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BINARY TREES AND THE n-CUTSET PROPERTY

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ABSTRACT. A partially ordered set P is said to have the n-cutset property if for every element x of P, there is a subset S of P all of whose elements are noncomparable to x, with $|S| \leq n$, and such that every maximal chain in P meets $\{x\} \cup S$. It is known that if P has the n-cutset property then P has at most 2^n maximal elements. Here we are concerned with the extremal case. We let $\operatorname{Max} P$ denote the set of maximal elements of P. We establish the following result. THEOREM: Let n be a positive integer. Suppose P has the n-cutset property and that $|\operatorname{Max} P| = 2^n$. Then P contains a complete binary tree T of height n with $\operatorname{Max} T = \operatorname{Max} P$ and such that $C \cap T$ is a maximal chain in T for every maximal chain C of P. Two examples are given to show that this result does not extend to the case when n is infinite. However the following is shown. THEOREM: Suppose that P has the ω-cutset property and that $|\operatorname{Max} P| = 2^\omega$. If $P - \operatorname{Max} P$ is countable then P contains a complete binary tree of height ω.

1. **Introduction and preliminaries.** Let P be a partially ordered set. A subset of P that intersects every maximal chain of P will be called a *cutset* for P. For $x \in P$, let $I(x) = \{p \in P : p \text{ is noncomparable to } x\}$. If $S \subseteq I(x)$ and $\{x\} \cup S$ is a cutset for P, we say that S is a *cutset for x in P*. If for all $x \in P$ there is a cutset S for x in P with $|S| \leq n$, the P is said to have the *n-cutset property*. Cutsets have been studied by several authors and other work can be found, for example, in [1,2,3,4,6,7,8].

If C is a chain in P then for $x \in P$ we will say that x *extends* C if $\{x\} \cup C$ is a chain in P. If S is a cutset for x in P, then if C is a chain in P, there is an element $p \in \{x\} \cup S$ such that p extends C. The set of maximal elements of P will be denoted by $\operatorname{Max} P$ and the set of maximal chains of P will be denoted by $M(P)$. The statement "x is noncomparable to y" will be denoted by $x \perp y$ and the statement that "x is comparable to y" will be denoted by $x \sim y$.

For $a \in P$ we let $[a, \rightarrow) = \{x \in P : a \leq x\}$. If $a, b \in P$ and $b > a$ and if there is no $c \in P$ such that $b > c > a$, then we say that b *covers* a and write $b \succ a$.

If k is a positive integer we let L_k denote the set of all 0-1 sequences of length k. Let $T_n = \bigcup_{k=1}^{n} L_k$. Order T_n as follows: for $\sigma, \tau \in T_n$, with $\sigma = (x_1, x_2, \ldots, x_j), \tau = (y_1, y_2, \ldots, y_k)$, we set $\sigma < \tau$ if $j < k$ and $x_i = y_i$ for all $i \in \{1, \ldots, j\}$. We say that T_n is the *complete binary tree of height n*. Let $T_n^* = T_n \cup \{\emptyset\}$ where for all $\sigma \in T_n, \emptyset < \sigma$. Then T_n^* will be referred to as the *complete rooted binary tree of height n* with root $\emptyset$. (see Figure 1).

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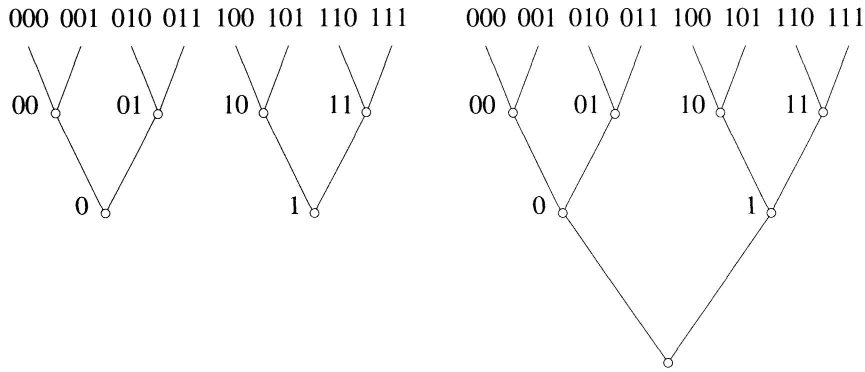

