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ON THE NUMBER OF SOLUTIONS OF THE DIOPHANTINE EQUATION $ax^m - by^n = c$

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Dedicated to Professor Paulo Ribenboim on his 80th birthday

Abstract

Let a, b, c, x and y be positive integers. In this paper we sharpen a result of Le by showing that the Diophantine equation

$$ax^m - by^n = c, \quad \gcd(ax, by) = 1$$

has at most two positive integer solutions (m, n) satisfying $\min(m, n) > 1$.

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1. Introduction

The Diophantine equation

$$ax^m - by^n = c, \quad a, b, c, x, y, m, n \in \mathbb{Z} \tag{1.1}$$

has a long and rich history. Philippe de Vitry asked the following question: '*Can* $3^m \pm 1$ *be a power of* 2*?*' The answer that $m = 2$ and $n = 3$ was given by Levi Ben Gerson (see, for example, [6]). Many authors (for example, Fermat, Euler, Lagrange, Gauss, etc.) were interested in the special case $a = b = 1$, $c = \pm 1$, particularly the Catalan equation that was solved by Mihailescu [12] in 2004. In general, for given a, b and c one can consider three cases. First, one can solve (1.1) for x, y assuming that m and n are fixed. Second, the equation can be solved for the exponents m and n when x, y are fixed. Finally, the difficult case consists of finding all of the variables x, y, m, n of (1.1). Chapter 7 of [17] is devoted to some particular cases of the problem. One can also see [6] for more details about the history of the equation.

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In this paper, we consider the second case, that is Equation (1.1) where a, b, c, x and y are given positive integers with $x > 1$, $y > 1$ and $\gcd(ax, by) = 1$. For the rest of the paper, we suppose that a, b, c, x and y are fixed positive integers. As we mentioned above, the first result was obtained by Levi Ben Gerson who proved the following theorem.

THEOREM 1.1. *If $m, n \geq 2$ and $3^m - 2^n = \pm 1$, then $m = 2$, $n = 3$.*

Other proofs of this theorem were given by Langevin and Franklin (see [17]). If $a = b = 1$, Pillai [14] studied the equation and conjectured that if $x = 3$ and $y = 2$ then $|c| > 13$. In 1982, this conjecture was solved by Stroeker and Tijdeman [19]. LeVeque [10] proved that if $a = b = c = 1$, then Equation (1.1) has at most one solution (m, n). If $a = b = 1$ and $c \leq 2$, Cao [4] showed that the number of solutions (m, n) of (1.1) is at most four. In the general case, Shorey [18] proved that (1.1) has at most nine solutions (m, n) with $ax^m > 953c^6$. Le [9] gave a series of results and in particular he proved the following theorem.

THEOREM 1.2. *If $\min(x, y) \geq e^e$ and $\min(m, n) > 1$, then (1.1) with $\gcd(ax, by) = 1$ has at most three solutions.*

The goal of this paper is to sharpen the above result. Here are the main results for the equation

$$ax^m - by^n = c, \quad \gcd(ax, by) = 1. \tag{1.2}$$

THEOREM 1.3. *If $\min(m, n) > 1$, then (1.2) has at most two positive integer solutions (m, n), except when*

$$\begin{aligned}(x, y) \text{ or } (y, x) \in \{&(2, 3), (2, 5), (2, 7), (2, 15), (2, 21), (3, 5), (3, 10),\\ &(3, 11), (3, 13), (3, 20), (3, 22), (3, 44), (3, 55), (3, 110), (3, 220),\\ &(5, 6), (6, 7), (7, 15), (7, 20), (7, 30), (11, 12), (13, 14), (19, 28)\}.\end{aligned}$$

Then, using Theorem 1.3 and considering the exceptional cases, we have the following result.

THEOREM 1.4. *Equation (1.2) has at most three positive integer solutions (m, n).*

Furthermore, the upper bound of number of solutions of (1.2) is lower for some special parameters a, b and c. For example, in [7], the authors sharpened a result of Bugeaud and Shorey [3] on the Goormaghtigh equation by proving the following result.

THEOREM 1.5. *Let $Y > X > 1$ be given integers. Then the equation*

$$\frac{X^m - 1}{X - 1} = \frac{Y^n - 1}{Y - 1}, \quad m > 1, n > 1 \tag{1.3}$$

has at most one solution (m, n).

