha^0 = \{R \mid R^0 \in \alpha\}$ and $\alpha_X = \{R \mid M_X(R) \in \alpha\}$ are also radicals.

PROPOSITION 6. (cf. [**1**]). *If $\mathcal{N}$ is a class of rings with zero multiplication, then $l_{\mathcal{N}} \subseteq l_{\mathcal{N}}^0$ and $l_{\mathcal{N}}$ is left and right strong.*

Proof. Clearly $\mathcal{N} \subseteq l_{\mathcal{N}}^0$. Hence, since $l_{\mathcal{N}}^0$ is a radical, $l_{\mathcal{N}} \subseteq l_{\mathcal{N}}^0$. In particular, if $L <_l R$ and $L \in l_{\mathcal{N}}$, then $L^0 \in l_{\mathcal{N}}$. Now, by Corollary 3 (ii), $(LR^\star)^0 \in l_{\mathcal{N}}$. Clearly $LR^\star \in \beta$, so that by Corollary 5 (ii), $LR^\star \in l_{\mathcal{N}}$. Therefore $l_{\mathcal{N}}$ is left strong. Dual arguments shows that $l_{\mathcal{N}}$ is right strong. $\square$

PROPOSITION 7. *Let $\alpha = l_{\mathcal{N}}$, where $\mathcal{N}$ is a class of rings with zero multiplication. Then, for every ring R and every set X, $\alpha(M_X(R)) = M_X(\alpha(R))$.*

Proof. Clearly $\mathcal{N} \subseteq \alpha_X$. Hence, since α_X is a radical, $\alpha \subseteq \alpha_X$. This shows that $M_X(\alpha(R)) \subseteq \alpha(M_X(R))$. Now $M_X(R)/M_X(\alpha(R)) \simeq M_X(R/\alpha(R))$ and so to get the result it suffices to prove that if a ring A is α-semisimple, then so is the ring $M_X(A)$. Suppose that $\alpha(M_X(A)) \neq 0$. Then $M_X(A)$ contains a nonzero accessible subring S which is a homomorphic image of a ring from $\mathcal{N}$. Obviously S is also an accessible subring of $M_X(A^\star)$. The ideal $\bar{S}$ of $M_X(A^\star)$ generated by S is nilpotent and, since $A^\star$ is a ring with identity, $\bar{S} = M_X(I)$ for some $I \lhd A$. Applying Corollary 3 (ii) we get that $\bar{S}^0 \in \alpha$. Clearly I^0 is a homomorphic image of $\bar{S}^0$, so that $I^0 \in \alpha$. However I is nilpotent and so, by Corollary 5 (ii), $I \in \alpha$. This is impossible as $\alpha(A) = 0$. $\square$

For a given ring R and a left R-module V, we denote by $\left(\begin{smallmatrix} R & V \\ 0 & 0 \end{smallmatrix}\right)$ the set of 2×2-matrices of the form $\left(\begin{smallmatrix} r & v \\ 0 & 0 \end{smallmatrix}\right)$, where $r \in R$, and $v \in V$. This set is a ring with respect to canonical matrix addition and multiplication. Similarly for every ring R and every right R-module V one defines the ring $\left(\begin{smallmatrix} R & 0 \\ V & 0 \end{smallmatrix}\right)$.

It is well known (cf. [**8**, Lemma 5]) that a radical α is left strong if and only if for arbitrary $L <_l R$, $L \in \alpha$ implies $LR^\star \in \alpha$.

In what follows for a given set X we denote by $|X|$ the cardinality of X.

