BULL. AUSTRAL. MATH. SOC. 46B20
VOL. 46 (1992) [67–79]

A CONTINUITY PROPERTY RELATED TO AN INDEX OF NON-SEPARABILITY AND ITS APPLICATIONS

WARREN B. MOORS

For a set E in a metric space X the *index of non-separability* is

$$\beta(E) = \inf\{r > 0 \colon E \text{ is covered by a countable-family of balls of radius less than } r\}.$$

Now, for a set-valued mapping Φ from a topological space A into subsets of a metric space X we say that Φ is β *upper semi-continuous* at $t \in A$ if given $\varepsilon > 0$ there exists a neighbourhood U of t such that $\beta(\Phi(U)) < \varepsilon$. In this paper we show that if the subdifferential mapping of a continuous convex function Φ is β upper semi-continuous on a dense subset of its domain then Φ is Fréchet differentiable on a dense G_δ subset of its domain. We also show that a Banach space is Asplund if and only if every weak* compact subset has weak* slices whose index of non-separability is arbitrarily small.

1. INTRODUCTION

For a bounded set E in a metric space X the *Kuratowski index of non-compactness* of E is

$$\alpha(E) \equiv \inf\{r > 0 \colon E \text{ is covered by a finite family of sets of diameter less than } r\}.$$

Recently, in a paper by Giles and Moors [2], a new continuity property related to Kuratowski's index of non-compactness was examined. In that paper they said that a set-valued mapping Φ from a topological space A into subsets of a metric space X is α *upper semi-continuous* at a point $t \in A$ if given $\varepsilon > 0$ there exists an open neighbourhood U of $t \in A$ such that $\alpha(\Phi(U)) < \varepsilon$. They showed that under suitable circumstances α upper semi-continuity characterises (metric) upper semi-continuity, and that a significant class of set-valued mappings which are α upper semi-continuous on a dense subset of their domain are single-valued and (metric) upper semi-continuous on a dense G_δ subset of their domain.

In this paper we consider a natural generalisation of α upper semi-continuity called β upper semi-continuity. This new upper semi-continuity condition is defined in terms of an index of non-separability.

Received 9th July, 1991.

In Section 2 we define the index of non-separability and prove some of its elementary properties which are analogous to those of the Kuratowski index of non-compactness. We also examine a property satisfied by the Kuratowski index of non-compactness, but which fails to be true for the index of non-separability.

In Section 3 we show that any minimal weak* cusco from a Baire space A into subsets of the dual of a Banach space X which is β upper semi-continuous on a dense subset of A, is single-valued and (norm) upper semi-continuous on a dense G_δ subset of A.

In Section 4 we use the index of non-separability to define β denting points of a set in a Banach space and establish a pleasing connection between β denting points and ordinary denting points. The single-valuedness property established in Section 3 suggests an application in determining conditions under which continuous convex functions on a Banach space are generically Fréchet differentiable.

In Section 5 we derive another characterisation of Asplund spaces. We also extend a recent result of Kenderov and Giles [**3**, Theorem 3.5] to show that on a Banach space X which can be equivalently renormed to have every point of the unit sphere of X a β denting point of the closed unit ball of X, a continuous convex function on an open convex subset of X^* is generically Fréchet differentiable provided that the set of points where the function has a weak* continuous subgradient is residual in its domian.

2. A MEASURE OF NON-SEPARABILITY

For a set E in a metric space X the *index of non-separability* is

$$\beta(E) = \inf\{r > 0\colon E \text{ is covered by a countable family of balls of radius less than } r\},$$

when E can be covered by a countable family of balls of fixed radii, otherwise, $\beta(E) = \infty$.

Throughout the rest of this paper all Banach spaces X are over the real numbers with dual X^*. The closed unit ball $\{x \in X\colon \|x\| \leqslant 1\}$ will be denoted by $B(X)$ and the unit sphere $\{x \in X\colon \|x\| = 1\}$ by $S(X)$. Consider a non-empty bounded subset K of X. Given $f \in X^* \setminus \{0\}$ and $\delta > 0$, the *slice* of K defined by f and δ is the set $S(K, f, \delta) \equiv \{x \in K\colon f(x) > \sup f(K) - \delta\}$. For a metric space (X, d), given $x_0 \in X$ and $r > 0$ we will denote by $B(x_0, r)$ the open ball $\{x \in X\colon d(x, x_0) < r\}$ and by $B[x_0, r]$ the closed ball $\{x \in X\colon d(x, x_0) \leqslant r\}$. For any set E in a topological space X we will denote by $C(E)$ the complement of E in X and $\overline{E}$ the closure of E in X. We will denote the interior of E by int E and the boundary of E by ∂E.

