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Elliptic polylogarithms: An analytic theory

To the memory of Roman Smolensky

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Abstract. In this article we introduce a natural 'elliptic' generalization of the classical polylogarithms, study the properties of these functions and their relations with Eisenstein series.

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Introduction

The notion of the elliptic polylogarithm functions as a natural generalization of the usual polylogarithms was introduced in [BL, 4.8]. In this article we study the properties of these functions. Some of these properties are equivalent to theorems of [BL] but I will prove them purely analytically.

The paper is organized as follows. In the first section we introduce some version of usual polylogarithms which are more convenvient for generalization and describe their properties. In the second section we define the elliptic polylogarithms and prove the simplest facts about them. The modular properties of elliptic polylogarithms are discussed in the third section. In the fourth section we relate elliptic polylogarithms with classical Eisenstein series.

I'll use some standard notations: $H := (\tau \in \mathbb{C}, \Im\tau > 0)$, $\mathbf{e}(t) := \exp(2\pi i t)$, $z = \mathbf{e}(\xi), q = \mathbf{e}(\tau), w = \mathbf{e}(\eta)$;

$$\begin{aligned}\theta(\xi,\tau) &= \sum_{j=-\infty}^{\infty} \mathbf{e}(\tfrac{1}{2}(j+\tfrac{1}{2})^2\tau + (j+\tfrac{1}{2})\xi) \\ &= q^{1/8}(z^{1/2} - z^{-1/2}) \prod_{j=1}^{\infty}(1-q^j)(1-q^j z)(1-q^j z^{-1}).\end{aligned}$$

1. Debye polylogarithms

DEFINITION 1.1. The nth Debye polylogarithm $\Lambda_n(\xi)$ is the multivalued analytic function on $\mathbb{C}\backslash\mathbb{Z}$ given by the integral

$$\Lambda_n(\xi) = \int_\xi^{i\infty} \frac{t^{n-1}}{(n-1)!} \frac{\mathrm{d}t}{\exp(-2\pi i t) - 1}.$$

For ξ in the upper half-plane this is a one-valued function by choosing the vertical path of integration from ξ to $i\infty$; it is clear that the integral converges and is bounded by

$$\Lambda_n(\xi) = O(|\xi|^{n-1} \exp(-2\pi \Im(\xi)), \quad (\Im(\xi) > 1). \tag{1}$$

We will use two single-valued branches $\Lambda_n^+(\xi)$ and $\Lambda_n^-(\xi)$ which are defined on the plane $\mathbb{C}$ without $(-\infty, 0) \cup (1, \infty)$ and $(-\infty, -1) \cup (0, \infty)$ respectively and are the analytic continuations to the lower half-plane across $(0, 1)$ or $(-1, 0)$.

Recall the definition of the classical (Euler) polylogarithms as the analytic continuations of the series

$$Li_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}.$$

It is clear that $Li_1(z) = -\log(1-z)$ and $Li_n(z) = \int_0^z Li_{n-1}(t) d\log(t)$.

The relation between $\Lambda_\cdot(*)$ and $Li_\cdot(*)$ is the following:

PROPOSITION 1.1.

(a) $$\Lambda_n(\xi) = \sum_{k=1}^{n} \frac{\xi^{n-k}}{(n-k)!} (-2\pi i)^{-k} Li_k(z),$$

(b) $$Li_n(z) = (-2\pi i)^n \sum_{k=1}^{n} \frac{(-\xi)^{n-k}}{(n-k)!} \Lambda_k(\xi).$$

Recall that $z = \mathbf{e}(\xi) = \exp(2\pi i \xi)$.

PROPOSITION 1.2.

(a) $$\Lambda_n^+(\xi) - \Lambda_n^-(\xi) = \begin{cases} 1 & \textit{if } n = 1 \\ 0 & \textit{if } n \neq 1 \end{cases} \quad \textit{if } \Im(\xi) < 0;$$

(b) $$\Lambda_n(\xi + j) = \sum_{k=0}^{n-1} \frac{j^k}{k!} \Lambda_{n-k}(\xi), \quad \textit{if } \Im(\xi) > 0, j \in \mathbb{Z};$$

(c) $$\Lambda_n^+(\xi) + (-1)^n \Lambda_n^-(-\xi) = \tfrac{1}{n!}(\xi^n - (-1)^n B_n);$$

(d) $$N^{n-1} \sum_{j=0}^{N-1} \sum_{k=0}^{n-1} \frac{(-j)^k}{k!} \Lambda_{n-k}\left(\xi + \frac{j}{N}\right) = \Lambda_n(N\xi), \quad \textit{if } \Im(\xi) > 0;$$

(e) $$(n-1)\, \mathrm{d}\Lambda_n(\xi) = \xi\, \mathrm{d}\Lambda_{n-1}(\xi);$$

Here the B_n are Bernoulli numbers.

To prove this proposition we introduce the generating function

$$\Lambda(\xi;K)=\sum_{n=1}^{\infty}\Lambda_n(\xi)K^{n-1}=\int_{\xi}^{i\infty}\frac{\exp(Kt)\,\mathrm{d}t}{\exp(-2\pi it)-1}$$

and similarly for $\Lambda^{\pm}(\xi;K)$. Then Proposition 1.2 can be reformulated in terms of this generating function in a very simple manner.

PROPOSITION 1.3.