Proposition 8. *If for a given ring A the radical $\alpha = l_A$ is left strong, then $\alpha \subseteq \alpha^0$; i.e, for every $R \in \alpha$, $R^0 \in \alpha$.*

Proof. Applying Corollary 2 one easily sees that it suffices to prove that if $R \neq 0$, then $\alpha(R^0) \neq 0$. Let X be a set with $|X| > |A|$ and let $V = \bigoplus_{x\in X} R_x{}^\star$, where for each $x \in X$, $R_x{}^\star = R^\star$. Identifying R with $\left(\begin{smallmatrix} R & 0 \\ 0 & 0 \end{smallmatrix}\right)$ we get that $R <_l \left(\begin{smallmatrix} R & V \\ 0 & 0 \end{smallmatrix}\right)$. The ideal of $\left(\begin{smallmatrix} R & V \\ 0 & 0 \end{smallmatrix}\right)$ generated by R is equal to $\left(\begin{smallmatrix} R & W \\ 0 & 0 \end{smallmatrix}\right)$, where $W = \bigoplus_{x\in X} R_x$ and for each $x \in X$, $R_x = R$. Since $R \in \alpha$ and α is left strong, $\left(\begin{smallmatrix} R & W \\ 0 & 0 \end{smallmatrix}\right) \in \alpha$. Consequently $\left(\begin{smallmatrix} R & W \\ 0 & 0 \end{smallmatrix}\right)$ contains a nonzero accessible subring S which is a homomorphic image of A. Obviously S is also an accessible subring of $\left(\begin{smallmatrix} R & V \\ 0 & 0 \end{smallmatrix}\right)$. Let I be the ideal of $\left(\begin{smallmatrix} R & V \\ 0 & 0 \end{smallmatrix}\right)$ generated by S. Then for a positive integer n, $J = I^n \subseteq S$. If there are $0 \neq r \in R$ and $v \in V$ such that $\left(\begin{smallmatrix} r & v \\ 0 & 0 \end{smallmatrix}\right) \in J$, then $\left(\begin{smallmatrix} 0 & rV \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} r & v \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & V \\ 0 & 0 \end{smallmatrix}\right) \subseteq J \subseteq S$. Now $|rV| = |\bigoplus_{x\in X} rR_x{}^\star| \geq |X| > |A|$. This is impossible as $|S| \leq |A|$. Consequently $J \subseteq \left(\begin{smallmatrix} 0 & V \\ 0 & 0 \end{smallmatrix}\right)$, which shows that I is nilpotent. Hence S is also nilpotent and Corollary 5 (i) implies that $S^0 \in \alpha$. Therefore $\alpha(\left(\begin{smallmatrix} R & W \\ 0 & 0 \end{smallmatrix}\right)^0) \neq 0$. However $\left(\begin{smallmatrix} R & W \\ 0 & 0 \end{smallmatrix}\right)^0$ is isomorphic to a direct sum of copies of R^0. These obviously imply that $\alpha(R^0) \neq 0$. □

Now we are ready to prove the main result of this paper.

Theorem 9. *For a ring A the following conditions are equivalent:*

(i) *l_A is left strong;*
(ii) *$l_A = l_{\mathcal{N}}$ for a class $\mathcal{N}$ of rings with zero multiplication;*
(iii) *$l_A = l_{A^0}$.*

Proof. Suppose that $\alpha = l_A$ is left strong. We shall prove that $\alpha = l_{\mathcal{N}}$, where $\mathcal{N} = \{R \in \alpha \mid R^2 = 0\}$. Obviously it suffices to show that if $0 \neq R \in \alpha$, then $l_{\mathcal{N}}(R) \neq 0$. Let X be a set with $|X| > |A|$ and, for every $x_0 \in X$, T_{x_0} the set of all matrices from $M_X(R)$ whose entries outside the x_0-column are equal to 0. Clearly $T_{x_0} <_l M_X(R)$ and $T_{x_0} \simeq \left(\begin{smallmatrix} R & 0 \\ V & 0 \end{smallmatrix}\right)$, where $V = \bigoplus_{x\in X} R_x$ and for each $x \in X$, $R_x = R$. Moreover $M_X(R) = \sum_{x\in X} T_x$. Obviously $\left(\begin{smallmatrix} 0 & 0 \\ V & 0 \end{smallmatrix}\right) \simeq \bigoplus_{x\in X} R_x^0$. Hence, applying Proposition 8, one gets that $\left(\begin{smallmatrix} 0 & 0 \\ V & 0 \end{smallmatrix}\right) \in \alpha$. Consequently, for each $x \in X$, $T_x \in \alpha$ and, since α is left strong, $M_X(R) \in \alpha$. This implies that $M_X(R)$ contains a nonzero accessible subring S which is a homomorphic image of A. Obviously S is also an accessible subring of $M_X(R^\star)$. If $\bar{S}$ is the ideal of $M_X(R^\star)$ generated by S, then for a positive integer n, $\bar{S}^n \subseteq S$ and $\bar{S} = M_X(I)$ for an ideal I of R. Obviously if $I^n \neq 0$, then $|S| \geq |\bar{S}^n| \geq |M_X(I^n)| \geq |X| > |A|$. This is impossible because S is a homomorphic image of A. Thus $\bar{S}^n = 0$, which implies that S is nilpotent. Now from Corollary 5 (i) it follows that $S^0 \in \alpha$ and, by Corollary 3 (ii), $\bar{S}^0 \in \alpha$. However $\bar{S}^0 \simeq M_X(I^0)$ and I^0 is a homomorphic image of $M_X(I^0)$. Therefore $I^0 \in \alpha$ and, from Corollary 5 (ii), we get that $I \in \alpha$. Consequently $\alpha(R) \neq 0$. Thus (i) implies (ii).

