BULL. AUSTRAL. MATH. SOC. 53C20, 53C21
VOL. 59 (1999) [133–138]

CONJUGATE RADIUS AND ISOMETRY GROUP OF A MANIFOLD WITH NEGATIVE RICCI CURVATURE

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It is known that the order of the isometry group on a compact Riemannian manifold with negative Ricci curvature is finite. We show by local nilpotent structures that a bound on the orders of the isometry groups exists depending only on the Ricci curvature, the conjugate radius and the diameter.

1. INTRODUCTION

It is a classical Bochner-theorem that if Ricci curvature $\mathrm{Ric}_M < 0$, then the group of isometries of M is finite [7]. But we do not have a bound on the orders of the isometry groups of manifolds with negative Ricci curvatures.

Yamaguchi found a bound on the isometry groups depending on the volume under negative sectional curvatures [10]. In [5], the following result is obtained for manifolds with $-K \leqslant \mathrm{Ric}_M \leqslant -k < 0$, the injectivity radius $\mathrm{inj}_M \geqslant i_0$ and the volume $\mathrm{vol}(M) \leqslant V$:

There exists a constant $N(n, K, k, i_0, V)$ such that for any n-dimensional Riemannian manifold M satisfying the above conditions, the order of the isometry group $\mathrm{Isom}(M)$ *is smaller than N.*

They used a $C^{1,\alpha}$-compactness theorem due to Anderson [1]. For applying this compactness theorem, $\mathrm{inj}_M \geqslant i_0$ is an essential assumption. We shall use the conjugate radius $\mathrm{conj}_M \geqslant c_0$ instead of the above injectivity radius condition. This generalises the above theorem. Let $\mathrm{diam}(M)$ be the diameter of M. We shall show the following theorem.

THEOREM 1.1. *Let M be an n-dimensional compact Riemannian manifold with $-K \leqslant \mathrm{Ric}_M \leqslant -k < 0$, $\mathrm{diam}(M) \leqslant d$ and $\mathrm{conj}_M \geqslant c_0$. Then there exists a constant $N(n, c_0, K, k, d)$ such that the order of the isometry group is bounded by N.*

As an analytic quantity, the injectivity radius and the conjugate radius are the same. The significant differences arise from the topology of manifold. A lower bound of the injectivity radii prevents a collapsing of manifolds so we can obtain compactness

Received 15th July, 1998
Supported by the Korea Institute for Advanced Study.

theorems [1] or [2]. But under the conjugate radius bounded below, a collapsing of manifolds may occur. Even if we assume that M is simply connected and $\mathrm{conj}_M \geqslant c_0$, we cannot obtain $\mathrm{inj}_M \geqslant c_0$. (See Berger's example for S^3 [4].)

Theorem 1.1 shows that the conjugate radius plays a similar role to the injectivity radius for negative Ricci curvature. Using local nilpotent (solvable) structures for the manifolds with $\mathrm{Ric}_M \geqslant -K$ and $\mathrm{conj}_M \geqslant c_0$ [3, 8], the proof of the above theorem will be reduced to linear algebra.

2. Local geometry under $\mathrm{Ric}_M \geqslant -K$ and $\mathrm{conj}_M \geqslant c_0$

As we state in section 1, the injectivity radius and conjugate radius have significant differences so we cannot use any compactness theorems. Hence we cannot apply the proof of [5] directly.

Let $\widetilde{B}(p, c_0/2)$ be the universal covering space of $B(p, c_0/2)$, the $c_0/2$-ball centred at p. Denote the deck transformation group, $\pi_1\big(B(p, c_0/2)\big)$ by Γ_p. We easily show that $\mathrm{inj}_{\widetilde{B}(p,c_0/2)} \geqslant c_0/2$. (For the more precise arguments, we may take the ε_0-ball which appears in section 3 instead of the $c_0/2$-ball.) Then we can apply the compactness theorem [1] to a bounded set of $\widetilde{B}(p, c_0/2)$. We also apply the proof of Theorem 1.3 of [5] to a bounded set of $\widetilde{B}(p, c_0/2)$. Let $\widetilde{p}$ be a lifting of p to $\widetilde{B}(p, c_0/2)$. Then we can obtain the following lemma.