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To see this, one can rewrite (1.3) in the form

$$(Y-1)X^m - (X-1)Y^n = Y - X.$$

Then the above equation becomes (1.2) with $(a, b, c) = (y-1, x-1, y-x)$. Bennett [1] and Bugeaud and Luca [2] have also studied some particular cases of (1.1) and proved that the equation has at most one solution. In fact, Bennett showed that if a, b and c are positive integers with $a, b \geq 2$ and $c \geq b^{2a^2 \log a}$ (or if a is prime, $c \geq b^a$), then the equation $a^x - b^y = c$ has at most one solution. (See [1, Theorem 1.3].) He also has another similar result and two other results where the equation has at most two solutions. One can refer to [1] for more details. For their part, Bugeaud and Luca considered a fixed, finite set of prime numbers $\mathcal{P} = \{p_1, \ldots, p_t\}$ and $\mathcal{S} = \{\pm p_1^{\alpha_1} \ldots p_t^{\alpha_t} \mid \alpha_i \geq 0,\ i = 1, \ldots, t\}$ the set of all nonzero integers whose prime factors belong to $\mathcal{P}$. They showed that if b is fixed, there exists a positive constant a_0 depending on b and $\mathcal{S}$ such that for any nonzero integer c, for any $a \geq a_0$, and for every positive integers A, B in $\mathcal{S}$, the more general equation $Aa^x - Bb^y = c$ has at most one solution. (See [2, Corollary 2.2].) Corollary 2.3 is a similar result. One result is more general but not the best.

We organize the paper as follows. In Section 2, we recall some useful results due to Le [9], Ribenboim [16], and Matveev [11]. The proof of Theorem 1.3 is given in Section 3. In fact, we suppose that (1.2) has three solutions. We use the result due to Le, cited in Section 2, and Baker's method to prove that the largest solution (n_3, m_3) verifies $\max(m_3, n_3) < 8.5 \cdot 10^{16}$. Then we use some congruence properties to obtain $2 \leq y \leq 439\,682$. To completely solve the equation, we ran a program written in PARI/GP [13] to obtain the exceptional solutions. In Section 4, we use a similar method to prove Theorem 1.4.

2. Some lemmas

The following result is contained in the proof of Theorem 3 by Le, see [9, Formulas (12) and (15)]. We write these properties as a lemma.

LEMMA 2.1. *If (1.2) possesses three positive integers solutions (m_i, n_i) for any $i = 1, 2, 3$ with $2 \leq m_1 < m_2 < m_3$ and $2 \leq n_1 < n_2 < n_3$, then we have*

$$y^{n_2 - n_1} \mid m_3 - m_2 \quad \textit{and} \quad x^{m_2 - m_1} \mid n_3 - n_2. \tag{2.1}$$

We recall the following result on linear forms in logarithms due to Matveev [11].

LEMMA 2.2. *Denote by $\alpha_1, \ldots, \alpha_n$ algebraic numbers, not 0 or 1, by $\log \alpha_1, \ldots, \log \alpha_n$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and by $b_1, \ldots, b_n$ rational integers. Define $B = \max\{|b_1|, \ldots, |b_n|\}$, and $A_i = \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}$ for all $1 \leq i \leq n$, where $h(\alpha)$ denotes the absolute logarithmic Weil height of α. Assume that the number $\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$ does not vanish; then*

$$|\Lambda| \geq \exp\{-C(n, \varkappa) D^2 A_1 \cdots A_n \log(eD) \log(eB)\},$$

where $\varkappa = 1$ *if* $\mathbb{K} \subset \mathbb{R}$ *and* $\varkappa = 2$ *otherwise and*

$$C(n, \varkappa) = \min\left\{\frac{1}{\varkappa}\left(\frac{1}{2}en\right)^{\varkappa} 30^{n+3} n^{3.5}, 2^{6n+20}\right\}.$$

Finally, we recall a result obtained by Ribenboim (see [16, (C6.5), pp. 276–278]). In fact, if a, b are two positive integers such that $\gcd(a, b) = 1$, we define $m(a, b)$ and $n(a, b)$ to be positive integers such that

$$b^{n(a,b)} = 1 + la^{m(a,b)} \tag{2.2}$$

with l an integer, $\gcd(l, a) = 1$, $m(a, b) \geq 2$ and $n(a, b)$ minimal. Such $m(a, b)$ and $n(a, b)$ exist and we have the following lemma.

LEMMA 2.3. *Suppose that* a *and* b *are relatively prime integers with* $a, b \geq 2$. *If* $N, M \geq 2$ *are positive integers with* $M \geq m(a, b)$ *and* $b^N \equiv 1 \bmod a^M$, *then* N *is divisible by* $n(a, b)a^{M-m(a,b)}$.