The Complete binary tree of height 3 The complete rooted binary tree of height 3

FIGURE 1

In this paper we give a new proof of the theorem proved in [1] which states that if n is a positive integer and if P has the n-cutset property, then $|\operatorname{Max} P| \leq 2^n$. It will be shown that in the case where $|\operatorname{Max} P| = 2^n$ where P has the n-cutset property, then P contains a complete binary tree T of height n such that $\operatorname{Max} T = \operatorname{Max} P$ and if $C \in M(P)$ then $C \cap T \in M(T)$. In the case where n is infinite it will be shown that generalizations of the finite case do not hold.

2. **Binary trees and the n-cutset property.**

In this section P will denote a poset and n will denote a positive integer.

LEMMA 1. *Let $p \in \operatorname{Max} P$ and $a \in P$ such that $p \perp a$. Suppose $x \in [a, \rightarrow)$ and x has a cutset S in P with $|S| = n$. Then $S \cap [a, \rightarrow)$ is a cutset for x in $[a, \rightarrow)$ and $|S \cap [a, \rightarrow)| \leq n - 1$.*

PROOF. Let p, a and x be as in the lemma. Then $x \perp p$. Since S is a cutset for x in p and $x \perp p$, there is a $y \in S$ such that y extends $\{p\}$ and since p is maximal, $y \leq p$. We must have $y \notin [a, \rightarrow)$ for if $y \in [a, \rightarrow)$ then $a \leq y$; but then $a \leq p$. Let $S' = S \cap [a, \rightarrow)$ and let $C \in M([a, \rightarrow))$, then $a \in C$ since a is minimum in $[a, \rightarrow)$. Either x extends C or some $z \in S$ extends C. If x extends C, then $x \in C$ since $C \in M([a, \rightarrow))$. And if z extends C then $a \leq z$, for $z \leq a$ implies, $z \leq x$ contradicting $z \in S$. And so $z \in C$. So $z \in S'$ and so S' is a cutset for x in $[a, \rightarrow)$. Since $y \notin S'$ we have $|S'| \leq |S - \{y\}| = n - 1$.

LEMMA 2. *Let P be a poset and let $X \subseteq \operatorname{Max} P$ with X finite. Let $P' = \{p \in P : p \leq x$ for some $x \in X\}$. If $p \in P'$ and S is a cutset for p in P, then $S \cap P'$ is a cutset for p in P'.*

PROOF. Let $C \in M(P')$. Let $X = \{x_1, x_2, \ldots, x_n\}$. Then there is an $x \in X$ such that $x \in C$. For, if not, then there is a $y_i \in C$ such that $y_i \perp x_i$ for all $i \in \{1, 2, \ldots, n\}$. Let y be the maximum element in $\{y_1, y_2, \ldots, y_n\}$. Then $y \leq x_k$ for some $k \in \{1, \ldots, n\}$. But then $y_k \leq x_k$, a contradiction. Now either p extends C or y extends C for some $y \in S$. If p extends C then $p \in C$. If y extends C then $y \leq x$ since x is maximal and so $y \in P'$.

We now give a new and shorter proof of the following theorem proved in [1].

THEOREM 1. *If P is a poset such that for all $x \in \operatorname{Max} P$ there is a cutset $S(x)$ for x in P with $|S(x)| \leq n$, then $|\operatorname{Max} P| \leq 2^n$.*

PROOF. For the case when $n = 1$: suppose that $|\operatorname{Max} P| > 2$. So there exist 3 maximal elements, a, b and c of P. Let $S(a)$ and $S(b)$ be cutsets for a and b respectively in P where $|S(a)| = |S(b)| = 1$. Let $\{x\} = S(a)$ and $\{y\} = S(b)$. Since $a \perp b$ and $a \perp c$, x extends $\{b\}$ and $\{c\}$ and so $x \leq b$ and $x \leq c$ since b and c are maximal. Similarly $y \leq a$ and $y \leq c$. Also y extends $\{x, c\}$ but $y \leq x$ implies $y \leq b$ and $x \leq y$ implies $x \leq a$, a contradiction, so $|\operatorname{Max} P| \leq 2$ when $n = 1$.