Lemma 2.1. *Let ϕ be an isometry of M. Let $\widetilde{\phi}$ be the lifting of ϕ to $\widetilde{B}(p, c_0)$ satisfying $d\big(\widetilde{p}, \widetilde{\phi}(\widetilde{p})\big) = d\big(p, \phi(p)\big)$. There exists a constant $\varepsilon(n, K, k, c_0) > 0$ depending on n, K, k, c_0 such that if ϕ satisfies that $d\big(x, \phi(x)\big) \leqslant \varepsilon$ for all x in the $c_0/2$-ball in $B(p, c_0/2)$ and $\widetilde{\phi} \circ \Gamma_p = \Gamma_p \circ \widetilde{\phi}$, then ϕ is the identity map.*

Proof: Since $\widetilde{\phi} \circ \Gamma_p = \Gamma_p \circ \widetilde{\phi}$, $d\big(\widetilde{x}, \widetilde{\phi}(\widetilde{x})\big) = d\big(x, \phi(x)\big)$ for all $\widetilde{x} \in \widetilde{B}(p, c_0/2)$. Theorem 1.3 of [5] is proved by analytic methods (the second variational formula and the Sobolev inequality, et cetera). If we apply the same proof to $\widetilde{B}(p, c_0/2)$, then we can prove the existence of $\varepsilon > 0$ such that if $d\big(\widetilde{x}, \widetilde{\phi}(\widetilde{x})\big) \leqslant \varepsilon$, then $\widetilde{\phi}$ is the identity map on the $c_0/2$-ball in $\widetilde{B}(p, c_0)$. Since the set of fixed points for an isometry is a totally geodesic submanifold, $\widetilde{M}$ is the set of fixed points for $\widetilde{\phi}$. ∎

If $\phi \in \mathrm{Isom}(M)$ is homotopic to the identity and $\mathrm{diam}\,(M) \leqslant \varepsilon$, then the conditions in Lemma 2.1 are satisfied. As an immediately consequence, we have the following corollary.

Corollary 2.2. *Let M be an n-dimensional compact Riemannian manifold with $-K \leqslant \mathrm{Ric}_M \leqslant -k < 0$ and $\mathrm{conj}_M \geqslant c_0$. Then there exists a constant $\varepsilon(n, c_0, K, k) > 0$ such that if $\mathrm{diam}\,(M) \leqslant \varepsilon$, then every isometry of M which is homotopic to the identity is the identity.*

Denote the displacement function $d(p,\phi(p))$ of an isometry ϕ by $\delta_\phi(p)$. We shall prove Theorem 1.1 by showing that the number of isometries satisfying $\delta_\phi(p) \leqslant \varepsilon$ for some $p \in M$ is uniformly bounded for sufficiently small $\varepsilon > 0$. Then we obtain Theorem 1.1 by the standard packing arguments.

Note that the conditions of almost nonnegative Ricci curvature and large conjugate radius imply that M is a nilmanifold up to finite cover [8]. By a rescaling of the metric, it follows that there exists $\varepsilon_0(n,c_0,K) > 0$ depending only on n, c_0, K such that if $\mathrm{Ric}_M \geqslant -K$ and $\mathrm{conj}_M \geqslant c_0$ and $\mathrm{diam}\,(M) \leqslant \varepsilon_0(n,c_0,K)$, then M is a nilmanifold up to finite cover. From this fact, if M satisfies the conditions in Theorem 1.1, we may assume that $B(p,\varepsilon_0) \simeq L \times R^m$ up to finite cover, where L is a nilmanifold. This fact follows from a splitting theorem of [3]. Let T^k be a k-dimensional torus. From [3] and [9], we can represent the above L as follows by a rescaling of the metric [8]:

(1) L is a fibre bundle over T^{n_1} with fibre $F^{(1)}$.

(2) $F^{(1)}$ is a fibre bundle over T^{n_2} with fibre $F^{(2)}$.