Suppose now that (ii) is satisfied. Then $A \in \beta$ and, applying Corollary 5 (ii), we get that $A \in l_{A^0}$. Consequently $l_A \subseteq l_{A^0}$. Applying Proposition 6, we obtain $A^0 \in l_A$, so that $l_{A^0} \subseteq l_A$ and (iii) holds.

The implication (iii) $\Longrightarrow$ (i) follows from Proposition 6. □

Note that the conditions (ii) and (iii) in Theorem 9 are left-right symmetric. Thus A satisfies Theorem 9 if and only if l_A is right strong.

As we have noted, for any set $\mathcal{N}$ of rings $l_{\mathcal{N}} = l_A$, where $A = \oplus_{R\in\mathcal{N}} R$. Hence from Theorem 9 we get that $l_{\mathcal{N}}$ is left (right) strong if and only if $l_{\mathcal{N}} = l_{A^0}$ or, equivalently, $l_{\mathcal{N}} = l_{\{R^0 \mid R\in\mathcal{N}\}}$.

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We conclude this section with a result on stable radicals. A radical α is called *left* (respectively, *right*) *stable* [**4**] if for arbitrary $I <_l R$ (respectively, $I <_r R$), $\alpha(I) \subseteq \alpha(R)$.

A ring will be called *torsion* or *divisible* if its additive group satisfies the respective property.

Corollary 10. *For a given ring A the radical l_A is left (right) stable if and only if A is torsion and divisible.*

Proof. It is not hard to check that $\bar{l}_A = \{R \in l_A \mid R$ is torsion and divisible$\}$ is a radical. Hence assuming that A is torsion and divisible we get that $l_A = \bar{l}_A$. Now if $L <_l R$, then $l_A(L)$ is torsion and divisible and so is the ideal I of R generated by $l_A(L)$. Hence $I^2 = 0$ and Corollary 3 implies that $I \in l_A$. This proves the "if" part.

Conversely, suppose that l_A is left stable. Clearly l_A is also left strong so that, by Theorem 9, $l_A = l_{A^0}$. By Proposition 8, for every $R \in l_A$, $R^0 \in l_A$. Let p be a prime and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. If $pR \neq R$, then R^0 can be homomorphically mapped onto ${\mathbf{Z}_p}^0$, so that ${\mathbf{Z}_p}^0 \in l_{A^0}$. Since l_{A^0} is left stable, this easily implies that the ring $M_2(\mathbf{Z}_p)$ of 2×2-matrices over $\mathbf{Z}_p$ is in l_{A^0}. However it is clear that $M_2(\mathbf{Z}_p)$ contains no nonzero accessible subrings with zero multiplication, and so $l_{A^0}(M_2(\mathbf{Z}_p)) = 0$. Thus $pR = R$, which shows that R is divisible. If R is not torsion, then R^0 can be homomorphically mapped onto Q^0, where Q is the field of rationals. However then stability of l_A implies that the ring $M_2(Q)$ of 2×2-matrices over Q is in l_{A^0}, which is impossible. Thus each ring in l_A is torsion and divisible. $\square$

3. On the class $\mathcal{S} = \{A \mid l_A = l_{A^0}\}$. From Theorem 9 we can deduce that $\mathcal{S} = \{A \mid l_A$ is left strong$\}$. In this section we study this class more closely.