PROPOSITION 2.1. *For a metric space (X, d), the index of non-separability β on X satisfies the following properties:*

1. $\beta(A) \geqslant 0$ *for any* $A \subseteq X$;

https://doi.org/10.1017/S0004972700011680 Published online by Cambridge University Press

2. $\beta(A) = 0$ *if and only if* A *is a separable subset of* X*;*
3. $\beta(A) \leqslant \beta(B)$ *if* $A \subseteq B \subseteq X$*;*
4. $\beta(A) = \beta(\overline{A})$ *for any* $A \subseteq X$*;*
5. $\beta\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup\{\beta(A_n) : n \in \mathbb{N}\}$ *where* $A_n \subseteq X$ *for all* $n \in \mathbb{N}$*;*
6. $\beta(A \cap B) \leqslant \min\{\beta(A), \beta(B)\}$ *for* $A, B \subseteq X$*.*

We omit the proofs of the properties (1) to (6) as they are straightforward.

Proposition 2.2. *For a normed linear space* $(X, \|\cdot\|)$*, the index of non-separability* β *on* X *satisfies the following additional properties:*

7. $\beta(A + B) \leqslant \beta(A) + \beta(B)$ *for* $A, B \subseteq X$*;*
8. $\beta(kA) = |k|\beta(A)$ *for* $A \subseteq X$ *and* $k \in \mathbb{R}$*;*
9. $\beta(coA) = \beta(A)$ *for* $A \subseteq X$ *where* coA *denotes the convex hull of* A*.*

Proof: The proofs of the properties (7) to (9) are straightforward, with the possible exception of (9), which we now prove. Clearly $\beta(A) \leqslant \beta(coA)$ as $A \subseteq coA$, so it is sufficient to show $\beta(A) \geqslant \beta(coA)$. Given $\varepsilon > 0$, choose a sequence $\{x_n\}_{n=1}^{\infty}$ and an $r > 0$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} B[x_n, r] \text{ and } \beta(A) < r < \beta(A) + \varepsilon.$$

But
$$A \subseteq \bigcup_{n=1}^{\infty} B[x_n, r] = \left(\bigcup_{n=1}^{\infty} \{x_n\}\right) + B[0, r] \subseteq co\left(\bigcup_{n=1}^{\infty} \{x_n\}\right) + B[0, r]$$

so
$$coA \subseteq co\left(\bigcup_{n=1}^{\infty} \{x_n\}\right) + B[0, r].$$

Therefore

$$\beta(coA) \leqslant \beta\left(co\bigcup_{n=1}^{\infty} \{x_n\}\right) + \beta(B[0, r])$$

by (7). But $co\bigcup_{n=1}^{\infty} \{x_n\}$ is separable so $\beta\left(co\bigcup_{n=1}^{\infty} \{x_n\}\right) = 0$ from which it follows that $\beta(coA) < \beta(A) + \varepsilon$. ∎

It is a well-known property of Kuratowski's index of non-compactness, that for a nested sequence of non-empty closed sets $\{F_n\}_{n=1}^{\infty}$ in a complete metric space X, if $\lim_{n\to\infty} \alpha(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, [4, p.303]. However, the analogous result for the index of non-separability is false, as is shown in the following example. Consider the Banach space c_0 and the element $\{1/2, 1/4, 1/8, \ldots, 1/2^n, \ldots\} \in S(\ell_1)$. For each

https://doi.org/10.1017/S0004972700011680 Published online by Cambridge University Press

$n \in \mathbb{N}$ let $F_n = \{\{x_1, x_2, x_3, \ldots, x_n, \ldots\} \in B(c_0)\colon \sum_{n=1}^{\infty} x_n/2^n \geqslant 1 - 1/n\}$. Then $\{F_n\}_{n=1}^{\infty}$ is a nested sequence of non-empty closed sets with $\lim_{n\to\infty} \beta(F_n) = 0$; however $\bigcap_{n=1}^{\infty} F_n = \emptyset$. We observe that for a normed linear space X either $\beta(B(X)) = 0$ or 1. In the first case, X is separable and in the second case, X is non-separable. The proof of this follows from properties (7) and (8) of Proposition 2.2.

3. β UPPER SEMI-CONTINUITY

Consider a set-valued mapping Φ from a topological space A into subsets of a topological space X. Φ is said to be *upper semi-continuous* at $t \in A$, if, given an open set W containing $\Phi(t)$ there exists an open neighbourhood U of t such that $\Phi(U) \subseteq W$. For brevity we call Φ an *usco* if it is upper semi-continuous and $\Phi(t)$ is a non-empty compact subset of X for each $t \in A$. If X is a linear topological space we call Φ a *cusco* if it is upper semi-continuous and $\Phi(t)$ is a non-empty convex compact subset of X for each $t \in A$.

