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LOCALLY FLAT VECTOR LATTICES

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1. Preliminaries. Let G be a lattice-ordered group (l-group). If $X \subseteq G$, then let

$$X' = \{g \in G: |g| \wedge |x| = 0, \text{ for all } x \text{ in } X\}.$$

Then X' is a convex l-subgroup of G called a *polar*. The set $P(G)$ of all polars of G is a complete Boolean algebra with $'$ as complementation and set-theoretic intersection as meet. An l-subgroup H of G is *large* in G (G is an *essential extension* of H) if each non-zero convex l-subgroup of G has non-trivial intersection with H. If these l-groups are archimedean, it is enough to require that each non-zero polar of G meets H. This implies that the Boolean algebras of polars of G and H are isomorphic. If K is a cardinal summand of G, then K is a polar, and we write $G = K \boxplus K'$.

A convex l-subgroup P of G is *prime* if $a \wedge b = 0$ implies that a or b is in P. The set of primes forms a root system; that is, the primes containing a given prime form a totally ordered set. Each prime contains at least one minimal prime. A prime P is minimal if and only if $a' \not\subseteq P$ whenever $a \in P$. A normal prime P is maximal if and only if G/P is o-isomorphic to a subgroup of the reals $\mathbf{R}$.

We denote the lattice of convex l-subgroups of G by $C(G)$. If $g \in G$, the smallest element of $C(G)$ containing g is denoted by $G(g)$. If H and K are in $C(G)$, then $H \vee K$ denotes the smallest element of $C(G)$ containing H and K.

For further information about l-groups, the reader may consult [**4**] or [**2**].

The topological notation and terminology of [**6**] will be used. All topological spaces referred to will be Tychonoff. If X is a topological space, $C(X)$ denotes the vector lattice of continuous real-valued functions on X. For f in $C(X)$, $Z(f) = \{x \in X: f(x) = 0\}$ and $\text{coz}(f) = X - Z(f)$. Sometimes, to emphasize the space on which f is defined, we will write $\text{coz}(f, X)$ instead. We denote by $\bar{1}$ the element of $C(X)$ which is equal to 1 at all x in X. If f is in $C(X)$ and A is a clopen subset of X, then $f|A$ is the continuous function f times the characteristic function of A. The Stone–Čech compactification of a space X is denoted by βX; its Hewitt

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realcompactification by νX. If $f: X \to K$ is continuous and K is compact (realcompact), $f^{\beta}: \beta X \to K$ ($f^{\nu}: \nu X \to K$) denotes its extension.

The Boolean algebras of regularly open and clopen subsets of a topological space X will be denoted by $\mathscr{R}(X)$ and $\mathscr{C}(X)$, respectively. A space X is *extremally disconnected* if $\mathscr{R}(X) = \mathscr{C}(X)$; it is *strongly zero dimensional* if $\mathscr{C}(X)$ is a base for open sets. The Boolean algebras $P(C(X))$ and $\mathscr{R}(X)$ are always isomorphic [**1**].

The following construction of the *projective cover*, or *absolute*, of a topological space X will prove useful (see [**7**]). Let EX be the set of all fixed open ultrafilters on X. For each open set U of X, let

$$\mathfrak{O}(U) = \{p \in EX: U \in p\}.$$

If the set of such $\mathfrak{O}(U)$ is used as a base for open sets, EX becomes an extremally disconnected space. The map $\pi: EX \to X$ that takes each ultrafilter to the point of X to which it converges is continuous, and the map

$$\mathfrak{O}: \mathscr{R}(X) \to \mathscr{C}(EX)$$

is a Boolean algebra isomorphism.

A *section* of EX is a subspace $\bar{X}$ contained in EX such that $\bar{X} \cap \pi^{-1}(x)$ is a singleton, for each x in X; such a space is dense in EX. If Y represents X equipped with a finer topology so that Y is extremally disconnected and $\mathscr{C}(Y)$ and $\mathscr{R}(X)$ are isomorphic in the natural way, then Y is homeomorphic to a section of EX [**8**]. Consequently, an extremally disconnected space X admits no such space Y.

Several of the topological proofs in this paper require that discrete spaces be realcompact, which is true if the cardinality of the space is non-measurable; since all cardinals obtainable from $\aleph_0$ by the standard processes of cardinal arithmetic are nonmeasurable [**6**], it is not a serious restriction to posit the following.

Axiom. All cardinals are nonmeasurable.