It is clear that $\mathcal{S} = \{A \mid$ for every radical α, $A \in \alpha$ if and only if $A^0 \in \alpha\}$ and $\mathcal{S} \subseteq \beta$. It is also clear that $\mathcal{S}$ contains the class $\mathcal{C} = \{R \mid$ for every radical α, $\alpha(R) = \alpha(R^0)\}$, where $\alpha(R) = \alpha(R^0)$ means the equality of the underlying additive groups. Note that, for every idempotent ring $R \in \beta$, $R \oplus \mathbf{Z}^0 \in \mathcal{S} \setminus \mathcal{C}$. The class $\mathcal{C}$ was studied in another context by Sands in [**9**, **10**]. In [**10**] he obtained the following properties of $\mathcal{C}$.

Theorem 11 (i) ([**10**, Corollary, p. 499]). *The class $\mathcal{C}$ is hereditary; i.e., if $I \triangleleft R$ and $R \in \mathcal{C}$, then $I \in \mathcal{C}$.*

(ii)([**10**, Theorem 8]). $\mathcal{C} = \{R \mid$ *for every* $I \triangleleft R$, $I \in l_{I/I^2}\}$.

From Theorem 11 (ii), it follows that $\mathcal{C} = \{R \mid$ for every $I \triangleleft R$ and arbitrary radical α, $I \in \alpha$ if and only if $I/I^2 \in \alpha\}$.

The following result can be obtained by extracting some arguments from the proof of Theorem 8 in [**10**]. We apply a slightly different method.

Proposition 12. $\mathcal{S} = \{A \mid$ *for every radical* α, $A \in \alpha$ *if and only if* $A/A^2 \in \alpha\}$.

Proof. Assume first that $A \in \mathcal{S}$ and α is a radical. Clearly, if $A \in \alpha$ then $A/A^2 \in \alpha$. Suppose now that $A/A^2 \in \alpha$. It is routine to check that $\bar{\alpha} = \{R \mid R/R^2 \in \alpha\}$ is a radical. Since $A \in \bar{\alpha}$ and $A \in \mathcal{S}$, $A^0 \in \bar{\alpha}$. Consequently $A^0 \simeq A^0/(A^0)^2 \in \alpha$. Hence the assumption $A \in \mathcal{S}$ gives $A \in \alpha$. This proves the inclusion $\subseteq$.

Assume that A belongs to the right hand class. Clearly $A \in \beta$. Hence if for a radical α, $A^0 \in \alpha$, then by Corollary 5 (ii), $A \in \alpha$. Suppose now that $A \in \alpha$. Then $(A/A^2)^0 \simeq A/A^2 \in \alpha$, so that $A/A^2 \in \alpha^0$. Consequently $A \in \alpha^0$, which means that $A^0 \in \alpha$. Therefore $A \in \mathcal{S}$ and the result follows. $\square$

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From Theorem 11 and Proposition 12, $\mathcal{C} = \{R \mid \text{for every } I \lhd R, I \in \mathcal{S}\}$ and $\mathcal{C}$ is the largest hereditary subclass of the class $\mathcal{S}$.

Given a ring R, $m(R) = \{x \in R \mid RxR = 0\}$ is called the *middle annihilator* of R. A ring R is called *M-nilpotent* [**9**] if for every nonzero homomorphic image R' of R, $m(R') \neq 0$. The class of M-nilpotent rings is contained in β and closed under homomorphic images and ideals.