3. Proof of Theorem 1.3

Suppose that (1.2) has at least three positive integer solutions (m_i, n_i) for any $i = 1, 2, 3$ with $2 \leq m_1 < m_2 < m_3$ and $2 \leq n_1 < n_2 < n_3$. Without loss of generality, we assume that x or y is not a perfect power.

From (1.2),

$$ax^{m_1} - by^{n_1} = ax^{m_2} - by^{n_2}. \tag{3.1}$$

This implies

$$ax^{m_1}(x^{m_2-m_1} - 1) = by^{n_1}(y^{n_2-n_1} - 1).$$

Since $\gcd(ax, by) = 1$, we have $ax^{m_1} \mid y^{n_2-n_1} - 1$. Lemma 2.1 implies $y^{n_2-n_1} \mid m_3 - m_2$. So we obtain

$$c < ax^{m_1} < y^{n_2-n_1} \leq m_3. \tag{3.2}$$

Let us consider the linear form

$$\Lambda = m_3 \log x - n_3 \log y + \log(a/b).$$

From (1.2) and as $c < y^{n_2-n_1}$,

$$\Lambda < e^{\Lambda} - 1 = \frac{c}{by^{n_3}} = \frac{c}{y^{n_2-n_1}} \cdot \frac{1}{by^{n_3-n_2+n_1}} < \frac{1}{by^2} \leq \frac{1}{4}.$$

Let $z = (c/ax^{m_3})$. We obtain $z = (c/(by^{n_3} + c)) \leq \frac{1}{5}$. Therefore,

$$|\Lambda| = |\log(1 - z)| < z(z + 1) < \frac{6}{5}z = \frac{1.2c}{ax^{m_3}}. \tag{3.3}$$

Then we deduce that

$$\log |\Lambda| < \log(1.2c) - m_3 \log x, \tag{3.4}$$

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and also

$$\log|\Lambda| < \log c - n_3 \log y. \tag{3.5}$$

Now we apply Lemma 2.2 with $D = 1$, $n = 3$, $\alpha_1 = x$, $\alpha_2 = y$, and $\alpha_3 = a/b$. Therefore, we take

$$A_1 = \log x, \quad A_2 = \log y, \quad A_3 = \max(a, b), \quad B = \max(m_3, n_3).$$

So we have

$$\log|\Lambda| > -1.391 \cdot 10^{11}(\log x)(\log y)(\log\max(a, b))(\log\max(em_3, en_3)). \tag{3.6}$$

We consider the upper bound for m_3 in two cases. First, if $m_3 \geq n_3$, then from (3.4) and (3.6)

$$m_3 < \frac{\log(1.2c)}{\log x} + 1.391 \cdot 10^{11}(\log y)(\log\max(a, b))(\log(em_3)).$$

Using (3.2), we have $\log(1.2c) < \log(1.2m_3)$. This and the fact that $1/\log x \leq 1/\log 2 < 1.45$ lead to

$$m_3 < 1.392 \cdot 10^{11}(\log y)(\log\max(a, b))(\log(em_3)). \tag{3.7}$$

Again (1.2) and (3.2) imply $\max(a, b) < ax^{m_1} < m_3$ and $y \leq y^{n_2 - n_1} < m_3$. Then (3.7) gives us

$$\frac{m_3}{(\log m_3)^2(\log(em_3))} < 1.392 \cdot 10^{11}.$$

It follows that

$$m_3 < 8.5 \cdot 10^{16}. \tag{3.8}$$

Second, if $m_3 < n_3$, then from (3.5) and (3.6),

$$n_3 < \frac{\log c}{\log y} + 1.391 \cdot 10^{11}(\log x)(\log\max(a, b))(\log(en_3)).$$

Using Lemma 2.1, we have $x^{m_2 - m_1} < n_3 - n_2 < n_3$. Notice that $c < m_3 < n_3$ and $\max\{a, b\} < m_3 < n_3$. Then we use a similar argument to obtain

$$n_3 < 8.5 \cdot 10^{16}. \tag{3.9}$$

Since $m_3 < n_3$, then m_3 is also bounded by above inequality.

From the above two cases, we have an upper bound for m_3 that is given by (3.8). Equation (3.1) and $ax^{m_1} > by^{n_1}$ imply $x^{m_2 - m_1} < y^{n_2 - n_1}$. Combining this with (3.2) and (3.8), we obtain

$$x^{m_2 - m_1} < y^{n_2 - n_1} < 8.5 \cdot 10^{16}. \tag{3.10}$$

As $x, y \geq 2$, we have $m_2 - m_1 < 57$ and $n_2 - n_1 < 57$.