For the sake of contradiction let n be the smallest positive integer such that the theorem doesn't hold. So there is a poset P with $|\operatorname{Max} P| > 2^n$ and such that for all $x \in \operatorname{Max} P$ there is a cutset $S(x)$ for x in P with $|S(x)| \leq n$.

Let $x \in \operatorname{Max} P$. Let $y \in S(x)$. Then if $X = \{z \in \operatorname{Max} P : y \leq z\}$, $|X| \leq 2^{n-1}$. This follows from Lemma 1 since $y \perp x$ and so for all $z \in X$, $S(z) \cap [y, \rightarrow)$ is a cutset for z in $[y, \rightarrow)$ and $|S(z) \cap [y, \rightarrow)| \leq n - 1$. Since $X = \operatorname{Max}[y, \rightarrow)$ with all $z \in X$ having a cutset of size at most $n - 1$, and since n is the smallest positive integer for which the theorem doesn't hold, $|X| \leq 2^{n-1}$.

Let y be such that $y \in S(x)$ for some $x \in \operatorname{Max} P$ and $|\{z \in \operatorname{Max} P : y < z\}|$ is maximal. Let $Y = \{z \in \operatorname{Max} P : y \leq z\}$ and let $Z \subseteq (\operatorname{Max} P) - Y$ be such that $|Z| = 2^{n-1} + 1$.

For all $z \in Z, y \perp z$. For any $w \in Z$, let $v \in S(w)$ such that v extends $\{y\}$. Then for all $z \in Z, v \perp z$. For if $v \sim z$ for some $z \in Z$ then $v \leq z$ since z is maximal. If $y \leq v$, then $y \leq z$ contradicting $y \perp z$ and if $v \leq y$ then $|\{z \in \operatorname{Max} P : v \leq z\}| > |Y|$ contradicting the maximum size of Y.

Let $P' = \{u \in P : u \leq z \text{ for some } z \in Z\}$. By Lemma 2, $S(z) \cap P'$ is a cutset for z in P' for all $z \in Z$. By the preceding argument, if $s_z \in S(z)$ such that s_z extends $\{y\}$, then $s_z \notin P'$ and so $|S(z) \cap P'| \leq n - 1$. So each maximal element in P' has a cutset of size at most $n - 1$. But $|\operatorname{Max} P'| > 2^{n-1}$ contradicting n being the smallest positive integer for which the theorem doesn't hold.

LEMMA 3. *Let P be a poset which contains a complete rooted binary tree T of height n with root a such that $\operatorname{Max} T \subseteq \operatorname{Max} P$. Then if $x \in \operatorname{Max} T$ and if S is any cutset for x in P, $|S \cap [a, \rightarrow)| \geq n$.*

PROOF. By induction on n. For $n = 1$, P contains a, b, c where $b > a$ and $c > a$ and $b \perp c$ and b and c are maximal. Let $S(b)$ be a cutset for b in P. Then there is a $b' \in S(b)$ such that b' extends $\{a, c\}$ and $b' > a$ since $b' \perp b$. So $b' \in [a, \rightarrow)$. Arguing in the same manner there is a $c' \in S(c)$ a cutset for c in P where c' extends $\{a, b\}$ and $c' > a$. So for all $x \in \operatorname{Max} T$ and any cutset $S(x)$ for x in P, $|S(x) \cap [a, \rightarrow)| \geq 1$.