(3) $F^{(2)}$ is a fibre bundle over T^{n_3} with fibre $F^{(3)}$.

$$\vdots$$

From the above fact, we know that $\pi_1(F^{(j)})/\pi_1(F^{(j+1)})$ is Abelian so $\pi_1(F^{(j+1)})$ contains $[\pi_1(F^{(j)}),\pi_1(F^{(j)})]$. So we easily show that M has a solvable structure. In fact, we can obtain a nilpotent structure from a commutator estimate [8], but we only use this solvable structure. We can take orthogonal basis of $\pi_1(F^l)/\pi_1(F^{l+1})$, $\{\gamma_{l1},\cdots,\gamma_{ln_l}\}$ which can be considered as a basis of T^{n_l}, that is, we may consider $\pi_1(F^l)/\pi_1(F^{l+1})$ as $\pi_1(T^{n_l})$. If $\mathrm{diam}\,(M) \leqslant \varepsilon_0$ for the above $\varepsilon_0 > 0$, we have that

$$0 < b_1(c_0,K) \leqslant \frac{|\gamma_{lj}|}{|\gamma_{lj'}|} \leqslant b_2(c_0,K) \tag{2.1}$$

for some constants b_1, b_2 and we may assume that

$$\frac{|\gamma_{lj}|}{|\gamma_{l'j'}|} \geqslant 100 b_2$$

for $j < j'$.

We consider the isometry of the k-dimensional flat torus T^k. Let $T^k = \mathbb{R}^k/\langle\gamma_1,\cdots,\gamma_k\rangle$, where we take $\gamma_i \in \pi_1(T^k)$ as an orthogonal basis of $\mathbb{R}^k$. Let ϕ be an isometry of T^k. Then we can lift ϕ to $\widetilde{\phi} : \mathbb{R}^k \to \mathbb{R}^k$ such that $d(\widetilde{\phi}(\widetilde{p}),\widetilde{p}) = d(\phi(p),p)$, where $\widetilde{p}$ is a lifting of some fixed p. We write

$$\widetilde{\phi}(x) = Ax + b,$$

where $A \in O(n)$ and $b \in \mathbb{R}^k$. A generator t_i of the deck transformation, can be written as

$$t_i(x) = x + \gamma_i.$$

Then we have

$$\widetilde{\phi} \circ t_i \circ \widetilde{\phi}^{-1} = x + A\gamma_i. \tag{2.2}$$

3. Proof of Theorem 1.1

We use the local solvable structure of M strongly. First we shall consider isometries of an almost flat manifold (a solvable manifold). Let $\varepsilon_0(n, K, k, c_0) > 0$ be the quantity which appears in section 2 and $\varepsilon_1(n, K, k, c_0) > 0$ be the $\varepsilon > 0$ which appears in Lemma 2.1. We assume that M satisfies the conditions in Lemma 2.1 with $\operatorname{diam}(M) \leqslant \varepsilon$, where $\varepsilon > 0$ is a number smaller than ε_0 and $\varepsilon_1/2$. Let ϕ be an isometry of M. Note that $\delta_\phi \leqslant \varepsilon$. By considering some finite covering space of M, we can consider M as a nilmanifold L as in Section 2 and ϕ can be lifted to L. We also write this lifting as ϕ. Let

$$\Lambda_l := \langle \gamma_{l1}, \cdots, \gamma_{ln_l} \rangle = \pi_1(T^{k_l}).$$

Since ϕ is an isometry of L, ϕ acts on each Λ_l. Precisely, define

$$\rho_\phi(\gamma) := \widetilde{\phi} \circ \gamma \circ \widetilde{\phi}^{-1}$$

for $\gamma \in \pi_1(L)$. In the case of a flat torus T^k, we know that $d(\gamma(x), x) = d(\rho_\phi(\gamma)(x), x)$ from (2.2). Since L is a nilmanifold as in Section 2, we obtain that for $\gamma \in \pi_1(M)$,

$$\left| \frac{d\big(\gamma(x), x\big)}{d\big(\rho_\phi(\gamma)(x), x\big)} - 1 \right| \leqslant b_1/100. \tag{3.1}$$

In fact, we can easily get the above inequality by restricting $\rho_\phi(\gamma)$ to T^{n_l} and applying (2.2) to T^{n_l}, since each $F^{(l-1)}$ converges to a flat torus T^{n_l} and isometries on M converge to isometries of T^{n_l} as $\varepsilon \to 0$. Rescaling the metric such that $\operatorname{diam}(T^{n_l}) = 1$, we know that ρ_ϕ acts on each $\pi_1(T^{n_l})$.

Now we shall prove Theorem 1.1. We shall use the methods in [**6**] and [**10**].