2. Locally flat *l*-groups. An l-group is *hyperarchimedean* if each of its l-homomorphic images is archimedean. We state here for future reference a theorem listing several characterizations of such groups. This theorem is due to several authors, as discussed in [**5**]. In particular, condition (d) is due to Bigard.

THEOREM 2.1. *Let G be an l-group. The following are equivalent:*

(a) *G is hyperarchimedean.*

(b) *Each prime subgroup of G is maximal and hence minimal.*

(c) *$G = G(g) \boxplus g'$, for all $g \in G$.*

(d) *G can be represented as a group of real-valued functions on a topological space, with pointwise addition and order, such that,*

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(i) *G separates points, and*
(ii) *the cozero set of each g in G is compact and open.*

Condition (b) can be weakened in a natural way to define a somewhat larger class of l-groups, as follows. Call a prime subgroup of an l-group a *minimax* prime if it is both maximal and minimal. Then, for any l-group G, let $M(G)$ be the set of all normal minimax primes of G. An l-group is *locally flat* if $\bigcap M(G) = 0$. The class of all locally flat l-groups is denoted by Φ.

THEOREM 2.2. *Let G be an l-group. Then G is locally flat if and only if G can be embedded as a large l-subgroup of $C(X)$, where $\{Z(f): f \in G\}$ is a clopen base for the closed sets of the topology.*

Proof. ($\Leftarrow$) Let $\pi_x: G \to \mathbf{R}$ be defined by $\pi_x(g) = g(x)$, where $x \in X$, $g \in G$, and G has been identified with its l-isomorphic copy in $C(X)$. Each such map is an l-homomorphism, and since $\pi_x(G) \subseteq \mathbf{R}$, its kernel P_x is either a maximal prime or G. Since G is embedded in $C(X)$, $\bigcap \{P_x\} = 0$. If $0 < g \in P_x \subset G$, then coz g is closed and $x \notin$ coz g. Thus there exists $Z(f)$ where $f \in G$ so that $Z(f) \supseteq$ coz g and $x \notin Z(f)$. Without loss of generality we may assume that $f \in G$. Then $f \wedge g = 0$, while $0 \neq f(x) = \pi_x(f)$, and so $g' \nsubseteq P_x$. This shows that P_x is a minimal prime, and so $P_x \in M(G)$.

($\Rightarrow$). Since G may be embedded into $\Pi\{G/P: P \in M(G)\}$, and each G/P is a subgroup of the real numbers (because P is a maximal prime), G is clearly archimedean. Any abelian l-group G admits a unique divisible hull G^d [**4**], and G^d is an a-extension of G (that is, the lattices $C(G)$ and $C(G^d)$ are isomorphic) [**3**]. Consequently, if $G \in \Phi$, then $G^d \in \Phi$. Thus, we may assume that G is divisible.

Choose a maximal disjoint collection $\{g_\gamma\} \subseteq G^+$. Let

$$X = \{P \in M(G): g_\gamma \notin P, \text{ for some } \gamma\}.$$

If $0 \leqq h \in \bigcap X$, then $h \wedge g_\gamma \in \bigcap M(G) = 0$, and so by maximality $h = 0$; thus $\bigcap X = 0$. We henceforth will refer to elements of X as P_x or x, depending on context. Let

$$\text{coz } g_\gamma = \{x \in X: g_\gamma \notin P_x\}.$$

Then $\{\text{coz } g_\gamma\}_\gamma$ is a set-theoretic partition of X. Since for all $x \in X$, G/P_x is (isomorphic to) a subgroup of the real numbers and there exists a unique γ such that $g_\gamma \notin P_x$, we may choose an automorphism $r_x: \mathbf{R} \to \mathbf{R}$ so that $r_x\pi_x(g_\gamma) = 1$, if $g_\gamma \in P_x$, where π_x is the usual map, $\pi_x: G \to G/P_x$. Therefore, we have l-embedded G into $\Pi\{R_x: x \in X\}$, where $R_x \subseteq \mathbf{R}$, and each g_γ is (identified with) the characteristic function on coz g_γ. We let $\{\text{coz } g: g \in G\}$ be a base for open sets on X, where coz $g = \{x \in X: g \in P_x\}$. Then $Z(f) = Z(|f|)$ is open, for all $f \in G$, for if $x \in Z(f)$,

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then $f \in P_x$ and thus there exists $g \in (f')^+ \backslash P_x$; consequently, $x \in \operatorname{coz} g$ and $\operatorname{coz} g \subseteq Z(f)$. (Furthermore, this topology is Hausdorff: given $x, y \in X$, choose $f \in P_x \backslash P_y$. Then $x \in Z(f)$, $y \in \operatorname{coz}(f)$, and both sets are open.)