(a) $\Lambda^{+}(\xi;K)-\Lambda^{-}(\xi;K)=\begin{cases}0 & \text{if } \Im(\xi)>0;\\ 1 & \text{if } \Im(\xi)<0;\end{cases}$

(b) $\Lambda(\xi+j;K)=\exp(jK)\Lambda(\xi;K),\quad \text{if } \Im(\xi)>0, j\in\mathbb{Z};$

(c) $\Lambda^{+}(\xi;K)=\Lambda^{-}(-\xi;-K)+\dfrac{\mathrm{e}^{\xi K}}{K}-\dfrac{\mathrm{e}^{K}}{\mathrm{e}^{K}-1};$

(d) $\displaystyle\sum_{j=0}^{N-1}\exp\left(\frac{-jK}{N}\right)\Lambda\left(\xi+\frac{j}{N};K\right)=\Lambda\left(N\xi;\frac{K}{N}\right)\quad \text{if } \Im(\xi)>0;$

(e) $\displaystyle\left(\frac{\partial}{\partial K}-\xi\right)\mathrm{d}_{\xi}\Lambda(\xi;K)=0.$

Proof. (a) The first statement is evident. The difference between the two continuations is equal to the residue of $-\frac{\exp(Kt)\,\mathrm{d}t}{\exp(-2\pi it)-1}$ at the point $t=0$, which is 1.

(b) Put $t'=t-j$. Then:

$$\begin{aligned}\Lambda(\xi+j;K) &= \int_{\xi+j}^{i\infty}\frac{\exp(Kt)\,\mathrm{d}t}{\exp(-2\pi it)-1}\\ &= \int_{\xi}^{i\infty}\frac{\exp(K(t'+j))\,\mathrm{d}t'}{\exp(-2\pi it')-1}\\ &= \exp(Kj)\int_{\xi}^{i\infty}\frac{\exp(Kt')\,\mathrm{d}t'}{\exp(-2\pi it')-1}\\ &= \exp(jK)\Lambda(\xi;K).\end{aligned}$$

The paths of integration are homotopical because of the restriction $\Im(\xi)>0$.

(c) We first check that the differentials of both sides of the equation are equal. Indeed

$$d\Lambda^{+}(\xi;K)=-\frac{\exp(K\xi)\,\mathrm{d}\xi}{\exp(-2\pi i\xi)-1},$$

$$d\left(\Lambda^{-}(-\xi;-K)+\frac{\exp(K\xi)}{K}\right)=\frac{\exp(K\xi)\,\mathrm{d}\xi}{\exp(2\pi i\xi)-1}+\exp(K\xi)\,\mathrm{d}\xi$$

and

$$-\frac{1}{\exp(-2\pi i\xi)-1}=\frac{1}{\exp(2\pi i\xi)-1}+1.$$

So our statement is valid with some function $F(K)$ instead of $-\frac{e^K}{e^K-1}$. To calculate this function we put $0>\Im K>-2\pi$. Then $\exp(K\xi)$ tends to 0 as ξ tends to $-i\infty$, so

$$F(K)=\int_{-i\infty}^{i\infty}\frac{\exp(Kt)\,\mathrm{d}t}{\exp(-2\pi it)-1}.$$

The integral converges at both limits according to the conditions on the imaginary part of K. The path of integration goes across $(0,1)$. Now consider the same integral with the path of integration going across $(-1,0)$. The change of t to $t+1$ shows that the second integral is equal to $\exp(-K)F(K)$. On the other hand, the difference between these integrals equals the residue of the differential form $-\frac{\exp(Kt)}{\exp(-2\pi it)-1}$ at the point $t=0$, which is 1, so $\exp(-K)F(K)-F(K)=1$.

(d) This follows from the identity

$$\frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{\exp(-2\pi i(t+\frac{j}{N}))-1}=\frac{1}{\exp(-2\pi iNt)-1}$$

together with the evident changes of variables.

(e) This is evident.

2. Elliptic polylogarithms

The elliptic polylogarithms are single-valued analytic functions on the universal covering of a punctured universal elliptic curve. We will define them as multivalued analytic functions on the partial covering $(\mathbb{C}\times H)\setminus L$, where L is the relative lattice $L:=(\{\xi=m+n\tau,\tau\}\,|\,m,n\in\mathbb{Z})$. It is well known that the universal covering of the universal elliptic curve is the product $\mathbb{C}\times H$ and its fundamental group is the semidirect product $\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, acting on $\mathbb{C}\times H$ in the usual way:

$$\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},(0,0)\right)(\xi,\tau)=\left(\frac{\xi}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right),$$

$$\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},(m,n)\right)(\xi,\tau)=(\xi+m+n\tau,\tau).$$

So the preimage of the zero section is L, and $(\mathbb{C}\times H)\setminus L$ covers the punctured universal elliptic curve.

DEFINITION 2.1. The (Debye) elliptic polylogarithm $\Lambda_{m,n}(\xi,\tau)$ with index (m,n) is the multivalued analytic function on $(\mathbb{C}\times H)\setminus L$ which is given in the strip $(0<\Im\xi<\Im\tau)$ by the series

$$\Lambda_{m,n}(\xi,\tau) = \frac{1}{m!}\left(\sum_{j=0}^{\infty} j^m\Lambda_n^+(\xi+j\tau)+(-1)^{m+n+1}\sum_{j=1}^{\infty} j^m\Lambda_n^-(-\xi+j\tau)\right.$$
$$\left.+\sum_{k=0}^{n}\frac{\xi^{n-k}\tau^k}{(n-k)!k!}\frac{B_{m+k+1}}{m+k+1}+(-1)^{n+1}\frac{B_nB_{m+1}}{n!(m+1)}\right).$$

The convergence of the infinite series is an evident consequence of (1).