For a set-valued mapping Φ from A into subsets of a metric space X we introduce another upper semi-continuity property defined in terms of the index of non-separability. We say that Φ is β *upper semi-continuous* at $t \in A$, if, given an $\varepsilon > 0$ there exists an open neighbourhood U of t such that $\beta(\Phi(U)) < \varepsilon$.

We notice then, that every set-valued mapping Φ from a topological space A into subsets of a separable metric space X is β upper semi-continuous on A. So it is clear that β upper semi-continuity is closely related to questions concerning the separability of the images of set-valued mappings.

PROPOSITION 3.1. *Let Φ be a set-valued mapping from a Lindelöf space A into subsets of a metric space X. Then Φ has a separable image $\Phi(A)$ if and only if Φ is β upper semi-continuous on A.*

PROOF: It is obvious that Φ is β upper semi-continuous on A when $\Phi(A)$ is separable.

Conversely, given $\varepsilon > 0$, for each $x \in A$ let U_x be an open neighbourhood of x in A such that $\beta(\Phi(U_x)) < \varepsilon$. But $A = \bigcup\{U_x \subseteq A\colon x \in A\}$ so there exists a countable subcover $\{U_{x_n}\}_{n=1}^{\infty}$ of A. So, $0 \leqslant \beta(\Phi(A)) = \beta\left(\bigcup_{n=1}^{\infty} \Phi(U_{x_n})\right) = \sup\{\beta(\Phi(U_{x_n})\colon n \in \mathbb{N})\} \leqslant \varepsilon$. It follows that $\beta(\Phi(A)) = 0$, and so $\Phi(A)$ is separable. ∎

One of the great advantages of the generalised continuity condition we have introduced, is that significant classes of β upper semi-continuous set-valued mappings from a Baire space into subsets of a metric space are single-valued and (metric) upper semi-continuous on a dense G_δ subset of their domain. This is a consequence of the countability arguments which are entailed by the index of non-separability.

An usco (cusco) Φ from a topological space A into subsets of a topological (linear topological) space X is said to be *minimal* if its graph does not strictly contain the graph of any other usco (cusco) with the same domain. We will need the follow characterisation of minimal uscos (cuscos); see [**2**, Lemma 2.5].

PROPOSITION 3.2. *Consider an usco (cusco) Φ from a topological space A into subsets of a Hausdorff space (separated linear topological space) X. Then Φ is a minimal usco (cusco) if and only if for any open set V in A and closed (closed and convex) set K in X where $\Phi(V) \not\subseteq K$ there exists a non-empty open subset $V' \subseteq V$ such that $\Phi(V') \cap K = \emptyset$.*

LEMMA 3.3. *Let A be a topological space and X a Hausdorff (separated linear topological) space. Consider Φ a minimal usco (cusco) from A into subsets of X. Let B be a closed (closed and convex) subset of X. If for each open subset U in A, $\Phi(U) \not\subseteq B$ then $\{x \in A : \Phi(x) \cap B = \emptyset\}$ is a dense open subset of A.*

PROOF: Let $W = \{x \in A : \Phi(x) \cap B = \emptyset\}$. Since Φ is upper semi-continuous and B is closed, W is open. So it is sufficient to show W is dense in A. Let V be a non-empty open set in A, then $\Phi(V) \not\subseteq B$ so from Proposition 3.2 there exists a non-empty open subset V' of V such that $\Phi(V') \cap B = \emptyset$ and so $\emptyset \neq V' \subseteq W \cap V$. Therefore W is dense in A. ∎

LEMMA 3.4. *Let U be a non-empty open subset of a Baire space A and X a metric (linear metric) space. Consider a minimal τ-usco (τ-cusco) Φ from A into subsets of X, where X is endowed with a topology τ such that the metric closed balls are also τ closed (and convex). If $\beta(\Phi(U)) < \varepsilon$ for some $\varepsilon > 0$, then there exists a non-empty open subset V of U such that diam $\Phi(V) \leqslant 2\varepsilon$.*

PROOF: Let Φ' be the restriction of Φ to U. It follows from Proposition 3.2 that Φ' is a minimal τ usco (τ-cusco). We note also that U is a Baire space. Since $\beta(\Phi(U)) < \varepsilon$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\Phi'(U) \subseteq \bigcup_{j=1}^{\infty} B[x_j \varepsilon]$. Now if $\Phi'(W) \subseteq B[x_1, \varepsilon]$ for some non-empty open set W contained in U write $V = W$, but if not, by Lemma 3.3 there exists a dense open set $O_1 \subseteq U$ such that $\Phi'(O_1) \cap B[x_1, \varepsilon] = \emptyset$. Now if $\Phi'(W) \subseteq B[x_2, \varepsilon]$ for some non-empty open set W contained in U write $V = W$, but if not, we have by Lemma 3.3 that there exists a dense open set $O_2 \subseteq U$ such that $\Phi'(O_2) \cap B[x_2, \varepsilon] = \emptyset$. Continue in this way, we will have defined V at some stage, because if not, there exists a dense G_δ subset $O_\infty \subseteq U$ where $O_\infty = \bigcap_{n=1}^{\infty} O_n$ and $\Phi'(O_\infty) \cap \bigcup_{j=1}^{\infty} B[x_j, \varepsilon] = \emptyset$ contradicting the fact that $\Phi'(U)$ is contained in $\bigcup_{j=1}^{\infty} B[x_j, \varepsilon]$.