Suppose the lemma is true for $m < n$. Let T be a complete rooted binary tree of height n with a the root of T and $\operatorname{Max} T \subseteq \operatorname{Max} P$ and let b and c be the two elements of T that cover a in T. Let $x \in \operatorname{Max} T$. Then x is comparable to one and only one of b, c. Without loss of generality let $b \leq x$ so $x \perp c$. Let $S(x)$ be a cutset for x in P. Let $y \in \operatorname{Max} T$ such that $y \geq c$. Then $y \perp b$. Let $x_1 \in S(x)$ extend $\{a, c, y\}$. Then $x_1 \geq a$ since $x_1 \perp x$ and $x_1 \leq y$ since y is maximal. Hence $x_1 \notin [b, \longrightarrow)$ for if $x_1 \geq b$ then $b \leq y$. Now $[b, \longrightarrow)$ contains a complete rooted binary tree T' of height $n-1$ with root b, $T' = T \cap [b, \longrightarrow)$, and $x \in \operatorname{Max} T' \subseteq \operatorname{Max} P$. So by the induction hypothesis $|S(x) \cap [b, \longrightarrow)| \geq n-1$. Since $x_1 \notin [b, \longrightarrow)$, $|S(x) \cap [a, \longrightarrow)| \geq n$.

THEOREM 2. *Let P be a poset with $|\operatorname{Max} P| = 2^n$ such that for all $x \in \operatorname{Max} P$ there is a cutset $S(x)$ for x in P with $|S(x)| \leq n$. Then there is a complete binary tree T of height n contained in P such that $\operatorname{Max} T = \operatorname{Max} P$. Furthermore, if $C \in M(P)$, then $C \cap T \in M(T)$.*

PROOF. By induction. For $n = 1$, $|\operatorname{Max} P| = 2$. So let $\operatorname{Max} P = \{a, b\}$ and by Theorem 1 there can be no more than 2 maximal elements. Let $C \in M(P)$. Either $a \in C$ or $b \in C$. For if not then there are $c, d \in C$ such that $c \perp a$ and $d \perp b$. Let $S(a) = \{x\}$ and $S(b) = \{y\}$. Then $x, y \in C$ and x extends $\{b\}$ and y extends $\{a\}$. Therefore $x \leq b$ and $y \leq a$ since a and b are maximal. But if $x \leq y$ then $x \leq a$ and if $y \leq x$, then $y \leq b$, both of which are impossible. So either $a \in C$ or $b \in C$. So $\{a, b\}$ forms the desired complete binary tree of height 1 satisfying the conditions of the theorem.

Now suppose the theorem is true for $m < n$. Let P be a poset with $|\operatorname{Max} P| = 2^n$ such that for all $x \in \operatorname{Max} P$, $S(x)$ is a cutset for x in P with $|S(x)| \leq n$. We claim there is an $a \in \operatorname{Max} P$ and $a_1 \in S(a)$ such that $|\{x \in \operatorname{Max} P : a_1 \leq x\}| = 2^{n-1}$. Now if $x \in \operatorname{Max} P$ and $y \in S(x)$ then Lemma 1 and Theorem 1 imply that $|\{z \in \operatorname{Max} P : y \leq z\}| \leq 2^{n-1}$. Suppose that for all $x \in \operatorname{Max} P$ and $y \in S(x)$, $|\{z \in \operatorname{Max} P : y \leq z\}| < 2^{n-1}$. Let $y \in S(x)$ for some $x \in \operatorname{Max} P$ be such that the size of $Y = \{z \in \operatorname{Max} P : y \leq z\}$ is greatest. Let $Z = (\operatorname{Max} P) - Y$. Then $|Z| > 2^{n-1}$. Consider $P' = \{p \in P : p \leq z \text{ for some } z \in Z\}$. By Lemma 2 if $z \in \operatorname{Max} P'$ and $S(z)$ is a cutset for z in P then $S(z) \cap P'$ is a cutset for z in P'. Since $\operatorname{Max} P' = Z$ then as in the proof of Theorem 1, $|S(z) \cap P'| \leq n-1$. But $|\operatorname{Max} P'| = |Z| > 2^{n-1}$ contradicting Theorem 1. Therefore our claim is verified. So let $a \in \operatorname{Max} P$ and $a_1 \in S(a)$ be such that $|\{x \in \operatorname{Max} P : a_1 \leq x\}| = 2^{n-1}$. Let $x \in \operatorname{Max}[a_1, \longrightarrow)$ and let $x_1 \in S(x)$ such that x_1 extends $\{a\}$. Then $x_1 < a$ and $x_1 \notin [a_1, \longrightarrow)$ otherwise $a_1 \leq a$. Now $S'(x) = S(x) \cap [a_1, \longrightarrow)$ is a cutset for x in $[a_1, \longrightarrow)$, and $|S'(x)| \leq n-1$ by Lemma 1. By the induction hypothesis $[a_1, \longrightarrow)$ contains a complete binary tree T_a of height $n-1$ with $\operatorname{Max} T_a = \operatorname{Max}[a_1, \longrightarrow)$ and every maximal chain of $[a_1, \longrightarrow)$ intersects T_a in a maximal chain of T_a.