Now let $f \in G$. We claim that f, considered as a function on X, is continuous. Since G is a group, we need only show that $f^{-1}(a, \infty)$ is open, for all $a \in \mathbf{R}$. But

$$f^{-1}(a, \infty) = \bigcup \{ f^{-1}(b, \infty) : a < b, b \in \mathbf{Q}\},$$

and so we may assume that $a \in \mathbf{Q}$. We claim that

$$f^{-1}(a, \infty) = \bigcup\{\operatorname{coz}((f - ag_\gamma) \vee 0) \cap \operatorname{coz} g_\gamma\}.$$

Since each cozero set is open, this would show that f is continuous. Suppose that $x \in f^{-1}(a, \infty)$. Then $f(x) > a$. Choose γ such that $g_\gamma(x) = 1$. Then

$$(f - ag_\gamma)(x) = f(x) - a > 0,$$

and so $x \in \operatorname{coz}((f - ag_\gamma) \vee 0) \cap \operatorname{coz} g_\gamma$. Conversely, if there exists γ such that $x \in \operatorname{coz}((f - ag_\gamma) \vee 0) \cap \operatorname{coz} g_\gamma$, then

$$g_\gamma(x) = 1 \text{ and } (f - ag_\gamma)(x) > 0;$$

that is, $f(x) > a$, and so $x \in f^{-1}(a, \infty)$.

If P is a polar of $C(X)$, then P is the set of all functions which live on some regularly open set U of X [**1**]; consequently, it is clear that G is large in $C(X)$.

We can now derive as a corollary a characterization of locally flat l-groups which generalizes Bigard's condition (d) of Theorem 2.1.

COROLLARY 2.3. *G is a locally flat l-group if and only if G can be represented as a group of real-valued functions on a topological space, with pointwise addition and order, such that*

(i) *G separates points, and*

(ii) *the support of each g in G is clopen.*

Proof. The implication ($\Rightarrow$) is clear from the theorem, and since the proof of ($\Leftarrow$) above does not use that each g in G is continuous, but only that $\operatorname{coz} g$ is clopen, the corollary is proved.

3. Locally flat vector lattices. Henceforth, we shall restrict our attention to locally flat vector lattices; that is, locally flat l-groups, which are also real vector spaces. The most important example of such may be defined as follows.

For a topological space X, we call f in $C(X)$ *locally flat* if, for all x in X, there exists a neighborhood U of x so that f is constant on U. Now let

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$F(X)$ be the set of all locally flat elements of $C(X)$. It is easily verified that $F(X)$ is an l-subgroup of $C(X)$. Furthermore, if X is strongly zero dimensional, then $F(X)$ is a large l-subgroup of $C(X)$, because it contains all real multiples of characteristic functions of clopen sets.

THEOREM 3.1. *Let G be a vector lattice. Then G is locally flat if and only if G can be embedded as a large l-subgroup of $F(X)$, where $\{Z(f): f \in G\}$ is a clopen base for the closed sets of the topology of X.*

Proof. It is easy to see that in the embedding of Theorem 2.2, G is included in $F(X)$.

In light of Theorem 3.1, it is natural to ask whether l-groups of the form $F(X)$ play a role as "maximal" locally flat vector lattices. It is in order to answer this question that we make the following definitions.

A vector lattice H is a Φ-*extension* of the vector lattice G if we have
(1) G is large in H,
(2) G is locally flat, and
(3) for all P in $M(G)$ there exists Q in $M(H)$ such that $Q \cap G = P$.
We say that a locally flat vector lattice is Φ-*closed* if it admits no proper Φ-extensions.

Note that if G is locally flat and H is a Φ-extension, then H is locally flat. Also, if G is a locally flat vector lattice with weak order unit e (that is, $e' = 0$), then the embedding of Theorem 3.1 can be chosen so that $X = M(G)$ and e is mapped to $\bar{1}$; in this case $F(M(G))$ is a Φ-extension of G, or rather, of an l-isomorphic copy of G. We shall regularly make this sort of identification.

We shall first examine the algebraic and topological properties possessed by $F(X)$ and $M(F(X))$.