This formula defines a single-valued branch of $\Lambda_{m,n}(\xi,\tau)$ on $\mathbb{C}\times H$ without the set

$$\{\xi = j\tau+s\,|\,j\in\mathbb{Z}, s\in(-\infty,0]\cup[1,\infty)\}$$

Remark 1. The origin of this definition is the following. The series

$$\sum_{j=-\infty}^{\infty} j^m\Lambda_n^+(\xi+j\tau)$$

diverges and one can regularise it by the following trick:

$$\sum_{j=-\infty}^{\infty} j^m\Lambda_n^+(\xi+j\tau)$$
$$=\sum_{j=0}^{\infty} j^m\Lambda_n^+(\xi+j\tau)+\sum_{j=-\infty}^{-1} j^m\Lambda_n^+(\xi+j\tau)$$
$$=\sum_{j=0}^{\infty} j^m\Lambda_n^+(\xi+j\tau)+\sum_{j=1}^{\infty}(-1)^{(m+n+1)}j^m\Lambda_n^-(-\xi+j\tau)$$
$$+\sum_{j=1}^{\infty}\frac{1}{n!}(-1)^m j^m((\xi-j\tau)^n-(-1)^nB_n),$$

where the last ‘equation’ follows from part (c) of Proposition 1.2. The first and second series converge according to the bound (1) and the third, which is a series of polynomials in the variable j, can be defined using the formal equality

$$\sum_{j=1}^{\infty} j^m = \zeta(-m) = (-1)^mB_{m+1}/(m+1).$$

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Remark 2. A one-valued version of the elliptical polylogarithms was introduced by Bloch [B] in the case $m+n=3$ and by Zagier [Z2] [Prop. 2(ii), (iii)] for arbitrary m, n (see also Theorem 4.2).

The generating function

$$\Lambda(\xi,\tau;X,Y)=\sum_{m\geqslant 0,n\geqslant 1}\Lambda_{m,n}(\xi,\tau)(-Y)^{n-1}X^m$$

is equal to

$$\sum_{j=0}^{\infty}\exp(jX)\Lambda^+(\xi+j\tau;-Y)+\sum_{j=1}^{\infty}\exp(-jX)\Lambda^-(-\xi+j\tau;Y)$$
$$+\frac{\exp(-Y\xi)}{-Y}\left(\frac{1}{\exp(-Y\tau+X)-1}-\frac{1}{-Y\tau+X}\right)$$
$$+\frac{1}{e^Y-1}\left(\frac{1}{e^X-1}-\frac{1}{X}\right).$$

Evidently $\Lambda_{0,1}(\xi,\tau)=\frac{1}{2\pi i}\log(\theta(\xi,\tau)/\eta(\tau))$, $\eta(\tau)=q^{1/24}\prod_{j=1}^{\infty}(1-q^j)$.

Many of the transformation properties of the $\Lambda_{m,n}(\xi,\tau)$ become simpler if we introduce the modified generating function

$$\underline{\Lambda}(\xi,\tau;X,Y)=\Lambda(\xi,\tau;X,Y)+\frac{\exp(-Y\xi)}{(-Y)(-Y\tau+X)}+\frac{1}{(\exp Y-1)X}$$
$$=\sum_{j=0}^{\infty}\exp(jX)\Lambda^+(\xi+j\tau;-Y)+\sum_{j=1}^{\infty}\exp(-jX)\Lambda^-(-\xi+j\tau;Y)$$
$$+\frac{\exp(-Y\xi)}{-Y}\frac{1}{\exp(-Y\tau+X)-1}+\frac{1}{\exp Y-1}\frac{1}{\exp X-1}.$$

The reason is that in the domain $\{0<\Re X,\quad 0<\Re(-Y\tau+X)<2\pi\Im\tau\}$ $\Sigma_{j\in\mathbb{Z}}e^{jX}\Lambda^+(\xi+j\tau;-Y)$ converges and equals $\underline{\Lambda}(\xi,\tau;X,Y)$.

PROPOSITION 2.1. (a) *Let* $0<\Im(\xi)<\Im(\tau)$. *Then*

$$\underline{\Lambda}(\xi+1,\tau;X,Y)=e^{-Y}\left(\underline{\Lambda}(\xi,\tau;X,Y)+\frac{1}{\exp(X)-1}\right).$$

(b) *Let* $0<\Re(\xi)\Im(\tau)-\Im(\xi)\Re(\tau)<\Im(\tau)$. *Then*

$$\underline{\Lambda}(\xi+\tau,\tau;X,Y)=e^{-X}\underline{\Lambda}(\xi,\tau;X,Y).$$

(c) *Let* $0<\Im(\xi)<\Im(\tau)$. *Then*

$$\sum_{j=0}^{N-1}\exp\left(\frac{jY}{N}\right)\underline{\Lambda}\left(\xi+\tfrac{j}{N},\tau;X,Y\right)=\underline{\Lambda}\left(N\xi,N\tau;X,\frac{Y}{N}\right).$$

(d)

$$\left(\frac{\partial}{\partial Y}+\tau\frac{\partial}{\partial X}+\xi\right)d_{\xi,\tau}\underline{\Lambda}(\xi,\tau;X,Y)=0.$$

Sketch of the proof. Statement (a) is obtained after a simple calculation by summing part (b) of Proposition 1.3 over the arguments $\xi+j\tau$ or $-\xi+j\tau$. Statement (b) is the result of substituting $\xi+\tau$ for ξ in the definition and changing the limits of summation in the first and second sums, part (c) of Proposition 1.3. Statement (c) is obtained after a simple calculation by summing part (d) of Proposition 1.3 over the arguments $\xi+j\tau$. Statement (d) is obtained after a simple calculation by summing part (e) of Proposition 1.3 over the arguments $\xi+j\tau$ or $-\xi+j\tau$.