https://doi.org/10.1017/S0004972700011680 Published online by Cambridge University Press

So U contains a non-empty open set V with diam $\Phi(V) \leqslant 2\varepsilon$. ∎

THEOREM 3.5. *Consider a Baire space A, a metric space (linear metric space) X, and X with topology τ where the metric closed balls are also τ closed. Consider a minimal τ usco (τ cusco) Φ from A into subsets of X. If Φ is β upper semi-continuous on a dense subset of A then there exists a dense G_δ subset of A on which Φ is single-valued and metric upper semi-continuous.*

PROOF: Given $\varepsilon > 0$, consider the set $O_\varepsilon = \bigcup\{\text{open sets } U \subseteq A\colon \text{diam } \Phi(U) \leqslant 2\varepsilon\}$. Clearly O_ε is open. We now show that O_ε is dense. Consider W a non-empty open subset of A. Now, there exists a $t \in W$ where Φ is β upper semi-continuous. So there exists an open neighbourhood U of t contained in W such that $\beta(\Phi(U)) < \varepsilon$. Therefore by Lemma 3.4 there exists a non-empty open subset V contained in U such that diam $\Phi(U) \leqslant 2\varepsilon$, and so $\emptyset \neq V \subseteq O_\varepsilon \cap W$. Since A is a Baire space, $\bigcup_{n=1}^{\infty} O_{1/n}$ is a dense G_δ subset of A on which Φ is single-valued and metric upper semi-continuous. ∎

An important application of our theory so far concerns the differentiability of convex functions. A continuous convex function ϕ on an open convex subset A of a Banach space X is said to be *Fréchet differentiable* at $x \in A$ if

$$\lim_{\lambda \to 0} \frac{\phi(x + \lambda y) - \phi(x)}{\lambda} \text{ exists and is approached uniformly for all } y \in S(X).$$

A *subgradient* of ϕ at $x_0 \in A$ is a continuous linear functional f on X such that $f(x - x_0) \leqslant \phi(x) - \phi(x_0)$ for all $x \in A$. The *subdifferential* of ϕ at x, denoted by $\partial\phi(x)$, is the set of subgradients of ϕ at x, and is non-empty for each $x \in X$. Now the subdifferential mapping $x \to \partial\phi(x)$ is a minimal weak* cusco from A into subsets of X^*, [**6**, p.100]. Further, ϕ is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \to \partial\phi(x)$ is single-valued and norm upper semi-continuous at $x \in A$, [**6**, p.18] so from Theorem 3.5 we have the following result.

COROLLARY 3.6. *A continuous convex function ϕ on an open convex subset A of a Banach space X whose subdifferential mapping $x \to \partial\phi(x)$ is β upper semi-continuous on a dense subset of A is Fréchet differentiable on a dense G_δ subset of A.*

The well-known result follows from this corollary that every continuous convex function on an open convex subset of a Banach space whose dual is separable is Fréchet differentiable on a dense G_δ subset of its domain.

We will require the following proposition in Section 5, which again concerns conditions which imply that the image of a set-valued mapping is separable.

PROPOSITION 3.7. *Let A be a separable topological space and X a normed linear space. If Φ is a minimal weak usco or cusco then the image $\Phi(A)$ is separable.*

PROOF: Let $\{x_n \in A\colon n \in \mathbb{N}\}$ be a dense subset of A. For each $n \in \mathbb{N}$, choose $y_n \in \Phi(x_n)$. Now, suppose $\Phi(A) \not\subseteq \overline{co} \bigcup_{n=1}^{\infty} \{y_n\}$; then by Proposition 3.2 there exists a non-empty open subset V of A such that $\Phi(V) \cap \overline{co} \bigcup_{n=1}^{\infty} \{y_n\} = \emptyset$. But for some $k \in \mathbb{N}$, $x_k \in V$ so, $\emptyset \neq (y_k) \subseteq \Phi(V) \cap \overline{co} \bigcup_{n=1}^{\infty} \{y_n\}$. Therefore we must have that $\Phi(A) \subseteq \overline{co} \bigcup_{n=1}^{\infty} \{y_n\}$. But $\overline{co} \bigcup_{n=1}^{\infty} \{y_n\}$ is separable, so $\Phi(A)$ is also separable. ∎