PROPOSITION 3.2. *Let X be a topological space. Then $F(X)$ is l-isomorphic to a direct limit of products of reals.*

Proof. Let $\Gamma(X)$ be the collection of all set-theoretic clopen partitions of X. For each α in $\Gamma(X)$ let

$$\Pi(\alpha) = \Pi\{\mathbf{R}_a : a \in \alpha\}.$$

Define $\rho_\alpha : \Pi(\alpha) \to F(X)$ by the following:

$$\rho_\alpha(- - r_a - -) = f,$$

where $f(x) = r_a$, if $x \in a$. Because α is a clopen partition, f is in $F(X)$. Clearly ρ_α is an l-monomorphism. Now $\Gamma(X)$ is partially ordered by $\gg$, where $\alpha \gg \beta$ means that β refines α. If α and β are in $\Gamma(X)$, define $\alpha \cap \beta$ to be

$$\{a \cap b = : a \in \alpha \text{ and } b \in \beta\};$$

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then $\alpha \cap \beta \in \Gamma(X)$, and $\Gamma(X)$ is lower-directed. If $\alpha \gg \beta$, define

$$\pi_{\alpha\beta}: \Pi(\alpha) \to \Pi(\beta) \text{ by}$$
$$\pi_{\alpha\beta}(f)(b) = f(a),$$

where $a \supseteq b$. This is clearly a well-defined l-monomorphism. But now

$$F(X) = \bigcup \{\rho_\alpha(\Pi(\alpha)) : \alpha \in \Gamma(X)\},$$

because if f is in $F(X)$, then

$$\{f^{-1}(r) : r \in \text{Range}(f)\}$$

is a clopen partition of X.

Let X be a strongly zero dimensional space. A filter $\mathfrak{F}$ on $\mathscr{C}(X)$ has the *countable intersection property* (CIP) if each countable subset of has nonvoid intersection. Such a filter has the *partition-meeting property* (PMP) if each clopen set-theoretic partition of X has nonvoid intersection with $\mathfrak{F}$. Let

$$mX = \{p \in \beta X : \text{there is a filter } \mathfrak{F} \text{ on } \mathscr{C}(X) \text{ such that } \mathfrak{F} \text{ has CIP and PMP and } \mathfrak{F} \to p\}.$$

THEOREM 3.3. *Let X be a strongly zero dimensional space. Then there is a one-to-one correspondence between $M(F(X))$ and mX. (This theorem and subsequent results depend on the axiom mentioned in Section* 1).

Proof. For P in $M(F(X))$, let $\mathfrak{F}$ be $\{Z(f) : f \in P\}$. If f and g are in P, then

$$Z(f) \cap Z(g) = Z(|f| \vee |g|),$$

and so is in $\mathfrak{F}$. Let K be a clopen set containing $Z(f)$, where f is in P. Because P is minimal, there exists g in $f'\backslash P$. But then

$$|g| \wedge \bar{1}|(X\backslash K) = 0$$

and so $\bar{1}|(X\backslash K)$ is in P and K is in $\mathfrak{F}$. Therefore $\mathfrak{F}$ is a filter on $\mathscr{C}(X)$.

To show that $\mathfrak{F}$ has CIP, suppose that $\bigcap_{i=1}^{\infty} Z(f_i) = \emptyset$, where we may assume that $Z(f_i)$ contains $Z(f_{i+1})$. Let

$$A_0 = X\backslash Z(f_1), \text{ and } A_i = Z(f_i)\backslash Z(f_{i+1}).$$

Then $\{A_i\}$ is a clopen partition of X and so if we define f by setting $f(A_i) = i$, then f is in $F(X)$. But $Z(f) = X\backslash Z(f_1)$ and so $f \notin P$. But

$$P + f \gg P + \bar{1},$$

which contradicts the fact that P is a maximal prime.

Finally, we show that $\mathfrak{F}$ has PMP. Suppose that α is a clopen partition of X, and let Π be $\rho_\alpha(\Pi(\alpha))$, with notation as in the proof of 3.2. Then $P \cap \Pi$ is a proper maximal prime of Π. Because the discrete space α is

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realcompact,

$$P \cap \Pi = \{f \in \Pi : f \mid a = 0, \text{some fixed } a \in \alpha\}.$$

Thus, $\bar{1}|(X\backslash a)$ is in $P \cap \Pi$ and so $a \in \mathfrak{J}$.

On the other hand, suppose that p is in mX, and let

$$P = \{f \in F(X) : f^{\beta}(p) = 0\},$$

where $f^{\beta} : \beta X \to \mathbf{R} \cup \{\infty\}$ is the unique extension of f. Because $p \in \nu X$, P is a maximal prime. To show that P is a minimal prime, suppose that f is in P, and let

$$\alpha = \{f^{-1}(r) : r \in \text{Range}(f)\}.$$

Then $a \in \mathfrak{J}$, for some a in α, and so p is in $\text{cl}_{\beta X}\, a$. But because f is constant on a and $f^{\beta}(p) = 0$, this means that $f\,|a = 0$. But then $\bar{1}|a$ is in $f'\backslash p$ and so P is a minimal prime.