Now, we suppose that $\min(m_1, n_1) = \min(m, n) > 1$ and we consider the equation

$$ax^{m_1}(x^{m_2-m_1} - 1) = by^{n_1}(y^{n_2-n_1} - 1) \quad \text{with } \gcd(ax, by) = 1.$$

Then there exist positive integers $m' = m_2 - m_1$ and $n' = n_2 - n_1$, such that

$$y^{n'} \equiv 1 \bmod x^2 \tag{3.11}$$

and

$$x^{m'} \equiv 1 \bmod y^2. \tag{3.12}$$

Since $n_1 \geq 2$, from $by^{n_1} < x^{m_2-m_1} < y^{n_2-n_1}$ we then have $n' = n_2 - n_1 \geq 3$. This result and (3.10) lead to

$$y < \sqrt[3]{8.5 \cdot 10^{16}} < 439\,683.$$

The congruences (3.11) and (3.12), with the upper bound given by (3.10), have a few solutions. To see this, we used PARI/GP [13] to write a short program for the computations. Here we give some details about the algorithm.

First, we searched for pairs (y, n') such that $y^{n'} - 1$ has a square factor with $2 \leq y \leq 439\,682$ and $n' \leq 56$. Also n' is bounded by $3 \leq n' < \log(8.5 \cdot 10^{16})/\log y$. For fixed y and n', the largest nonsquare-free divisor of $y^{n'} - 1$ has the form $X = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$ ($s_i \geq 2$). Then every possible x in (3.11) must be a divisor of X.

Second, for each $x \geq 2$, a fixed divisor of X, we searched for integers m' such that $1 \leq m' < \log(8.5 \cdot 10^{16})/\log x$. If y^2 is a divisor of $x^{m'} - 1$, then we output the pairs (x, y).

It took about 6 minutes to run the program. In all cases, we obtain $x, y \leq 220$. The pairs (x, y) that satisfy (3.11) and (3.12) are

$$\begin{aligned}(x, y) \text{ or } (y, x) \in \{&(2, 3), (2, 5), (2, 7), (2, 15), (2, 21), (3, 5), (3, 10),\\ &(3, 11), (3, 13), (3, 20), (3, 22), (3, 44), (3, 55), (3, 110), (3, 220),\\ &(5, 6), (6, 7), (7, 15), (7, 20), (7, 30), (11, 12), (13, 14), (19, 28)\}.\end{aligned} \tag{3.13}$$

This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

If (x, y) is not on the list (1.3), since there was at most one solution satisfying $\min(m, n) = 1$, then the theorem holds by Theorem 1.3. Therefore, we need only to consider (x, y) in (1.3). Suppose that (1.2) has at least four positive integer solutions (m_i, n_i) for all $i = 1, 2, 3, 4$ with $1 \leq m_1 < m_2 < m_3 < m_4$ and $1 \leq n_1 < n_2 < n_3 < n_4$. Also, we assume that x or y is not a perfect power.

From (1.2),

$$ax^{m_1}(x^{m_j-m_1} - 1) = by^{n_1}(y^{n_j-n_1} - 1) \quad \text{for } j = 2, 3, 4.$$

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Eliminating ax^{m_1} and by^{n_1}, we obtain

$$\frac{x^{m_k-m_1}-1}{x^{m_2-m_1}-1} = \frac{y^{n_k-n_1}-1}{y^{n_2-n_1}-1} \quad \text{for } k = 3, 4. \tag{4.1}$$

If $m_2 - m_1 = 1$, then it is obvious to see that $(x^{m_2-m_1} - 1) \mid (x^{m_k-m_1} - 1)$. Thus, we have also $(y^{n_2-n_1} - 1) \mid (y^{n_k-n_1} - 1)$. Then we obtain $(n_2 - n_1) \mid (n_k - n_1)$. Therefore, the equation

$$\frac{X^M - 1}{X - 1} = \frac{Y^N - 1}{Y - 1}, \quad M > 1,\ N > 1,$$

with $(X, Y) = (x^{m_2-m_1}, y^{n_2-n_1})$, has two positive integer solutions

$$(M, N) = \left(\frac{m_3 - m_1}{m_2 - m_1}, \frac{n_3 - n_1}{n_2 - n_1}\right) \quad \text{and} \quad \left(\frac{m_4 - m_1}{m_2 - m_1}, \frac{n_4 - n_1}{n_2 - n_1}\right).$$

This and Theorem 1.5 lead to a contradiction. Similarly, $n_2 - n_1$ cannot be equal to 1. Thus, we assume

$$m_2 - m_1 \geq 2 \quad \text{and} \quad n_2 - n_1 \geq 2. \tag{4.2}$$

It follows that $m_2, n_2 \geq 3$.