3. Modular properties of elliptic polylogarithms

Consider the following function $F(\xi,\eta,\tau)$

$$\begin{aligned}F(\xi,\eta,\tau) &= 1-\frac{1}{1-z}-\frac{1}{1-w}\\ &\quad -\sum_{m,n=1}^{\infty}(z^m w^n - z^{-m}w^{-n})q^{mn} \quad \Im\tau>\Im\xi>0, \Im\tau>\Im\eta>0.\end{aligned}$$

Then [Z1] F can be continued to a meromorphic function with poles at divisors $\xi=m+n\tau$ and $\eta=m'+n'\tau$ and

$$F(\xi,\eta,\tau)=\frac{\theta'(0,\tau)\theta(\xi+\eta,\tau)}{\theta(\xi,\tau)\theta(\eta,\tau)}. \tag{2}$$

The transformation properties of $F(\xi,\eta,\tau)$ are very simple:

$$F(\xi+1,\eta,\tau)=F(\xi,\eta,\tau), \tag{3}$$

$$F(\xi+\tau,\eta,\tau)=\exp(-2\pi i\eta)F(\xi,\eta,\tau), \tag{4}$$

$$\begin{aligned}&F\left(\frac{\xi}{c\tau+d},\frac{\eta}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\\ &\quad=(c\tau+d)\exp\left(2\pi i\frac{c\xi\eta}{c\tau+d}\right)F(\xi,\eta,\tau).\end{aligned} \tag{5}$$

$F(\xi,\eta,\tau)$ can be expressed as the exponential of the generating function of Eisenstein functions $E_k(\xi,\tau)=\Sigma_{e_{w\in L}}(1/(w+\xi)^k)$ and Eisenstein series $e_k(\tau)=\Sigma'_{e_{w\in L}}(1/(w)^k)$ (Σ_e denotes Eisenstein summation [W]):

$$F(\xi,\eta,\tau)=\frac{1}{\eta}\exp\left(-\sum_{k=1}^{\infty}\frac{(-\eta)^i}{i}(E_k(\xi,\tau)-e_k(\tau))\right). \tag{6}$$

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This statement is a simple corollary of Zagier's 'Logarithmic Formula' for $F(\xi,\eta,\tau)$ [Z1, Section 3, Theorem (viii)] and the power series for E_n [W, Chapter III, formula (10)].

PROPOSITION 3.1.

$$\text{(a)}\quad \frac{\partial}{\partial\xi}\underline{\Lambda}(\xi,\tau;X,Y)=\mathrm{e}^{-Y\xi}F\left(\xi,\frac{-Y\tau+X}{2\pi i},\tau\right), \tag{7}$$

$$\text{(b)}\quad \frac{\partial}{\partial\tau}\underline{\Lambda}(\xi,\tau;X,Y)=\mathrm{e}^{-Y\xi}\frac{\partial}{\partial X}F\left(\xi,\frac{-Y\tau+X}{2\pi i},\tau\right). \tag{8}$$

The proof is a direct simple calculation.

This proposition together with (6) gives an expression for the derivatives of elliptic polylogarithms as polynomials in Eisenstein functions.

We now consider the transformation of $\Lambda(\xi,\tau;X,Y)$ under the group $\mathrm{SL}_2(\mathbb{Z})$, acting on the variables ξ,τ in the standard way and on the variables X,Y as on a column-vector $\binom{X}{Y}$.

THEOREM 3.1. *Let* $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ *belong to* $\mathrm{SL}_2(\mathbb{Z})$. *Then*

$$\underline{\Lambda}\left(\frac{\xi}{c\tau+d},\frac{a\tau+b}{c\tau+b};aX+bY,cX+dY\right)=\underline{\Lambda}(\xi,\tau;X,Y)+c_M(X,Y),$$

where $c_M(X,Y)$ *is a Laurent series in* X *and* Y *with rational coefficients.*

Sketch of the proof. We first prove that $c_M(X,Y)$, which can be defined as the difference

$$\underline{\Lambda}\left(\frac{\xi}{c\tau+d},\frac{a\tau+b}{c\tau+b};aX+bY,cX+dY\right)-\underline{\Lambda}(\xi,\tau;X,Y),$$

doesn't depend on ξ and τ. This means that the differential $d_{\xi,\tau}\underline{\Lambda}(\xi,\tau;X,Y)$ satisfies the following property:

$$d_{\xi,\tau}\underline{\Lambda}\left(\frac{\xi}{c\tau+d},\frac{a\tau+b}{c\tau+b};aX+bY,cX+dY\right)-d_{\xi,\tau}\underline{\Lambda}(\xi,\tau;X,Y)=0.$$

One can deduce this equality from (5) using the expressions for derivatives from Proposition 3.1. So we have proved that $c_M(X,Y)$ is a formal function in X and Y with complex coefficients $c_M(X,Y)\in\mathbb{C}((X,Y))\widehat{}$ ($\mathbb{C}((X,Y))\widehat{}$ denote the completion of $\mathbb{C}((X,Y))$).