Note. It is clear that the topology on $M(F(X))$ induced by $F(X)$ is contained in the topology it inherits from βX. The latter topology, however, may be finer (see Example 4.2).

COROLLARY 3.4. *Let X be an extremally disconnected space. Then $M(F(X))$ is homeomorphic to νX, and so $F(X)$ and $F(M(F(X)))$ are l-isomorphic.*

Proof. Because νX consists of the z-ultrafilters on X with CIP [**6**], it is clear that mX is contained in νX. Suppose that there exists p in $\nu X\backslash mX$. Then the clopen ultrafilter $\mathfrak{J}$ which converges to p does not have PMP. So, we can choose a clopen partition α which is disjoint from $\mathfrak{J}$. Let $\mathfrak{U}$ be the set of all families $\mathfrak{T}$ of α such that p is in $\text{cl}_{\nu X} \cup \mathfrak{T}$. Because the discrete space α is realcompact, $\mathfrak{U}$ has a countable subset $\{\mathfrak{D}_i\}$ such that $\cap\, \mathfrak{D}_i = \emptyset$, where without loss of generality $\mathfrak{D}_i \supset \mathfrak{D}_{i+1}$. If we let $A_0 = X\backslash \mathfrak{D}_1$ and $A_i = \mathfrak{D}_i\backslash \mathfrak{D}_{i+1}$ and define f as in the proof of Theorem 3.3, we obtain an element of $F(X)$. But there exists an extension f^{ν}: $\nu X \to \mathbf{R}$, which is impossible, because $f^{\nu}(p)$ cannot be finite. Thus, $\mathfrak{J}$ has PMP. Now, we may identify νX and $M(F(X))$ as sets; then νX and $M(F(X))$ are extremally disconnected spaces with the same Boolean algebra of clopen sets, and are thus homeomorphic.

We can now characterize vector lattices of the form $F(X)$, where X is an extremally disconnected space. In order to do this we make use of the absolute introduced in Section 1.

THEOREM 3.5. *Let G be a vector lattice with weak order unit. Then G is Φ-closed if and only if G is l-isomorphic to $F(X)$, where X is an extremally disconnected space.*

Proof. ($\Rightarrow$) Let G be Φ-closed, with weak order unit e. Consider the l-embedding $G \to F(M(G))$ which takes e to $\bar{1}$, given by 3.1, and the

l-embedding $F(M(G)) \to F(EM(G))$ given by $f \to f \circ \pi$. The composition of these maps makes $F(EM(G))$ a Φ-extension of G, and so G is l-isomorphic to $F(EM(G))$.

($\Leftarrow$). Let X be an extremally disconnected space, and suppose that H is a Φ-extension of $F(X)$. We may assume that X is in a one-to-one correspondence with $M(F(X))$, because of Theorem 3.4. Choose $\bar{X}$, a dense subspace of $M(H)$, so that for each x in X, there is a unique $\bar{x}$ in $\bar{X}$ such that,

$$\bar{x} \cap F(X) = \{ f \in F(X) : f(x) = 0\}.$$

Now l-embed H into $F(\bar{X})$, so that $\bar{1}$ is mapped to $\bar{1}$. We may identify X and $\bar{X}$ set-theoretically; but then they are extremally disconnected spaces with the same clopen sets, and so identical topologically. Thus, $F(X)$ is l-isomorphic to H.

We see from the proof of Theorem 3.5 that each locally flat vector lattice G with weak order unit admits a Φ-closed Φ-extension $F((EM(G))$. However, Φ closed Φ-extensions need not be unique (see Example 4.3). In order to identify $F(EM(G))$ algebraically, we need to consider more restrictive classes of extensions.

If H is a Φ extension of the locally flat vector lattice G and, for each P in $M(H)$, $P \cap G$ is in $M(G)$, we say that H is a *strong* Φ-*extension* of G.

Unfortunately, if G is a locally flat vector lattice with weak order unit, $F(M(G))$ need not be a strong Φ-extension of G (see Example 4.1). We consequently define $M^2(G)$ to be the set of all primes of G of the form $Q \cap G$, where Q is in $M(F(M(G)))$. Then a Φ-extension H of G is called an *intermediate* Φ-*extension* if, for each P in $M(H)$, $P \cap G$ is in $M^2(G)$. Clearly $F(M(G))$ is such an extension. We will now identify $M^2(G)$ algebraically.