Note that $2 \leq m_2 < m_3 < m_4$ and $2 \leq n_2 < n_3 < n_4$, according to the proof of Theorem 1.3, see (3.10),

$$x^{m_3-m_2} < y^{n_3-n_2} < 8.5 \cdot 10^{16}, \tag{4.3}$$

and then $m_3 - m_2 < 57$ and $n_3 - n_2 < 57$. From (1.2), we obtain

$$ax^{m_2}(x^{m_3-m_2} - 1) = by^{n_2}(y^{n_3-n_2} - 1).$$

As $\gcd(ax, by) = 1$, this implies that ax^{m_2} divides $y^{n_3-n_2} - 1$ and by^{n_2} divides $x^{m_3-m_2} - 1$. There exist positive integers $m'' = m_3 - m_2$ and $n'' = n_3 - n_2$ such that

$$y^{n''} \equiv 1 \bmod x^3, \quad x^{m''} \equiv 1 \bmod y^3. \tag{4.4}$$

Again, we use PARI/GP [13] to write a short program for the computations. We found that only $(x, y) = (2, 3), (3, 2)$ satisfy congruences (4.4).

Let us consider the two remaining cases. When $(x, y) = (2, 3)$, Equation (1.2) becomes

$$a \cdot 2^m - b \cdot 3^n = c. \tag{4.5}$$

Using Lemma 2.3 and knowing that $2^6 = 1 + 7 \cdot 3^2$ and $2^{m_3-m_2} \equiv 1 \bmod 3^{n_2}$, then we have

$$6 \cdot 3^{n_2-2} \mid m_3 - m_2. \tag{4.6}$$

Since $m_3 - m_2 < 57$, then $n_2 - 2 \leq 2$. As $n_2 \geq 3$, we have $n_2 = 3$ or 4. As $1 \leq n_1 \leq n_2 \leq n_3$,

$$(n_1, n_2) = (1, 3),\ (1, 4) \text{ or } (2, 4).$$

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If $(n_1, n_2) = (1, 3)$ or $(2, 4)$, then one can see that $n_2 - n_1 = 2$. Thus, from $a \cdot 2^{m_1} \mid 3^{n_2 - n_1} - 1 = 8$, we have $a \cdot 2^{m_1} = 2^k$ for all $1 \le k \le 3$. Since $b \cdot 3^{n_1} < a \cdot 2^{m_1} \le 8$, then $b = 1$. Equation (4.5) becomes

$$2^m - 3^n = c.$$

Using the well-known theorem of Stroeker and Tijdeman [19] for Pillai's conjecture [15], the above equation has at most two positive integer solutions (m, n).

Now we consider $(n_1, n_2) = (1, 4)$. From $a \cdot 2^{m_1} \mid 3^{n_2 - n_1} - 1$, we have $a \le 13$. Moreover, the inequalities $a \cdot 2^{m_1} \le 26$ and $b \cdot 3^{n_1} < a \cdot 2^{m_1}$ give us $b \cdot 3^{n_1} \le 25$, then $b \le 25/3$. Therefore, one has $b \le 8$. Now we apply Lemma 2.2 with

$$A_1 = \log 2, \quad A_2 = \log 3, \quad A_3 = 13, \quad B = \max\{m_4, n_4\}.$$

This is similar to what we have done previously. If $m_4 \ge n_4$, from (3.7) we obtain

$$m_4 < 3.923 \cdot 10^{11} \log(em_4).$$

This implies that $n_4 < m_4 < 1.2 \cdot 10^{13}$. If $n_4 > m_4$, then we obtain the same result. Using the new bound, as $2^{m_3 - m_2} < 1.2 \cdot 10^{13}$, we obtain $m_3 - m_2 < 44$. Therefore, (4.6) implies $n_2 - 2 \le 1$. However, $n_2 = 4$, so this contradicts the hypothesis.

Finally, when $(x, y) = (3, 2)$, we use the same argument to the equation

$$a \cdot 3^m - b \cdot 2^n = c$$

and we obtain a contradiction on m_2. This completes the proof of Theorem 1.4.

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