In [**9**], Sands proved that the class of M-nilpotent rings is contained in the class $\mathcal{C}$. Hence Theorem 11 (ii) gives the following extension of Corollary 5 (i): if R is an M-nilpotent ring and, for a radical α, $R/R^2 \in \alpha$, then $R \in \alpha$. To get this result one can also apply the following arguments. By Corollary 5 (ii) it suffices to prove that $R^0 \in \alpha$. If not then, applying Corollary 2, we can assume that $\alpha(R^0) = 0$. Let $m_1(R) = \{x \in R \mid R^2xR^2 = 0\}$. Clearly $m_1(R)/m(R) = m(R/m(R))$. Applying Proposition 4 (ii) to the R-Q−bimodule $m_1(R)$ we get that $(Rm_1(R)R)^0 \subseteq \alpha(R^0) = 0$. Hence $m_1(R) = m(R)$ and, since R is M-nilpotent, $R = m(R)$. Thus $R^3 = 0$ and it suffices to apply Corollary 5 (i).

Note that if $\mathcal{N}$ is a class of rings contained in $\mathcal{S}$, then $l_{\mathcal{N}} = l_{\{R^0 \mid R \in \mathcal{N}\}}$. Hence, by Proposition 6, $l_{\mathcal{N}}$ is left and right strong. In particular, the lower radical determined by any class of M-nilpotent rings is left and right strong. This result was proved in another context in [**1**].

In [**10**] Sands asked whether the class $\mathcal{C}$ is closed under extensions. Now we shall answer this question. For this we need the following lemma.

Lemma 13. *For a given radical α and $I \lhd R$, the following conditions are equivalent:*
(i) $(R/RIR)^0 \in \alpha$;
(ii) $(R/RI)^0 \in \alpha$;
(iii) $(R/I)^0 \in \alpha$ *and* $R/R^2 \in \alpha$.

Proof. Since $(R/I)^0$ and R/R^2 are homomorphic images of $(R/RIR)^0$, we get that (i) implies (iii).

Suppose now that (iii) is satisfied. Applying Proposition 4 (i) to the right R/I-module R/RI, we obtain $(R^2/RI)^0 \in \alpha$ and, since $R/R^2 \in \alpha$, also $(R/RI)^0 \in \alpha$. Thus (ii) follows. Assuming (ii) and applying Proposition 4 (i) to the left R/RI-module R/RIR we get that $(R^2/RIR)^0 \in \alpha$. This and $R/R^2 \in \alpha$ imply that $(R/RIR)^0 \in \alpha$ and so (i) holds. □

Proposition 14. *Suppose that $I \lhd R$. If $RI \in \mathcal{S}$ and $R/I \in \mathcal{S}$, then $R \in \mathcal{S}$.*

Proof. Clearly $R \in \beta$. Hence if, for a radical α, $R^0 \in \alpha$, then by Corollary 5 (ii), $R \in \alpha$. Suppose now that $R \in \alpha$. Then $R/I \in \alpha$ and, since $R/I \in \mathcal{S}$, $(R/I)^0 \in \alpha$. Clearly $R/R^2 \in \alpha$. Consequently, by Lemma 13, $(R/RIR)^0 \in \alpha$. Applying Proposition 4 (ii) to the left R/RIR−module $I/(RI)^2$ we get that $RI/(RI)^2 \in \alpha$. Since $RI \in \mathcal{S}$, by Proposition 12, $RI \in \alpha$ and $(RI)^0 \in \alpha$. By Lemma 13, $R^0/(RI)^0 \simeq (R/RI)^0 \in \alpha$. Consequently $R^0 \in \alpha$. The result follows. □

The following corollary shows in particular that the question of Sands' quoted above has a positive answer.

Corollary 15. *Suppose that $I \lhd R$.*
(i) *If $I \in \mathcal{C}$ and $R/I \in \mathcal{S}$, then $R \in \mathcal{S}$.*
(ii) *If $I \in \mathcal{C}$ and $R/I \in \mathcal{C}$, then $R \in \mathcal{C}$.*

Proof. (i) By Theorem 11 (i), $RI \in \mathcal{S}$. Hence, applying Proposition 14, we obtain $R \in \mathcal{S}$.