Let G be a locally flat vector lattice. A polar K of G is a Φ-*summand* if

$$K \vee K' \nsubseteq P,$$

for all minimax primes P. The following proposition, which is easy to prove, identifies Φ-summands as coming from cardinal summands of $F(M(G))$:

PROPOSITION 3.6. *For a locally flat vector lattice G with weak order unit, the following are equivalent:*

(a) *K is a Φ-summand of G.*

(b) *K^{**} is a cardinal summand of $F(M(G))$ (where $*$ is the polar operation for $F(M(G))$).*

(c) *$\{P \in M(G): K \nsubseteq P\}$ is a clopen subset of $M(G)$.*

Note. If M is a collection of primes of an arbitrary l-group and $\bigcap M = 0$, then it is clearly possible to define M-summand in an analogous way. In particular, if M is the set of all primes (or all minimal primes), then an M-summand is just a cardinal summand. For further discussion of this case, see [**9**].

PROPOSITION 3.7. *Let G be a locally flat vector lattice with weak order unit. Then $M^2(G)$ is the set of maximal primes P of G such that if $\{K_\alpha\}$ is a partition of $P(G)$ consisting of Φ-summands, then K_α is not contained in P, for some α. Furthermore, if $M^2(G)$ is equipped with the topology induced by G, then $F(M^2(G))$ and $F(M(G))$ are l-isomorphic.*

Proof. Suppose that F is a minimax prime of $F(M(G))$. Then $\bar{1} \in G\backslash Q$ and so $Q \cap G$ is a proper maximal prime of G. If $\{K_\alpha\}$ is a partition of Φ-summands and $K_\alpha \subseteq P$ for all α, then $K_\alpha' \nsubseteq P$, for all α. But then $K_\alpha^{**} \subseteq Q$. Let

$$\Pi = \{f \in F(M(G)) \colon f \mid \text{coz } K_\alpha \text{ is constant for all } \alpha\},$$

an l-subgroup of $F(M(G))$ isomorphic to a product of reals. Because $Q \cap \Pi$ is a maximal prime of Π,

$$Q \cap \Pi = \{f \in \Pi \colon f \mid \text{coz } K_\alpha = 0, \text{ some fixed } \alpha\},$$

which is a contradiction. Therefore, $Q \cap G$ is as required.

On the other hand, suppose that P is a maximal prime of G so that if $\{K_\alpha\}$ is a partition of Φ-summands, then some $K_\alpha \nsubseteq P$. Let Q be a maximal prime of $C(M(G))$ such that $Q \cap G = P$. Then,

$$Q \cap F(M(G)) = \{f \in F(M(G)) \colon f^\nu(p) = 0\},$$

for some fixed p in $\nu M(G)$. But if α is a clopen partition of $M(G)$, then $\{K \in P(G) \colon \text{coz } K \in \alpha\}$ is a partition of Φ-summands and so some such K is not contained in P. That is, coz $K \in p$. Thus, by Theorem 3.3, Q is in $M(F(M(G)))$.

Because $M(G)$ is a dense subspace of $M^2(G)$, we have the natural l-embedding $F(M^2(G)) \to F(M(G))$. If α is a clopen partition of $M(G)$, and $a \in \alpha$, let

$$K_a = \{f \in G \colon \text{coz} f \subseteq a\}.$$

Then $\{K_a\}$ is a partition of Φ-summands and so if P is in $M^2(G)$, then some K_a is not contained in P. Thus, $\{\text{coz}(K_a, M^2(G))\}$ is a clopen partition of $M^2(G)$. This means that the l-embedding above is onto.

PROPOSITION 3.8. *Let X be a strongly zero dimensional space. Then $F(EX)$ is a strong Φ-extension of $F(X)$.*

Proof. We here identify $F(X)$ with its image under the l-embedding $f \to f \circ \pi$. Let P be a minimax prime of $F(X)$. The proof of 3.3 shows that

$$P = \{f \in F(X) \colon f^\nu(p) = 0\},$$

for some p in νX. If we consider π as a map from EX into νX, then we

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have the map $\pi^\nu : \nu EX \to \nu X$. Choose r in $(\pi^\nu)^{-1}(p)$. Then r is in νEX, which is homeomorphic to $M(F(EX))$, because EX is extremally disconnected. Thus, $\{f \in F(\nu EX): f(r) = 0\}$ is a minimax prime of $F(EX)$ which cuts down to P, and so $F(EX)$ is a Φ-extension of $F(X)$.