Now $\{a_1\} \cup T_a$ is a complete rooted binary tree of height $n-1$. Let $b \in \operatorname{Max}[a_1, \longrightarrow)$. By Lemma 3, $|S(b) \cap [a_1, \longrightarrow)| \geq n-1$. We note that $|\operatorname{Max} P - \operatorname{Max}[a_1, \longrightarrow)| = 2^{n-1}$.

Let $b_1 \in S(b)$ such that b_1 extends $\{z\}$ for all $z \in \operatorname{Max} P - \operatorname{Max}[a_1, \longrightarrow)$. Therefore $b_1 \leq z$ for all $z \in \operatorname{Max} P - \operatorname{Max}[a_1, \longrightarrow)$. And for all $x \in [a_1, \longrightarrow)$, $b_1 \perp x$. For if $b_1 \leq x$, then since for all $C \in M([a_1, \longrightarrow))$, $C \cap T_a \in M(T_a)$, there is a $y \in \operatorname{Max} T_a = \operatorname{Max}[a_1, \longrightarrow)$ such that $x \leq y$. So if $b_1 \leq x$ then $|\operatorname{Max}[b_1, \longrightarrow)| > 2^{n-1}$, contradicting a previous argument. If $x \leq b_1$ then $a_1 \leq a$.

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Arguing as above, since $|\operatorname{Max}[b_1,\longrightarrow)| = 2^{n-1}$, then by the induction hypothesis $[b_1,\longrightarrow)$ contains a complete binary tree T_b of height $n-1$ such that if $C \in M([b_1,\longrightarrow))$ then $C \cap T_b \in M(T_b)$. Furthermore if $x \in [b_1,\longrightarrow)$ then $a_1 \perp x$. Hence $[a_1,\longrightarrow) \cap [b_1,\longrightarrow) = \emptyset$. So there are no comparabilities between $\{a_1\} \cup T_a$ and $\{b_1\} \cup T_b$.

Therefore $T = \{a_1\} \cup T_a \cup \{b_1\} \cup T_b$ is a complete binary tree of height n.

If $C \in M(P)$, to show that $C \cap T \in M(T)$ it suffices to show that either $a_1 \in C$ or $b_1 \in C$.

For if $a_1 \in C$ then $C \cap [a_1,\longrightarrow) \in M([a_1,\longrightarrow))$ and so $C' = (C \cap [a_1,\longrightarrow)) \cap T_a \in M(T_a)$ by induction. Thus $C' \cup \{a_1\} = C \cap T \in M(T)$. Similarly if $b_1 \in C$.