COROLLARY 3.8. *A continuous convex function ϕ on an open convex subset A of a Banach space X whose subdifferential mapping $x \to \partial\phi(x)$ is a weak cusco is Fréchet differentiable on a dense G_δ subset of A.*

PROOF: In Phelps, [**6**, p.23], it is shown that ϕ is Fréchet differentiable on a dense G_δ subset of A if $\phi|_Y$ is Fréchet differentiable on a dense G_δ subset of $A \cap Y$ for every closed separable subspace Y of X. Let $i\colon Y \to X$ be the inclusion map and $i^*\colon X^* \to Y^*$ be the conjugate map of i. Then $\partial\phi|_Y = i^*\partial\phi i$. Therefore if $\partial\phi$ is a weak cusco, then so is $\partial\phi\,|_Y$. So by Proposition 3.7, Corollary 3.6 and Proposition 3.1 we have that $\partial\phi\,|_Y$ is Fréchet differentiable on a dense G_δ subset of $Y \cap A$. Hence ϕ is Fréchet differentiable on a dense G_δ subset of A. ∎

4. Denting point structure for closed bounded convex sets

Consider a closed convex set K with $0 \in \operatorname{int} K$ in a Banach space X. The gauge p of K is defined by $p(x) = \inf\{\lambda > 0\colon x \in \lambda K\}$ and is a continuous sublinear functional on X. We define the polar of K as the set $K^0 \equiv \{f \in X^*\colon f(x) \leqslant 1 \text{ for all } x \in K\}$. Then K^0 is weak* compact convex and $0 \in K^0$. We denote by K^{00} the polar of K^0 in X^{**}.

If K is bounded we say that a point $x \in \partial K$ is a *denting point* of K, if given $\varepsilon > 0$, there exists a slice $S(K, f, \delta)$ containing the point x and whose diameter is less than ε. We now introduce the notion of an α-denting point. An extreme point $x \in \partial K$ is called an *α denting point* of K if given $\varepsilon > 0$, there exists a slice $S(K, f, \delta)$ containing the point x such that $\alpha(S(K, f, \delta)) < \varepsilon$. Similarly for a closed bounded convex subset of X^* we can define weak* denting points and weak* α-denting points, where the slices are generated by weak* continuous linear functionals.

Generalising further, we say that an extreme point $x \in \partial K$ is a *β denting point* of K if given $\varepsilon > 0$, there exists a slice $S(K, f, \delta)$ containing the point x such that $\beta(S(K, f, \delta)) < \varepsilon$. Similarly for a closed bounded subset of X^* we can define weak* β denting points, where again the slices are generated by weak* continuous linear functionals.

We now proceed towards establishing a close relationship between weak* denting points and weak* β denting points for weak* compact convex subsets of the dual of a

https://doi.org/10.1017/S0004972700011680 Published online by Cambridge University Press

Banach space.

PROPOSITION 4.1. *Consider a weak* compact convex set K with $0 \in K$ in the dual of a Banach space X. Define the functional p on X by $p(x) = \sup\{f(x)\colon f \in K\}$; then p is a continuous sublinear functional and for every $x \in X$ and $\delta > 0$, $\overline{S(K, \widehat{x}, \delta^2)} \subseteq \partial p(B(x, \delta)) + \delta B(X^*)$.*

PROOF: Given $x \in X$ and $\delta > 0$, consider $f \in \overline{S(K, \widehat{x}, \delta^2)}$. Then $f(x) \geqslant \sup \widehat{x}(K) - \delta^2 = p(x) - \delta^2$ and since $f \in K$, $f(y) \leqslant p(y)$ for all $y \in X$, we have that $f(y - x) \leqslant p(y) - p(x) + \delta^2$ for all $y \in X$. By the Brøndsted-Rockafellar Theorem, [**6**, p.51], there exist an $x_0 \in X$ and $f_0 \in \partial p(x_0)$ such that $\|x - x_0\| < \delta$ and $\|f - f_0\| < \delta$. So $f_0 \in \partial p(B(x, \delta))$ and $\overline{S(K, \widehat{x}, \delta^2)} \subseteq \partial p(B(x, \delta)) + \delta B(X^*)$. ∎

LEMMA 4.2. *Let K be a non-empty weak* compact convex subset of the dual of a Banach space X. If K is the weak* closed convex hull of its weak* β denting points then every weak* slice of K contains a closed weak* slice of arbitrarily small diameter.*