To prove the rationality of these coefficients we observe that $M\to c_M(X,Y)$ is a cocycle of $\mathrm{SL}_2(\mathbb{Z})$ with coefficients in $\mathbb{C}((X,Y))\widehat{}$. Indeed, this is a cocycle because this is a coboundary. So it is enough to check the rationality for a set of generators of $\mathrm{SL}_2(\mathbb{Z})$:

$$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad T=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

The calculation of $c_T(X,Y)$ is a simple exercise like the proof of part (a) of Proposition 2.1

$$c_T(X,Y) = \frac{1}{\exp(Y)-1}\left(\frac{1}{\exp(X+Y)-1} - \frac{1}{\exp(X)-1}\right).$$

To calculate $c_S(X,Y)$ we use the following trick (in which we use that $\mathrm{SL}_2(\mathbb{Z})$ intertwines the action of $\mathbb{Z}^2$):

$$\begin{aligned} c_S(X,Y) &= \underline{\Lambda}\left(\frac{\xi-1}{\tau}, -\frac{1}{\tau}; -Y, X\right) - \underline{\Lambda}(\xi-1,\tau;X,Y) \\ (\text{use Prop. 2.1.(b)}) &= \mathrm{e}^Y\left(\underline{\Lambda}\left(\frac{\xi}{\tau}, -\frac{1}{\tau}; -Y, X\right)\right) \\ (\text{use Prop. 2.1.(a)}) &\quad -\left(\mathrm{e}^Y\underline{\Lambda}(\xi,\tau;X,Y) - \frac{1}{\exp(X)-1}\right) \\ &= \mathrm{e}^Y\left(\underline{\Lambda}(\xi,\tau;X,Y) + c_S(X,Y)\right) \\ &\quad -\left(\mathrm{e}^Y\underline{\Lambda}(\xi,\tau;X,Y) - \frac{1}{\exp(X)-1}\right) \\ &= \mathrm{e}^Y c_S(X,Y) + \frac{1}{\exp(X)-1}. \end{aligned}$$

We have deduced an equation for $c_S(X,Y)$ with an evident solution which is a Laurent series with rational coefficients:

$$c_S(X,Y) = -\left(\frac{1}{\exp(Y)-1}\right)\left(\frac{1}{\exp(X)-1}\right). \tag{9}$$

We now describe the action of isogenies on the elliptic polylogarithms. First we mention that the parts (a) and (b) of Proposition 2.1 imply the following result for $m,n \in \mathbb{Z}$

$$\exp(mY+nX)\underline{\Lambda}(\xi+m+n\tau,\tau;X,Y) - \underline{\Lambda}(\xi,\tau;X,Y) \in \mathbb{Q}(X,Y).$$

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belong to $M_2(\mathbb{Z})$, $N = \det\ M = ad - bc \neq 0$. Let $\mathcal{K}_M \subset \mathbb{Q}^2$ be a set of representatives of the kernel of the map $M\colon (\mathbb{Q}/\mathbb{Z})^2 \to (\mathbb{Q}/\mathbb{Z})^2$. We denote elements of $\mathcal{K}_M$ by $v = \left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)$. Consider two representatives v_1 and v_2 of some element in the kernel. The difference $v_1 - v_2$ belongs to $\mathbb{Z}$ and consequently

$$\begin{aligned} &\mathrm{e}^{(r_1Y - s_1X)}\underline{\Lambda}(\xi + r_1 - s_1\tau, \tau; X, Y) \\ &\quad -\mathrm{e}^{(r_2Y - s_2X)}\underline{\Lambda}(\xi + r_2 - s_2\tau, \tau; X, Y) \end{aligned}$$

belongs to $\mathbb{Q}((X,Y))\hat{}$.

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THEOREM 3.2. *Let* $M \in M_2(\mathbb{Z})$, $N = \det M = ad - bc \neq 0$. *Than, with notations as above,*

$$\sum_{v\in\mathcal{K}_M} \exp(rY - sX)\underline{\Lambda}(\xi + r - s\tau, \tau; X, Y)$$
$$-\underline{\Lambda}\left(N\frac{\xi}{c\tau+d}, \frac{a\tau+b}{c\tau+b}; \frac{aX+bY}{N}, \frac{cX+dY}{N}\right) \in \mathbb{Q}((X,Y))$$

for some and, consequently, for any choice of $\mathcal{K}_M$.

Sketch of proof. It is well-known [L] that every element M of $M_2(\mathbb{Z})$, $\det M \neq 0$ can be decomposed in a product of 'standard' elements $\left(\begin{smallmatrix} N_i & 0 \\ 0 & 1\end{smallmatrix}\right)$ and elements of $\mathrm{SL}_2(\mathbb{Z})$. For 'standard' elements the statement of this Theorem was proved in the part (c) of Proposition 2.1 and for elements of $\mathrm{SL}_2(\mathbb{Z})$ this statement is equal to the previous theorem. Hence we must show that if our statement is valid for two matrices, it is valid for their product. Introduce some notation. Let

$$\left\langle \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}, \begin{pmatrix} r_2 \\ s_2 \end{pmatrix} \right\rangle = r_1 s_2 - s_1 r_2$$

be the usual skewsymmetric pairing. It is easy to check that for $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in M_2(\mathbb{Z})$, $ad - bc = N \neq 0$, $\langle Mv_1, Mv_2\rangle = N\langle v_1, v_2\rangle$. Clearly

$$M\begin{pmatrix} \tau \\ 1 \end{pmatrix} = (c\tau + d)\begin{pmatrix} \dfrac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix}.$$

Put $M_3 = M_1 M_2$ and $\mathcal{K}_i = \mathcal{K}_{M_i}$; $M_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i\end{smallmatrix}\right)$, $N_i = a_i d_i - b_i c_i$. We can choose for $\mathcal{K}_3$ the set

$$\left\{ \begin{pmatrix} r_2 + \dfrac{r_1 d_2 - s_1 b_2}{N_2} \\ s_2 + \dfrac{r_1 c_2 - s_1 a_2}{N_2} \end{pmatrix} = v_2 + M_2^{-1} v_1 \,\middle|\, \begin{pmatrix} r_i \\ s_i \end{pmatrix} = v_i \in \mathcal{K}_i \right\}.$$