Suppose $C \in M(P)$ and $a_1 \notin C$ and $b_1 \notin C$. Then $a \in C$ or $b \in C$ or neither belong to C. If $a \in C$ then there is a $b_2 \in S(b)$ such that $b_2 \in C$. Now $b_2 \geq a_1$ since, as argued above, $|S(b) \cap [a_1,\longrightarrow)| = n-1$. But then $a_1 \leq a$, a contradiction. Similarly if $b \in C$ then $b_1 \leq b$. If neither belong to C then there are $a_2 \in S(a)$ and $b_2 \in S(b)$ such that $a_2, b_2 \in C$ and $a_2 \neq a_1$ and $b_2 \neq b_1$. But then $a_2 \geq b_1$ and $b_2 \geq a_1$. Since $b_2 \sim a_2$ then $[a_1,\longrightarrow) \cap [b_1,\longrightarrow) \neq \emptyset$, a contradiction. Therefore either $a_1 \in C$ or $b_1 \in C$.

We note that in the above theorem, since $C \cap T \in M(T)$ for all $C \in M(P)$, it follows that any cutset in T is a cutset in P. Now in a complete binary tree of height n, T, for any $x \in T$ the set $\{y \in I(x) : y \succ z \text{ for some } z < x\} \cup \{y \in I(x) : y \text{ is minimal in } T\}$ is a cutset for x in T. Hence this is a cutset for x in P.

3. **On the case when n is infinite.** As proved in [1], the inequality $|\operatorname{Max} P| \leq 2^n$ for an ordered set P with the n-cutset property holds for infinite cardinals n as well. Here we will consider whether the result of the preceding section can be extended to the case when n is infinite. We will present two examples to show that the answer is negative. We are able to obtain a positive theorem, in the case $n = \omega$, for ordered sets of a special type.

As above, for each positive integer k, we let L_k denote the set of all 0-1 sequences of length k; $L_k = \{(x_1, x_2, \ldots, x_k) : x_i \in \{0,1\} \text{ for all } i = 1, 2, \ldots, k\}$. We also let L_ω denote the set of all infinite 0-1 sequences; $L_\omega = \{(x_1, x_2, \ldots) : x_i \in \{0,1\} \text{ for all } i = 1, 2, \ldots\}$. We let $T_\omega = \bigcup_{k=1}^{\infty} L_k$ and we let $T_{\omega+1} = T_\omega \cup L_\omega$. If $\sigma = (x_1, x_2, \ldots, x_k) \in L_k$ and $\tau = (y_1, y_2, \ldots, y_j) \in L_j$ we set $\sigma < \tau$ if $k < j$ and $x_i = y_i$ for all $i = 1, 2, \ldots k$. Similarly if $\sigma = (x_1, x_2, \ldots, x_k) \in L_k$ and $\tau = (y_1, y_2, \ldots) \in L_\omega$, we set $\sigma < \tau$ if $x_i = y_i$ for all $i = 1, 2, \ldots k$. With this standard ordering, T_ω is referred to as the *complete binary tree of height* ω. $T_{\omega+1}$ is of course just T_ω together with the ωth level adjoined.

The set L_ω can also be regarded as the Cantor set, or equivalently as $\{0,1\}^\omega$, the infinite product of countably many copies of the set $\{0,1\}$. As such, L_ω carries its natural topology (the product topology), which has a basis for the open sets all sets of the form $G_\sigma = \{\tau \in L_\omega : \sigma < \tau\}$, where $\sigma \in L_k$ for some integer k. The sets G_σ are open and closed in this topology.

It is well-known and easy to show that there is a subset X of L_ω, of cardinality 2^ω, such that X contains no closed subset of L_ω having cardinality 2^ω. A so-called *Bernstein set* has this property, as described for example in [5]. For the unfamiliar reader, here is the standard argument for obtaining such a set: since L_ω has a countable basis, there are exactly 2^ω closed subsets of L_ω. Let $\{C_\alpha : \alpha < 2^\omega\}$ be an ennumeration of all the closed subsets of L_ω which have cardinality 2^ω. Now inductively choose elements a_α and b_α, for all $\alpha < 2^\omega$, such that $a_\alpha \in C_\alpha, b_\alpha \in C_\alpha, a_\alpha \neq b_\alpha$ and $a_\alpha \notin \{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}, b_\alpha \notin \{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$. We then take $X = \{a_\alpha : \alpha < 2^\omega\}$. Such a set will be useful in our first example below.

