On a Generalisation of the Laplace Transformation

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1. On reading a recent paper by R. S. Varma (Varma 1949) I recalled that in May 1942 I investigated an integral transformation which is very similar to Varma's. Varma has

$$\phi_m^k(st) = \int_0^\infty (2st)^{-\frac{1}{4}} W_{k,m}(2st) f(t)\, dt \tag{1}$$

and points out that this reduces to a Laplace integral for $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$. Instead of (1), one could consider the integral

$$\int_0^\infty e^{-\frac{1}{2}st} W_{k,m}(st)\ (st)^{-k} f(t)\, dt \tag{2}$$

which was introduced by C. S. Meijer (Meijer 1940 b); this integral reduces to a Laplace integral whenever $k = m + \frac{1}{2}$. Now, apart from comparatively unimportant factors, the nucleus of (2) is a fractional derivative or integral, as the case may be, of e^{-st}, and on carrying out a fractional integration by parts, it appears that (2) is essentially the Laplace transform of a fractional integral or derivative of f. Thus, the whole theory of the transformation (2), including inversion formulae, representation theorems, etc., can be deduced from the well-known theory of the Laplace transformation. It is not quite clear that a similar reduction is possible for (1), although it is certainly possible when $k = 0$.

My work of 1942 remained unpublished, and I still hope to describe it in more detail on some future occasion. Meanwhile, in view of the reviving interest in the subject,[1] I should like to establish briefly the connection between (2) and the Laplace transformation.

2. I shall use the operators of fractional integration and differentiation whose theory has been developed by H. Kober and myself. For the sake of brevity, I shall formulate all results for the class $L_2(0, \infty)$ and merely remark that corresponding results are known for the classes L_p, $1 \leq p \leq \infty$. The definition of the operators in the simplest case is (Kober 1940)

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[1] Cf. for instance, a series of papers by S. K. Bose, a pupil of Dr Varma's.

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$$I^{+}_{\eta,\alpha} f(x) = \{\Gamma(\alpha)\}^{-1} x^{-\eta-\alpha} \int_0^x (x-u)^{\alpha-1} u^{\eta} f(u)\, du \tag{3}$$

$$K^{-}_{\eta,\alpha} f(x) = \{\Gamma(\alpha)\}^{-1} x^{\eta} \int_x^{\infty} (u-x)^{\alpha-1} u^{-\eta-\alpha} f(u)\, du \tag{4}$$

where $f(x)$ is in $L_2(0, \infty)$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\eta > -\frac{1}{2}$.

First, the definitions can be extended to other values of η, as long as $\operatorname{Re}\eta - \frac{1}{2}$ is not a negative integer (Erdélyi 1940); next follows the extension to $\operatorname{Re}\alpha = 0$ which is much more difficult (Kober 1941). The domain of these extensions is still the full class L_2. Lastly, the extension to $\operatorname{Re}\alpha < 0$ is given by the definition

$$I^{+}_{\eta,\alpha} = (I^{+}_{\eta+\alpha,-\alpha})^{-1}, \qquad K_{\eta,\alpha} = (K_{\eta+\alpha,-\alpha})^{-1}.$$

In this last extension it is necessary to contract the domain of definition from the full L_2 to a class $L_2^{(\alpha)}$ which coincides with L^2 if $\operatorname{Re}\alpha \geqq 0$. Here the operators will be used in the extended sense (Erdélyi 1940).

We define the Mellin transform as

$$\mathfrak{M}_t f(x) = \underset{X\to\infty}{\text{l. i. m.}} \int_{X^{-1}}^{X} x^{-\frac{1}{2}+it} f(x)\, dx,$$

where the right-hand side is a limit in mean square. For the extended operators we then have

$$\mathfrak{M}_t I^{+}_{\eta,\alpha} f = \frac{\Gamma(\eta+\frac{1}{2}-it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)} \mathfrak{M}_t f \tag{5}$$

$$\mathfrak{M}_t K^{-}_{\eta,\alpha} f = \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} \mathfrak{M}_t f. \tag{6}$$

Moreover, we have the formula for fractional integration by parts

$$\int_0^{\infty} dx\, \phi(x)\, I^{+}_{\eta,\alpha} f(x) = \int_0^{\infty} dx\, f(x)\, K^{-}_{\eta,\alpha} \phi(x), \tag{7}$$

valid if both $f(x)$ and $\phi(x)$ belong to $L_2^{(\alpha)}$.

3. After these preliminaries we define the nucleus of our transform

$$k(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\eta+\frac{1}{2}+it)\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} z^{-\frac{1}{2}-it}\, dt \tag{8}$$

where we assume that neither $\operatorname{Re}\eta - \frac{1}{2}$ nor $\operatorname{Re}\rho - \frac{1}{2}$ is a negative integer. The evaluation of (8) as the sum of residues leads to an expression in terms of confluent hypergeometric functions or "cut" confluent hypergeometric functions; and from Mellin's inversion formula we have

$$\mathfrak{M}_t k(z) = \frac{\Gamma(\eta+\frac{1}{2}+it)\,\Gamma(\rho+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} = \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)} \mathfrak{M}_t (z^{\rho} e^{-z}),$$

and thus $$k(z) = K^{-}_{\eta, \alpha}(z^{\rho} e^{-z}). \tag{9}$$

We can now integrate by parts according to (7) and find on account of (9) that

$$g(x) = \int_0^{\infty} k(xy) f(y)\, dy = \int_0^{\infty} e^{-xy}(xy)^{\rho} I^{+}_{\eta, \alpha} f(y)\, dy \tag{10}$$

and hence the reduction of the k-transform of f (in $L_2^{(\alpha)}$) to x^{ρ} times the Laplace transform of $y^{\rho} I^{+}_{\eta, \alpha} f(y)$. It is also possible to prove (although this requires a justification, by absolute convergence, of the interchange of the order of integrations) that for the function defined by (10)

$$K^{-}_{\eta+\alpha, -\alpha}\, g(x) = \int_0^{\infty} e^{-xy} (xy)^{\rho} f(y)\, dy \tag{11}$$

for all functions f in L_2. This latter form enables one to invert the k-transformation by means of any of the numerous inversion formulae of the Laplace transformation. For representation theorems, (10) is the more suitable form.

In my unpublished work, I developed the theory of the more general transformation whose nucleus is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\{(\eta + \frac{1}{2} + it)/n\}\, \Gamma(\rho + \frac{1}{2} + it)}{\Gamma\{\alpha + (\eta + \frac{1}{2} + it)/n\}} (xy)^{-\frac{1}{2} - it}\, dt$$

where n is any positive number, not necessarily integer. $n = 1$ is the nucleus (8), $n = 2$ leads to the particular case $k = 0$ of (1): this particular case has been studied in some detail (Meijer 1940a, Boas 1942a, b).

[*Added 20th September* 1954. Since this note was submitted for publication, a further instalment has appeared in *Rend. Sem. Mat. Università e Politecnico di Torino* 10 (1950/51), 217-234. A transformation which is equivalent to (2) has also been investigated by K. P. Bhatnagar, *Ganita* 3 (1952), 13-18, who refers to unpublished work by R. S. Varma.]

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