Then

$$\sum_{v_3\in\mathcal{K}_3} \exp\left(\left\langle v_3, \begin{pmatrix} X \\ Y \end{pmatrix}\right\rangle\right) \underline{\Lambda}\left(\xi + \left\langle v_3 \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right\rangle, \tau; X, Y\right)$$
$$= \sum_{v_2\in\mathcal{K}_2}\sum_{v_1\in\mathcal{K}_1} \exp\left(\left\langle v_2 + M_2^{-1}v_1, \begin{pmatrix} X \\ Y \end{pmatrix}\right\rangle\right)$$
$$\times\underline{\Lambda}\left(\xi + \left\langle v_2 + M_2^{-1}v_1, \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right\rangle, \tau; X, Y\right) =$$

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$$= \sum_{v_1 \in \mathcal{K}_1} \exp\left(N_2^{-1}\left\langle M_2 M_2^{-1} v_1, M_2 \begin{pmatrix} X \\ Y \end{pmatrix}\right\rangle\right)$$

$$\times \underline{\Lambda}\left(\frac{N_2\xi + \left\langle M_2 M_2^{-1} v_1, M_2 \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right\rangle}{c_2\tau + d_2},\right.$$

$$\left.\frac{a_2\tau + b_2}{c_2\tau + b_2}; \frac{a_2 X + b_2 Y}{N_2}, \frac{c_2 X + d_2 Y}{N_2}\right) + c_{M_2}(X, Y)$$

$$\equiv \sum_{v_1 \in \mathcal{K}_1} \exp\left(N_2^{-1}\left\langle v_1, M_2 \begin{pmatrix} a_2 X + b_2 Y \\ c_2 X + d_2 Y \end{pmatrix}\right\rangle\right)$$

$$\times \underline{\Lambda}\left(\frac{N_2\xi +}{c_2\tau + d_2} + \left\langle v_1, \begin{pmatrix} \frac{a_2\tau + b_2}{c_2\tau + d_2} \\ 1 \end{pmatrix}\right\rangle,\right.$$

$$\left.\frac{a_2\tau + b_2}{c_2\tau + b_2}; \frac{a_2 X + b_2 Y}{N_2}, \frac{c_2 X + d_2 Y}{N_2}\right)$$

$$\equiv \underline{\Lambda}\left(N_3 \frac{\xi}{c_3\tau + d_3}, \frac{a_3\tau + b_3}{c_3\tau + b_3}; \frac{a_3 X + b_3 Y}{N_3}, \frac{c_3 X + d_3 Y}{N_3}\right).$$

Here $\equiv$ denotes congruence mod $\mathbb{Q}((X, Y))\hat{}$.

4. Eisenstein–Kronecker series and elliptic polylogarithms

We recall a classical result of Kronecker [W]: Denote by L the lattice generated by 1 and τ. Any $\eta \in \mathbb{C}$ determines a character χ_η on L

$$\chi_\eta(\xi) = \exp\left(2\pi i \frac{\xi\overline{\eta} - \overline{\xi}\eta}{\tau - \overline{\tau}}\right).$$

Then [W, Z1] the Eisenstein–Kronecker series of weight 1 which is given by

$$K_1(\xi, \eta, 1) = {\sum_{w \in L}}_e \frac{\chi_\eta(w)}{w + \xi}$$

(where Σ_e denotes Eisenstein summation; see [W]) expresses as

$$K_1(\xi, \eta, 1) = 2\pi i \exp\left(2\pi i \xi \frac{\eta - \overline{\eta}}{\tau - \overline{\tau}}\right) F(\xi, \eta, \tau), \tag{10}$$

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(F was defined in the beginning of Section 3). One can represent this function as the generating function with respect to the variable ξ of the usual Eisenstein series $e_k(\eta,\tau)=\Sigma'_{w\in L}\,{}_e(\chi_\eta(w)/w^k)$:

$$\sum_{w\in L}{}_e\frac{\chi_\eta(w)}{w+\xi} = \frac{1}{\xi}+\sum_{w\in L}{}'_e\sum_{j=0}^{\infty}(-\xi)^j\frac{\chi_\eta(w)}{w^{j+1}}$$
$$= \frac{1}{\xi}+\sum_{j=0}^{\infty}(-\xi)^j\sum_{w\in L}{}'_e\frac{\chi_\eta(w)}{w^{j+1}}. \tag{11}$$

Put $r=\frac{\tau\overline{\xi}-\overline{\tau}\xi}{\tau-\overline{\tau}}$, $s=\frac{\xi-\overline{\xi}}{\tau-\overline{\tau}}$, then $\xi=r+s\tau,\overline{\xi}=r+s\overline{\tau}$. Define the function:

$$\Xi(\xi,\tau;X,Y)=\mathrm{e}^{sX+rY}\underline{\Lambda}(\xi,\tau;X,Y).$$

Now we describe the relation between elliptic polylogarithms and indefinite Eichler–Shimura integrals of Eisenstein series.

Let Γ be a congruence subgroup and $G(\tau)$ a modular form of weight k with respect to Γ. Then the vector-valued differential form $G(\tau)(-Y\tau+X)^{k-2}\,\mathrm{d}\tau$ is Γ-invariant. The indefinite Eichler–Shimura integral $\mathcal{G}(\tau,X,Y)$ of $G(\tau)$ is the indefinite integral of this form:

$$d_\tau\mathcal{G}(\tau,X,Y)=G(\tau)(-Y\tau+X)^{k-2}\,\mathrm{d}\tau.$$

Let r and s be rational. Then the Eisenstein series $e_k^{r,s}(\tau)=e_k(r+s\tau,\tau)$ is a modular form of weight k for some congruence subgroup.