The natural extension of the theorem in the preceding section to the case $n = \omega$ would be this: if P is an ordered set containing 2^ω maximal elements, and if every element of P has a countable cutset in P, then P contains a copy of the tree $T_{\omega+1}$. We can give a simple example where this fails, using a Bernstein set.

EXAMPLE 1. Let X be a Bernstein set in L_ω and let $P = T_\omega \cup X$, with the ordering induced from $T_{\omega+1}$. Then P has 2^ω maximal elements, every element of P has a countable cutset in P, but P contains no subset isomorphic to $T_{\omega+1}$.

The countable cutset condition is verified by noting that, for each $\tau \in X$, the set $\{\sigma \in T_\omega : \sigma \text{ is noncomparable to } \tau\}$ is a cutset for τ in P. Since X contains no closed subset of L_ω of cardinality 2^ω, the last assertion follows from the following observation: let S be a subset of T_ω which is isomorphic to T_ω, and let $S^* = \{\tau \in L_\omega\text{: there are infinitely many } \sigma \in S \text{ with } \sigma < \tau\}$. Then S^* is a closed subset of L_ω. To establish this latter fact, note that if S is isomorphic to T_ω, we can write $S = \bigcup_{k=1}^\infty S_k$, where S_k is the kth level of S. Therefore, we have, for $\tau \in L_\omega, \tau \in S^* \leftrightarrow$ for all $k = 1, 2, \ldots$ there exists $\sigma \in S_k$ such that $\sigma < \tau$. In other words, $S^* = \bigcap_{k=1}^\infty \left(\bigcup_{\sigma \in S_k} G_\sigma\right)$ where G_σ is the open and closed set described above. As an intersection of closed sets, $S^*$is itself closed in L_ω.

We now present a second example which further shows how unsatisfactory things can be in the infinite case. As usual, we let ω_1 denote the first uncountable ordinal number.

EXAMPLE 2. Let $P = \{x_\alpha : \alpha < \omega_1\} \cup \{y_\alpha : \alpha < \omega_1\}$ with the ordering $<$ described as follows: $x_\alpha \leq x_\beta \leftrightarrow \alpha \leq \beta$ and $x_\alpha \leq y_\beta \leftrightarrow \alpha \leq \beta$ (see Figure 2). P has ω_1 maximal elements, every element of P has a countable cutset in P, but P does not contain a complete binary tree of height 2.

The elements y_α are all maximal elements of P. A countable cutset for y_α in P is the set $\{y_\beta : \beta < \alpha\} \cup \{x_{\alpha+1}\}$, and a countable cutset for x_α is the set $\{y_\beta : \beta < \alpha\}$.

So there is no straightforward generalization of the result in the preceding section that applies to the infinite case: assuming the continuum hypothesis; the ordered set P in Example 2 has 2^ω maximal elements, every element has a countable cutset, and yet P does not even contain a complete binary tree of height 2. However, for a special kind of ordered set P, one for which $P - \operatorname{Max} P$ is countable, such a result can be obtained, as we next show.

THEOREM 3. *Let P be an ordered set with uncountably many maximal elements. Suppose that every maximal element of P has a countable cutset in P and that $P - \operatorname{Max} P$ is countable. Then P contains a complete binary tree of height ω.*

PROOF. Let a be any maximal element of P, and let S be a countable cutset for a in

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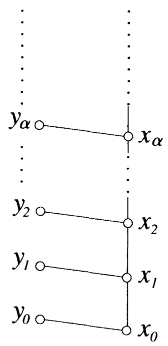

Figure 2

P. Every other maximal element of P is comparable with some element of S. Since there are uncountably many maximal elements it follows that there is some element $p \in S$ for which the set $A = \{x \in \operatorname{Max} P : p < x\}$ is uncountable. The desired binary tree can be obtained inductively, choosing one level after another, using the following lemma repeatedly.