(ii) Let $J \lhd R$. Then $J \cap I \lhd I$ and $(J+I)/I \lhd R/I$. Since $I \in \mathcal{C}$ and $R/I \in \mathcal{C}$, by Theorem 11 (i), $J \cap I \in \mathcal{C}$ and $J/(J \cap I) \simeq (J+I)/I \in \mathcal{S}$. Hence (i) implies that $J \in \mathcal{S}$. Consequently $R \in \mathcal{C}$. □

Now we obtain some results which allow us to answer another question raised by Sands in [**10**].

For a given class M of rings, let ls_M be the lower left strong radical determined by M; i.e., the smallest left strong radical containing M. In [**3**, p. 379], it was proved that if $R \in ls_M$, then every nonzero homomorphic image of R contains a nonzero left accessible subring which is a homomorphic image of a ring from M. Observe that this implies that if $R^2 = 0$ then $R \in ls_M$ if and only if $R \in l_M$.

Proposition 16. $\mathcal{S} = \{R \mid$ *for every left strong radical* α, $R \in \alpha$ *if and only if* $R^0 \in \alpha\}$.

Proof. The inclusion $\subseteq$ is clear. Suppose now that R belongs to the right hand class. Since β is left strong and $R^0 \in \beta$, we get that $R \in \beta$. Consequently from Corollary 5 (ii) it follows that if, for a radical α, $R^0 \in \alpha$, then $R \in \alpha$. Assume now that $R \in \alpha$. Let ls_R be the lower left strong radical determined by $\{R\}$. Then $R^0 \in ls_R$ and from the remark made before the proposition, $R^0 \in l_R \subseteq \alpha$. This proves the inclusion $\supseteq$. □

Proposition 17. $\{R \mid$ *for every left strong radical* α, $\alpha(R) = \alpha(R^0)\} = \{R \mid$ *for every left strong radical* α *and arbitrary* $I \lhd R$, $I \in \alpha$ *if and only if* $I^0 \in \alpha\}$.

Proof. Denote the first of these classes by $\mathcal{C}_1$ and the second by $\mathcal{C}_2$. By Proposition 6, for every ring A, the radical l_{A^0} is left strong. Hence it is clear that $\mathcal{C}_2 = \{R \mid$ for arbitrary $I \lhd R$, $ls_I = l_{I^0}\}$. Take any $R \in \mathcal{C}_2$ and let α be any left strong radical. By Corollary 2, $\alpha(R^0) = I^0$ for some $I \lhd R$. Now $ls_I = l_{I^0} \subseteq \alpha$, so that $I \in \alpha$. Consequently $\alpha(R^0) \subseteq \alpha(R)$. Now let $I = \alpha(R)$. Then $l_{I^0} = ls_I \subseteq \alpha$ and so $I^0 \in \alpha$. Thus $\alpha(R) \subseteq \alpha(R^0)$, which shows that $R \in \mathcal{C}_1$.

Suppose now that $R \in \mathcal{C}_1$ and $I \lhd R$. Since β is left strong and $\beta(R) = \beta(R^0) = R^0$, we get that $R \in \beta$. Hence, by Corollary 5 (ii), $I \in l_{I^0}$ and, since l_{I^0} is left strong, $ls_I \subseteq l_{I^0}$. Conversely let $A = ls_I(R)$. Since $R \in \mathcal{C}_1$, $A^0 = ls_I(R^0)$. Thus $A^0 \in ls_I$ and, since A^0 is a ring with trivial multiplication, $A^0 \in l_I$. Applying now Proposition 4 (i) to the left $A-$module I, we obtain $(AI)^0 \in l_I$. Obviously $I \subseteq A$, so that $I^2 \subseteq AI \subseteq I$. Since $(I/AI)^2 = 0$, we obtain $I^0/(AI)^0 \simeq I/AI \in l_I$. Thus $I^0 \in l_I$, so $l_{I^0} \subseteq ls_I$ and the result follows. □

In [**10**, p. 503], Sands posed the problem of describing the class $\{R \mid \alpha(R) = \alpha(R^0)$ for every left strong radical $\alpha\}$. Combining Theorem 11 and Propositions 12, 16 and 17 one obtains the following solution to this problem.

Corollary 18. $\{R \mid \alpha(R) = \alpha(R^0)$, *for every left strong radical* $\alpha\} = \mathcal{C}$.

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