PROOF: We may assume that $0 \in K$. Given $\varepsilon > 0$ consider $S(K, \widehat{x}, \delta_0)$ an arbitrary weak* slice of K where $x \in S(X)$ and $\delta_0 > 0$. Then the slice $S(K, \widehat{x}, \delta_0/2)$ must contain at least one weak* β denting point f (say). Choose a weak* slice $S(K, \widehat{y}, \delta_1)$ of K containing f such that $\beta(S(K, \widehat{y}, \delta_1)) < r = \min\{\delta_0/4, \varepsilon/4\}$ and consider the following weak* compact subset $H \equiv \operatorname{co}(K \backslash S(K, \widehat{x}, \delta_0/2) \cup K \backslash S(K, \widehat{y}, \delta_1))$. Clearly $f \notin H$, as f is an extreme point of K. Define the functional p on X by $p(x) \equiv \max\{f(x)\colon f \in K\}$ for each $x \in X$. Now, choose $z \in S(X)$ such that $\widehat{z}(f) > \max \widehat{z}(H)$. Therefore $\partial p(z) = \{g \in K\colon g(z) = p(z)\} \subseteq K \backslash H$, and so by the upper semi-continuity of ∂p, there exists an open neighbourhood W of z, such that $\partial p(W) \subseteq K \backslash H$ so clearly $\beta(\partial p(W)) < r$. But from Lemma 3.4 there exists a non-empty open subset V of W such that $\operatorname{diam} \partial p(V) < 2r$. Choose $x_0 \in V$ and $0 < \delta < r$ such that $B(x_0, \delta) \subseteq V$. Now

$$\begin{aligned}\overline{S(K, \widehat{x}, \delta^2)} &\subseteq [\partial p(B(x_0, \delta)) + \delta B(X^*)] \cap K \\ &\subseteq [S\Big(K, \widehat{x}, \frac{\delta_0}{2}\Big) + \delta B(X^*)] \cap K \subseteq S(K, \widehat{x}, \delta_0)\end{aligned}$$

and $\operatorname{diam} S(K, \widehat{x}_0, \epsilon^2) < 2r + 2\delta < \varepsilon$. ∎

THEOREM 4.3. *Let K be a non-empty weak* compact convex subset of the dual of a Banach space X. Then K is the weak* closed convex hull of its weak* β denting points if and only if K is the weak* closed convex hull of its weak* denting points.*

PROOF: Clearly if K is the weak* closed convex hull of its weak* denting points then it is the weak* closed convex hull of its weak* β denting points.

Conversely, suppose that $\overline{co}^{w^*} D \neq K$, (where D is the set of all weak* denting points of K) then there exists an $x_0 \in S(X)$ and a $\delta_0 > 0$ such that $S(K, \widehat{x}_0, \delta_0) \cap \overline{co}^{w^*} D = \emptyset$. But from Lemma 4.2 there exists a slice $\overline{S(K, \widehat{x}_1, \delta_1)} \subseteq S(K, \widehat{x}_0, \delta_0)$ such that $\operatorname{diam} S(K, \widehat{x}, \delta_1) < 1/2$. We can now reapply Lemma 4.2 to the slice $S(K, \widehat{x}_1, \delta_1)$ and get another slice $S(K, \widehat{x}, \delta_2) \subseteq S(K, \widehat{x}_1, \delta_1) \subseteq S(K, \widehat{x}_0, \delta_0)$ with $\operatorname{diam} S(K, \widehat{x}_2, \delta_2) < 1/4$. We can continue in this way, and at the nth iteration we will have a slice $\overline{S(K, \widehat{x}_n, \delta_n)} \subseteq S(K, \widehat{x}_{n-1}, \delta_{n-1}) \subseteq \ldots \subseteq S(K, \widehat{x}_1, \delta) \subseteq S(K, \widehat{x}_0, \delta_0)$ with $\operatorname{diam} S(K, \widehat{x}_n, \delta_n) < 1/2^n$. Then $\bigcap_{n=1}^{\infty} S(K, \widehat{x}_n, \delta_n) = \bigcap_{n=1}^{\infty} \overline{S(K, \widehat{x}_n, \delta_n)} = \{f\}$, for some $f \in K$. Clearly f is a weak* denting point, but $f \notin \overline{co}^{w^*} D$. Therefore we have reached a contradiction, so in fact we must have had $K = \overline{co}^{w^*} D$. ∎

Theorem 4.4. *For a closed bounded convex set K of a Banach space X, K is the closed convex hull of its denting points if and only if K is the closed convex hull of its α denting points.*

Proof: Clearly if K is the closed convex hull of its denting points then it is the closed convex hull of its α denting points.