Proof of Theorem 1.1: We use the same notation as above. We can define a homomorphism from $\operatorname{Isom}(M)$ to $\operatorname{Aut}(\Gamma_p)$, the automorphism group of Γ_p, as follows:

$$\rho : \operatorname{Isom}(M) \to \operatorname{Aut}(\Gamma_p)$$
$$\phi \mapsto \rho_\phi.$$

By considering the kernel of ρ, we obtain that

$$\ker(\rho) = \{\phi \mid \phi \circ \gamma = \gamma \circ \phi,\ \gamma \in \Gamma_p\}.$$

It follows from [6] and Lemma 2.1 that the order of $\ker(\rho) \leqslant C_1(n, K, k, c_0, d)$. Precisely, take $\{x_i \mid i = 1, \cdots, s\}$ in M such that M can be covered by $\bigcup_{i=1}^{s} B(x_i, \varepsilon/4)$ and $B(x_i, \varepsilon/8)$'s are pairwise disjoint. We define $F(\phi)(i)$ as the smallest j such that $\phi(x_i) \in B(x_j, \varepsilon/4)$. Then F is a map from $\mathrm{Isom}(M)$ to $S^S = \{f \mid f : S \to S\}$ where $S = \{1, \cdots, s\}$. Also F is an injective map, as follows [6]. Let $x \in B(x_i, \varepsilon/4)$. For $F(\phi) = F(\psi)$, we obtain that

$$\begin{aligned} d\big(\phi(x), \psi(x)\big) \leqslant d\big(\phi(x), \phi(x_i)\big) + d\big(\phi(x_i), x_{F(\phi)(i)}\big) \\ + d\big(\psi(x_i), x_{F(\psi)(i)}\big) + d\big(\psi(x_i), \psi(x)\big) \leqslant \varepsilon. \end{aligned}$$

From Lemma 2.1, we know that $\psi^{-1} \circ \phi$ is the identity map so F is injective. The cardinality of S is bounded by $C_1(n, K, k, c_0, d)$ so the order of $\ker(\rho)$ is bounded by $C = C_1^{C_1}$.

Now we only need to compute the order of $\mathrm{Im}(\rho)$. We consider the following two cases.

CASE I. The cardinality of $I_1 = \big\{\phi \mid \delta_\phi(p) \leqslant \varepsilon \text{ for some } p \in M\big\}$.

We consider $B(p, 10\varepsilon)$ and we may regard this ball as $L \times \mathbb{R}^k$, where L is an almost flat manifold. From (3.1), for $\gamma \in \Gamma_p$,

$$\left| \frac{d(\gamma(p), p)}{d(\rho_\phi(\gamma)(p), p)} - 1 \right| \leqslant b_1/100$$

on an R-ball in $\widetilde{B}(p, c_0/2)$ as above. So ρ_ϕ maps the points in the lattice generated by $\{\gamma_{ls}\}$ in a $b_2(c_0, K)\,|\gamma_{l1}|$-ball to those in a $2b_2(c_0, K)\,|\gamma_{l1}|$-ball. The number of such maps is uniformly bounded by some constant $D(c_0, K)$ since the number of lattices in a $2b_2$-ball is bounded. Then the cardinality of $\{\rho_\phi\}$ is bounded by $\prod_1^N D \leqslant D^n$ since $N \leqslant n$, where L is a N-step nilmanifold, that is, $F^{(N+1)}$ in section 2 is a point. Then the cardinality of I_1, $|I_1| \leqslant CD^n$.

CASE II. The cardinality of $I_2 = \big\{\phi \mid \delta_\phi(p) \geqslant \varepsilon \text{ for all } p \in M\big\}$.

Let $I_2' = \{\phi_1, \cdots, \phi_L\}$ be a maximal subset of I_2 such that $d\big(\phi_i(x), \phi_j(x)\big) \geqslant \varepsilon$ for all $p \in M$ and all $\phi_i, \phi_j \in I_2'$. From the above arguments, we know that $L \leqslant C$. For any $\phi \in I_2$, we obtain that $\phi\phi_i^{-1} \in I_1$ for some $\phi_i \in I_2'$. Then $I_2 \subset I_1 I_2'$. Hence, $|I_2| \leqslant C^2 D^n$.

Consequently, the total number of isometries is bounded by $CD^n(1 + C)$. Since ε depends only on n, K, k, c_0, d so $(1 + C)CD^n$ also depends on n, K, k, c_0, d. This completes the proof of Theorem 1.1. ∎

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