THEOREM 4.1. *If $(r,s)\neq(0,0)$ then $\Xi(\xi,\tau;X,Y)|_{\xi=r+s\tau}$ is the modified generating function for the indefinite Eichler–Shimura integrals of Eisenstein series $e_k^{r,s}(\tau)$:*

$$\Xi(\xi,\tau;X,Y)=\frac{-\tau}{X(-Y\tau+X)}+\sum_{k=2}^{\infty}(-1)^{k-1}(k-1)(2\pi i)^{-k}\mathcal{E}_k^{r,s} \tag{12}$$

and $\Xi^*(0,\tau;X,Y)=(\Xi(\xi,\tau;X,Y)-\frac{1}{2\pi i}\log(2\pi i\xi))|_{\xi=0}$ is the modified generating function of Eisenstein series $e_k^{0,0}(\tau)$ of the level 1:

$$\Xi^*(0,\tau;X,Y)=\frac{-\tau}{X(-Y\tau+X)}+\sum_{k=2}^{\infty}(-1)^{k-1}(k-1)(2\pi i)^{-k}\mathcal{E}_k^{0,0},$$

$$d_\tau\mathcal{E}_k^{r,s}=e_k^{r,s}(\tau)(-Y\tau+X)^{k-2}\,\mathrm{d}\tau.$$

The proof is a direct calculation of $d_\tau \Xi^{r,s}(\tau, X, Y)$ using Proposition 3.1 and (11).

Now we have expressed the nonholomorphic Eisenstein series

$$e_{k,l}(\xi,\tau) = \sum_{w\in L}{}'_e \frac{\chi_\eta(w)}{w^k \overline{w}^l}$$

in terms of elliptic polylogarithms. Evidently

$$K_0(\eta,\xi,1) = \sum_{w\in L}{}_e \frac{\chi_\xi(w)}{|w+\eta|^2}$$

is the generating function of $e_{k,l}(\xi,\tau)$ with respect to variables $\eta, \overline{\eta}$:

$$\begin{aligned}\sum_{w\in L}{}_e \frac{\chi_\xi(w)}{|w+\eta|^2} &= \frac{1}{|\eta|^2} + \sum_{w\in L}{}'_e \sum_{k,l=0}^{\infty} (-\eta)^k(-\overline{\eta})^l \frac{\chi_\xi(w)}{w^{k+1}\overline{w}^{l+1}} \\ &= \frac{1}{|\eta|^2} + \sum_{k,l=0}^{\infty} (-\eta)^k(-\overline{\eta})^l \sum_{w\in L}{}'_e \frac{\chi_\xi(w)}{w^{k+1}\overline{w}^{l+1}}. \end{aligned} \tag{13}$$

Let us consider X and Y as imaginary objects, this means that they are antiinvariant with respect to the action of the complex conjugation, so if we put $\eta = \frac{-Y\tau+X}{2\pi i}$, then $\overline{\eta} = \frac{-Y\overline{\tau}+X}{2\pi i}$ and conversely $X = 2\pi i \frac{\tau\overline{\eta}-\overline{\tau}\eta}{\tau-\overline{\tau}}$, $-Y = 2\pi i \frac{\eta-\overline{\eta}}{\tau-\overline{\tau}}$.

THEOREM 4.2. *Consider X and Y as expressions in η and $\overline{\eta}$ as described above. Then*

$$\Xi(\xi,\tau;X,Y) - \overline{\Xi(\xi,\tau:-X,-Y)} = -\frac{\tau-\overline{\tau}}{(2\pi i)^2} K_0(\eta,\xi,1).$$

Sketch of the proof. A direct calculation shows that both sides of the equality satisfy the following conditions:

(i) They are solutions of the system of differential equations

$$\left\{\begin{aligned} \left(\frac{\partial}{\partial\xi} - \frac{2\pi i}{\tau-\overline{\tau}}\overline{\eta}\right) f &= \underline{\frac{1}{2\pi i} K_1(\eta,\xi,1)} \\ \left(\frac{\partial}{\partial\overline{\xi}} + \frac{2\pi i}{\tau-\overline{\tau}}\eta\right) f &= -\frac{1}{2\pi i} K_1(-\eta,\xi,1) \\ \left(\frac{\partial}{\partial\tau} + \frac{\xi-\overline{\xi}}{\tau-\overline{\tau}}\frac{\partial}{\partial\xi} + \frac{\eta-\overline{\eta}}{\tau-\overline{\tau}}\frac{\partial}{\partial\eta}\right) f &= \underline{\frac{1}{(2\pi i)^2}\frac{\partial K_1(\eta,\xi,1)}{\partial\eta}} \\ \left(\frac{\partial}{\partial\overline{\tau}} + \frac{\xi-\overline{\xi}}{\tau-\overline{\tau}}\frac{\partial}{\partial\overline{\xi}} + \frac{\eta-\overline{\eta}}{\tau-\overline{\tau}}\frac{\partial}{\partial\overline{\eta}}\right) f &= -\frac{1}{(2\pi i)^2}\frac{\partial K_1(-\eta,\xi,1)}{\partial\eta}. \end{aligned}\right. \tag{14}$$

(ii) They are modular invariant.