LEMMA 4. *Let $p \in P$ and let A be an uncountable set of maximal elements of P with $p < x$ for all x in A. Then there exist elements p_0 and p_1 in P and uncountable subsets A_0 and A_1 of A such that*

(i) $p < p_0$ and $p < p_1$,
(ii) $p_0 < x$ for all $x \in A_0$ and $p_0 \not< x$ for all $x \in A_1$,
(iii) $p_1 < x$ for all $x \in A_1$ and $p_1 \not< x$ for all $x \in A_0$.

To prove the lemma we first establish the following statement $(*)$: there is some element q with $p < q$ such that both the sets $\{x \in A : q < x\}$ and $\{x \in A : q \not< x\}$ are uncountable. For the sake of contradiction, assume $(*)$ is false. We claim that we can inductively select, for each $\alpha < \omega_1$, elements p_α of A and co-countable subsets H_α of P such that $p_0 = p, H_0 = A$, and such that $\alpha < \beta \longrightarrow p_\alpha < p_\beta$, and $p_\alpha < x$ for all x in H_α. For, suppose we have selected elements p_α and co-countable subsets H_α of A for all $\alpha < \beta$, satisfying these conditions. Then the set $B = \bigcup_{\alpha<\beta}(A - H_\alpha)$ is countable. We have $p_\alpha < x$ for all $\alpha < \beta$ and for all $x \in A - B$. Let x_0 be any element of $A - B$, and let S_0 be a countable cutset for x_0 in P. For each element $x \in A - B - \{x_0\}$, the chain $\{p_\alpha : \alpha < \beta\} \cup \{x\}$ is extended by some element c_x of S_0. Since c_x is noncomparable to x_0 we must have $p_\alpha < c_x$ for all $\alpha < \beta$, and of course $c_x \leq x$. Since S_0 is countable, there is an uncountable subset T of $A - B - \{x_0\}$ such that $c_x = c_y$ for all $x, y \in T$. Let $p_\beta = c_x$ for any (all) x in T. Then we have $p_\alpha < p_\beta$ for all $\alpha < \beta$, and $p_\beta < x$ for all x in T. Since by assumption $(*)$ is false, the set $\{x \in A : p_\beta \not< x\}$ is countable. We let

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$H_\beta = \{x \in A : p_\beta < x\}$. This completes the induction step. In particular, if $(*)$ fails, there is an ω_1-sequence $\{p_\alpha : \alpha < \omega_1\}$ in P. But this contradicts the assumption that $P - \operatorname{Max} P$ is countable.

We return to the proof of the lemma. We start with an element p and an uncountable set A of maximal elements with $p < x$ for all $x \in A$. By $(*)$ there is an element $p_0 > p$ and uncountable subsets B_0 and B_1 of A such that $p_0 < x$ for all x in B_0 and $p_0 \not< x$ for all x in B_1. Let x be an element of B_0 and let S_x be a countable cutset for x in P. Each of the chains $\{p, y\}$, for $y \in B_1$ is extended by some element of S_x. Since S_x is countable there is an element c_x of S_x and an uncountable subset B_x of B_1 such that $p < c_x < y$ for all $y \in B_x$. Now $c_x \in P - \operatorname{Max} P$ and $P - \operatorname{Max} P$ is countable. Therefore there is an uncountable subset A_0 of B_0 and an element $c \in P$ such that $c_x = c$ for all x in A_0. We have that c is noncomparable to x for every x in A_0, since c_x belongs to a cutset for x. Let $p_1 = c$ and let $A_1 = B_x$ for any x in A_0. We have $p_1 < y$ for all y in A_1 and $p_1 \not< x$ for all x in A_0. Furthermore, $p_0 < x$ for all x in A_0 and $p_0 \not< y$ for all y in A_1. This completes the proof.

We do not know the extent to which the condition on $P - \operatorname{Max} P$ can be weakened in Theorem 3. The counterexample in Example 2 suggests attempting to replace the countability of $P - \operatorname{Max} P$ by the condition that P contains no uncountable chains, a prospect we have been unable to settle.

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