Now, suppose Q is a minimax prime of $F(EX)$. Then Q corresponds to a clopen ultrafilter $\mathscr{G}$ on EX with CIP and PMP. Let $\mathfrak{J}$ be $\pi(\mathscr{G}) \cap \mathscr{C}(X)$. It is easy to check that $\mathfrak{J}$ is a clopen ultrafilter on X with CIP and PMP, which corresponds to the minimax prime $Q \cap F(X)$. Thus, $F(EX)$ is a strong Φ-extension of $F(X)$.

THEOREM 3.9. *Let G be a locally flat vector lattice with weak order unit. Then $F(EM(G))$ is the unique minimal Φ-closed intermediate Φ-extension of G.*

Proof. It is clear that $F(EM(G))$ is a Φ-closed intermediate Φ-extension of G, because $F(EM(G))$ is a strong Φ-extension of $F(M(G))$. Suppose then that $F(X)$ is a Φ-closed intermediate Φ-extension of G. We may assume that the same weak order unit e of G is mapped to $\bar{1}$ in both $F(X)$ and $F(EM(G))$. We may also assume that each minimax prime of $F(X)$ is the form P_x, where

$$P_x = \{f \in F(X): f(x) = 0\}.$$

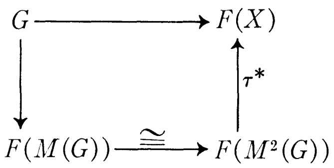
Define $\tau : X \to M^2(G)$ by $\tau(x) = P_x \cap G$. Let g be in G; then

$$\tau^{-1}(\operatorname{coz}(g, M^2(G))) = \operatorname{coz}(g, X),$$

and so τ is continuous. We then define

$$\tau^* : F(M^2(G)) \to F(X)$$

by $\tau^*(g) = g \circ \tau$. This is an l-monomorphism which makes the following diagram commute:



We now show that $F(EM(G))$ can be l-embedded into $F(X)$. We do this by showing that each set-theoretic clopen partition of $\nu EM(G)$ induces one on X, in a way that preserves the elements of G. Note that we may define a continuous map

$$\pi : \nu EM(G) \to M^2(G)$$

in the same way that we defined τ. (This map is called π because $\pi | EM(G)$ is the usual map from $EM(G)$ to $M(G)$). If α is a clopen set-theoretic partition of $\nu EM(G)$, then $\{\operatorname{Int} \operatorname{cl} a : a \in \alpha\}$ is a maximal disjoint collection of regularly open subsets of $M^2(G)$, whose closures cover $M^2(G)$.

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But then $\{\tau^{-1}(\text{Int cl } a)\}$ is a clopen set-theoretic partition of X. Consequently, we may l-embed $F(EM(G))$ into $F(X)$ in a way which preserves the elements of G. If $F(X)$ were another minimal Φ-closed intermediate Φ-extension, then we could in turn l-embed $F(X)$ into $F(EM(G))$; the composition of these maps would be the identity. Thus $F(EM(G))$ is the unique minimal Φ-closed intermediate Φ-extension of G.

It is also possible to identify certain maximal Φ-closed Φ-extensions, in the following sense. A Φ-extension H of a locally flat vector lattice G with weak order unit is a Φ-closure of G if whenever $G \subseteq H \subseteq K$, and K is a Φ-extension of G, then $H = K$. It is of course clear that a Φ-closure is Φ-closed.

THEOREM 3.10. *Let G be a locally flat vector lattice with weak order unit. Then G has Φ-closures, and each is of the form $F(X)$, where X is a section of $EM(G)$.*

Proof. If X is a section of $EM(G)$, then the l-embedding $f \rightarrow f|X$ of $F(EM(G))$ into $F(X)$ clearly makes $F(X)$ a Φ-extension of G. If $F(X) \subseteq K$, a Φ-extension of G, then we choose $\bar{X} \subseteq M(K)$, so that X and $\bar{X}$ are in a one-to-one correspondence. Then we can l-embed K into $F(\bar{X})$. But X and $\bar{X}$ are both extremally disconnected and so homeomorphic; thus $K = F(X)$.

Suppose now that H is a Φ-closure of G. Then we may assume that H is of the form $F(Y)$, where Y is extremally disconnected and realcompact. For each P in $M(G)$, choose y in Y so that $P_y \cap G = P$. Then the set X of such y is a dense subspace of Y, and so there exists the l-embedding of $F(Y)$ into $F(X)$ taking f to $f|X$. Now $F(X)$ is a Φ-extension of G, and so $F(Y)$ is l-isomorphic to $F(X)$. But X is homeomorphic to a section of $EM(G)$ [**8**].