For the right-hand side this statement is the result of a simple calculation. For the left hand side (i) is the corollary of Proposition 3.1 and (10) and (ii) is the corollary of the modular properties of elliptic polylogarithms (Theorem 3.1); the additional term $c_M(X,Y) - \overline{c_M(-X,-Y)}$ vanishes, because c_M is a real function and X and Y are imaginary.

So the difference between the rhs and the lhs of the conclusion of the theorem is modular invariant and is a solution of the homogeneous version of (14). Any solution of the homogeneous equations is given by

$$g\left(\frac{\eta-\overline{\eta}}{\tau-\overline{\tau}}\right)\exp\left(-2\pi i\frac{\eta\overline{\xi}-\overline{\eta}\xi}{\tau-\overline{\tau}}\right)$$

with an arbitrary function g. The modular invariance show that g is some constant C. The simplest way to show the vanishing of C is to consider the constant term in the $\eta,\overline{\eta}$ expansion and to apply the Kronecker limit formula. We will prove the vanishing of C in another way.

Suppose ξ and η are real, τ is imaginary. Consider the asymptotic of both sides as τ tends to $i\infty$. The first terms are proportional to τ, the second terms don't depend on τ and other terms tend to zero. We will prove that first and second terms of both sides coincide, so the limit of the difference vanishes and consequently the difference is zero.

If η an ξ are real then $Y=0$ and $s=0$, so

$$\begin{aligned}
\Xi(\xi,\tau,X,0) &= \underline{\Lambda}(\xi,\tau;X,0)\\
&= \frac{1}{2\pi i}\left(\sum_{j=0}^{\infty}\exp(jX)\log(1-zq^j) + \sum_{j=1}^{\infty}\exp(-jX)\log(1-z^{-1}q^j)\right)\\
&\quad -\tau\frac{\exp X}{(\exp X-1)^2} + (\xi-\tfrac{1}{2})\frac{1}{\exp X-1}
\end{aligned}$$

and

$$\begin{aligned}
&\Xi(\xi,\tau,X,0) - \overline{\Xi(\xi,\tau,-X,0)}\\
&\quad = \frac{1}{2\pi i}\left(\sum_{j=0}^{\infty}\exp(jX)(\log(1-zq^j)+\log(1-z^{-1}q^j))\right.\\
&\qquad\qquad \left.+\sum_{j=1}^{\infty}\exp(-jX)(\log(1-z^{-1}q^j)+\log(1-zq^j))\right)\\
&\qquad -2\tau\frac{\exp X}{(\exp X-1)^2}\\
&\quad = -2\tau\frac{\exp X}{(\exp X-1)^2}\frac{1}{2\pi i}(\log(1-z)+\log(1-z^{-1})) + O(\tau^{-1}).
\end{aligned}$$

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On the other hand $\chi_\xi(m+n\tau) = \exp(-2\pi i n\xi)$ and if $n \neq 0$

$$\frac{1}{|m+n\tau+\eta|^2} = \frac{1}{2n\tau}\left(-\frac{1}{m+n\tau+\eta} + \frac{1}{m-n\tau+\eta}\right).$$

We used above our conditions on ξ, η and τ. So

$$\begin{aligned}
&K_0(\eta,\xi,1)\\
&\quad= \sum_{m=-\infty}^{\infty} \frac{1}{(m+\eta)^2}\\
&\qquad+ \left(\sum_{n>0} + \sum_{n<0}\right) \frac{\exp(2\pi i n\xi)}{2n\tau} \sum_m \left(-\frac{1}{m+n\tau+\eta} + \frac{1}{m-n\tau+\eta}\right)\\
&\quad= (2\pi i)^2 \frac{\exp(2\pi i\eta)}{(\exp(2\pi i\eta)-1)^2} + \left(\sum_{n>0} + \sum_{n<0}\right)\\
&\qquad\times \frac{\exp(2\pi i n\xi)}{2n\tau}\\
&\qquad\times \left(-\pi i \frac{\exp(2\pi i(\eta+n\tau))+1}{\exp(2\pi i(\eta+n\tau))-1} + \pi i \frac{\exp(2\pi i(\eta-n\tau))+1}{\exp(2\pi i(\eta-n\tau))-1}\right).
\end{aligned}$$

In this computation we use the classical representation of cotangent as an infinite sum, which is equivalent to the Euler product formula for sine, see e.g. [W]. According to our restriction $2\pi i\eta = X$. If τ tends to $i\infty$, $\pi i \frac{\exp 2\pi i(\eta+n\tau)+1}{\exp 2\pi i(\eta+n\tau)-1}$ tends to $-$ sign $(n)\pi i$ and we get:

$$\begin{aligned}
&K_0(\eta,\xi,1)\\
&\quad= (2\pi i)^2 \frac{\exp(2\pi i\eta)}{(\exp(2\pi i\eta)-1)^2} + \frac{1}{2\tau}\\
&\qquad\times \left(\sum_{n>0} \frac{\exp(2\pi i n\xi)}{n}(\pi i - (-\pi i)) + \sum_{n<0} \frac{\exp(2\pi i n\xi)}{n}(-\pi i - \pi i)\right)\\
&\qquad+O(\tau^{-2})\\
&\quad= (2\pi i)^2 \frac{\exp(2\pi i\eta)}{(\exp(2\pi i\eta)-1)^2} - \frac{2\pi i}{2\tau}\\
&\qquad\times(\log(1-z) + \log(1-z^{-1})) + O(\tau^{-2}).
\end{aligned}$$

We proved that the asymptotics of both sides of equation are equal. This ends the proof.

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Remark. The statement of the previous Theorem means that the elliptic analogs of the Bloch–Wigner–Zagier polylogarithms are nonholomorphic Eisenstein series.

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