Conversely, denoting by D the set of all denting points of K, suppose that $\overline{co} D \neq K$. Then there exists a slice $S(K, f, \delta)$ of K such that $\overline{co} D \cap S(K, f, \delta) = \emptyset$. It follows from [**2**, Lemma 3.1(ii)] that if x is an α denting point of K then $\widehat{x}$ is a weak* α denting point K^{00}. However, $\widehat{K}$ is weak* dense in K^{00}, so K^{00} is the weak* closed convex hull of the natural embedding of the α denting points of K from which it follows that K^{00} is the weak* closed convex hull of its weak* β denting points. Now Theorem 4.3 implies that K^{00} is the weak* closed convex hull of its weak* denting point. But every weak* denting point of K^{00} is a member of $\widehat{K}$. So $K^{00} = \overline{co}^{w^*} \widehat{D} \subseteq K^{00} \backslash S\left(K^{00}, \widehat{f}, \delta\right)$ which is a contradiction. Therefore K must be the closed convex hull of its denting points. ∎

5. β UPPER SEMI-CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF CONVEX FUNCTIONS

A Banach space X is called an *Asplund space* if every continuous convex function on an open convex domain in X is Fréchet differentiable on a dense G_δ subset of its domain. In Theorem 3.5 we showed that minimal weak* uscos (or weak* cuscos) from a Baire space into subsets of the dual of a Banach space which are β upper semi-continuous on a dense subset of their domain are single-valued and norm upper semi-continuous on a dense G_δ subset of their domain. In Corollary 3.6 we showed that this had differentiability implications for convex functions on open convex subsets of a Banach space whose subdifferential mapping is β upper semi-continuous on a dense subset of their domain. This suggests that we explore further β upper semi-continuity

https://doi.org/10.1017/S0004972700011680 Published online by Cambridge University Press

and related properties in determining conditions under which a Banach space is an Asplund space or has similar differentiability properties.

Given a Banach space X, for each $x \in S(X)$ we denote by $D(x)$ the set $\{f \in S(X^*) : f(x) = 1\}$. The set-valued mapping $x \to D(x)$ from $S(X)$ into subsets of $S(X^*)$ is called the *duality mapping* on $S(X)$. Part of the study of Asplund spaces is to determine norm properties which imply that a Banach space is an Asplund space. We show that if the duality mapping is β upper semi-continuous on $S(X)$ then X is an Asplund space.

Lemma 5.1. *Let X be a normed linear space, and Y a closed subspace of X. If the duality mapping D on $S(X)$ is β upper semi-continuous on $S(X)$ then D_Y, (the restriction of D to $S(Y)$), is β upper semi-continuous on $S(Y)$.*

Proof: Given $\varepsilon > 0$ and $x \in S(Y)$ there exists an open neighbourhood U of x in $S(X)$ such that $\beta(D(U)) < \varepsilon$. So there exists a sequence $\{f_n\}_{n=1}^{\infty}$ contained in X^* such that $D(U) \subseteq \bigcup_{n=1}^{\infty} \{f_n\} + \varepsilon B(X^*)$. Now, $D_Y = i^* D i$ where i is the inclusion map from Y into X and i^* is its conjugate. Therefore,

$$D_Y(U \cap Y) \subseteq i^*(D(U)) \subseteq i^* \left(\bigcup_{n=1}^{\infty} \{f_n\} + \varepsilon B(X^*) \right) \subseteq \bigcup_{n=1}^{\infty} i^*(f_n) + \varepsilon B(Y^*).$$

So $\beta(D_Y(U \cap Y)) \leqslant \varepsilon$, from which it follows that D_Y is β upper semi-continuous on $S(Y)$. ∎

Theorem 5.2. *Let X be a Banach space. Then X is Asplund if the duality mapping D is β upper semi-continuous on $S(X)$.*

Proof: It is well-known that a Banach space X is Asplund if and only if every closed separable subspace Y has a separable dual Y^*, (see [**6**, p.34]). We will proceed by showing that every closed separable subspace Y has a separable dual Y^*. But this is immediate from Lemma 5.1 and Proposition 3.1. Since Lemma 5.1 tells us that D_Y is β upper semi-continuous on $S(Y)$ and then Proposition 3.1 tells us that $D_Y(S(Y))$ is separable. But we know from the Bishop-Phelps theorem that $D_Y(S(Y))$ is dense in $S(Y^*)$. Therefore $S(Y^*)$ is separable, and hence Y^* is separable. ∎

We will now prove the following well-known result for comparison with Theorem 5.2.

Theorem 5.3. *Let X be a Banach space. Then X is Asplund if the duality mapping D is a weak cusco on $S(X)$.*

Proof: Again, as in Theorem 5.2, it is sufficient to show that every closed separable subspace Y of X has a separable dual Y^*. It follows from the proof of Corollary 3.8 that D_Y is a weak cusco on $S(Y)$. Therefore from Proposition 3.7, $D_Y(S(Y))$ is separable, which, as in Theorem 5.2 implies $S(Y^*)$ is separable, which in turn implies that Y^* is separable. $\square$

We notice that for a separable Banach space X, if the duality mapping D is a weak cusco on $S(X)$ then it is β upper semi-continuous on $S(X)$. However if the duality mapping D is β upper semi-continuous on $S(X)$ then it is not necessarily a weak cusco. In fact, it may not even be weak upper semi-continuous . For example let X be the James space, [**1**, p.92]; then X^{**} is separable but X is not reflexive. The duality mapping D on $S(X^*)$ is β upper semi-continuous but not weak upper semi-continuous because if D were weak upper semi-continuous then X would be reflexive.