4. Examples, remarks and open questions.

4.1. A locally flat vector lattice G where $F(M(G))$ is not a strong Φ-extension of G.

Let W be the topological space consisting of all countable ordinals, and let W^* be $W \cup \{\omega_1\}$, where ω_1 is the first uncountable ordinal. Then

$$W^* = \beta W = \nu W.$$

Define X to be the topological sum of countably many copies W_i of W. If x is the point ω_1 in W_1, and $P = \{f \in F(X): f(x) = 0\}$, then P is a minimax prime of $F(X)$. Let

$$\Sigma = \left(\sum_{i=1}^{\infty} F(W_i)\right) \cap P,$$

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a convex l-subgroup of $F(X)$. Define the element h of $F(X)$ by the following:

$$h|W_n = 0 \text{ if } n \text{ is odd;}$$
$$h|W_n = 1/n \text{ if } n \text{ is even.}$$

Let G be $\Sigma \oplus \mathbf{R}\bar{1} \oplus \mathbf{R}h$, a large l-subgroup of $F(X)$. Note that $P \cap G$ is not a minimax prime of G because $h \in P \cap G$, while $h' \subseteq P$. Thus, there is a one-to-one correspondence between $M(G)$ and $X\backslash\{x\}$. However,

$$F(X) = F(X\backslash\{x\}) = F(M(G)),$$

and so $P \cap G$ is in $M^2(G)$.

4.2. A strongly zero-dimensional space X where the topology on $M(F(X))$ induced by $F(X)$ is coarser than the topology it inherits from νX.

In problems 16M, 16N, and 16P of [**6**], the topological spaces Δ_1 and Δ are defined so that

$$\Delta_1 \subset \Delta \subset W^* \times [0, 1].$$

They show that Δ_1 has a clopen base, but $\Delta_1 \cup \{p\}$ does not, for any p in $\Delta\backslash\Delta_1$. Furthermore, Δ_1 is C-embedded in Δ. If p is chosen in $\{\omega_1\} \times$-$\{$irrationals$\}$ it is easy to check that

$$F(\Delta_1) = F(\Delta_1 \cup \{p\}).$$

But $\Delta_1 \cup \{p\}$ given the subspace topology from $M(F(\Delta_1))$ has a clopen base, and so the topology on $\Delta_1 \cup \{p\}$ from $\nu\Delta_1$ must be finer.

4.3. A locally flat vector lattice G with distinct Φ-closed Φ-extensions H and K. Also, H is contained in K, and so K is not a Φ-extension of H.

Let G be $F(\beta\mathbf{Q})$, where $\mathbf{Q}$ is the space of rationals. Then let H be $F(E\beta\mathbf{Q})$, clearly a proper Φ-closed Φ-extension of G. Now $E\beta\mathbf{Q}$ is compact, and so each clopen partition of it is finite. However, it is easy to choose a section Y of $E\beta\mathbf{Q}$ which has an infinite clopen partition, and so $K = F(Y)$ is a Φ-closed Φ-extension of G so that $K \supset H$.

4.4. In Theorem 5.3 of [**9**], Šik in effect claims that if X is an extremally disconnected space, then $C(X)$ is locally flat. However, it can be shown that $C(X)$ is locally flat if and only if νX contains a dense subset of P-points (a point at which every continuous function is locally flat). If X has nonmeasurable cardinality, however, an extremally disconnected space has no P-points (see exercise 12 H in [**6**]).

4.5. Suppose that G is an arbitrary l-group, and the intersection of the collection of all (not necessarily normal) minimax primes is zero. Is G

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locally flat? The answer is of course yes if this condition implies representability.

4.6. Is Theorem 3.1 true if we don't assume that G is a vector lattice? In other words, if G is a locally flat l-group, is the minimal vector lattice which contains G locally flat? If the answer is yes, all of the theory of Section 3 applies to locally flat l-groups.

4.7. How much of the theory of Section 3 can be extended to locally flat vector lattices without weak order unit?

4.8. The class of locally flat l-groups is closed under taking convex l-subgroups, but not under taking l-subgroups. Is it closed under taking large l-subgroups?

4.9. Does a locally flat vector lattice with weak order unit admit distinct Φ-closures? For an example it is only necessary to obtain a strongly zero dimensional space X and sections Y and Z of EX so that $F(Y)$ and $F(Z)$ are not l-isomorphic.

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