The classical characterisation theorem for Asplund spaces was given by Namioka and Phelps, [**5**, p.737]. We present an extended characterisation using Theorem 3.5.

Theorem 5.4. *For a Banach space X the following are equivalent:*

(i) *every continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A;*

(ii) *every non-empty bounded set in X^* has weak* slices of arbitrarily small diameter;*

(iii) *every non-empty bounded set in X^* has weak* slices whose index of non-separability is arbitrarily small.*

Proof: In view of the classical characterisation and because it is obvious that (ii) $\Rightarrow$ (iii), it will be sufficient to prove (iii) $\Rightarrow$ (i). Consider a continuous convex function ϕ on an open convex subset A in X and given $\varepsilon > 0$, consider $0_\varepsilon \equiv \bigcup\{\text{open sets } V \text{ in } A\colon \beta(\partial\phi(V)) < \varepsilon\}$. Now 0_ε is open; we show that 0_ε is dense in A. Consider a non-empty open set W in A. Since the subdifferential mapping $x \to \partial\phi(x)$ is locally bounded, [**6**, p.29], there exists a non-empty open subset U of W for which $\partial\phi(U)$ is bounded. Now by the hypothesis in (iii) there exists a $z \in X$ and a $\delta > 0$ such that $\beta(S(\partial\phi(U), z, \delta)) < \varepsilon$. Now $\partial\phi(U) \not\subseteq \{f \in X^*\colon f(z) \leqslant \sup \widehat{z}(\partial\Phi(U)) - \delta\}$ so from Proposition 3.2 there exists a non-empty open subset V of U and so of W such that $\partial\phi(V) \subseteq S(\partial\phi(U), z, \delta)$. Then $\beta(\partial\phi(V)) < \varepsilon$. So $\emptyset \neq V \subseteq W \cap O_\varepsilon$, from which we conclude that O_ε is dense and the subdifferential mapping $x \to \partial\phi(x)$ is β upper semi-continuous on the dense G_δ subset $\bigcap_1^\infty 0_{1/n}$ of A. It follows again from Corollary 3.6 that ϕ is Fréchet differentiable on a dense G_δ subset of A. $\square$

It has recently been shown, [**3**, Theorem 3.5], that there is a large class of Banach spaces, including the separable spaces, where every continuous convex function on an open convex subset of the dual is Fréchet differentiable on a dense G_δ subset of its domain provided that the set of points were the function has a weak* continuous subgradient is residual in its domain. Such spaces are those which can be equivalently renormed to have every point of the unit sphere a denting point of the closed unit ball. We generalise this result still further using the index of non-separability and Theorem 3.5.

We will need the following property of minimal weak* cuscos, [**3**, Lemma 3.4(iii)].

LEMMA 5.5. *Given a minimal weak* cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X, there exists a dense G_δ subset D of A such that at each $t \in D$ the real-valued mapping defined on A by*

$$\rho(t) = \inf\{\|f\| : f \in \Phi(t)\}$$

is continuous and $\Phi(t)$ lies in the face of a sphere of X^* of radius $\rho(t)$.

THEOREM 5.6. *Consider a Banach space X which can be equivalently renormed to have every point of $S(X)$ a β denting point of $B(X)$. Then every minimal weak* cusco Φ from a Baire space A into subsets of X^{**} for which the set $G \equiv \{t \in A\colon \Phi(t) \cap \widehat{X} \neq \emptyset\}$ is residual in A, is single-valued and norm upper semi-continuous on a dense G_δ subset of A. In particular, every continuous convex function ϕ on an open convex set A in X^* for which the set $G \equiv \{f \in A\colon \partial\phi(f) \cap \widehat{X} \neq \emptyset\}$ is residual in A, is Fréchet differentiable on a dense G_δ subset of A.*

PROOF: Consider X so renormed. Given $\varepsilon > 0$, consider $O_\varepsilon \equiv \bigcup\{$ open sets U in $A\colon \operatorname{diam}\Phi(U) \leqslant 2\varepsilon\}$. Now O_ε is open; we will show O_ε is dense in A. From Lemma 5.5 there exists a dense G_δ subset G_1 of A such that at every point $t \in G_1$ the mapping ρ where $\rho(t) = \inf\{\|f\| : f \in \Phi\}$ is continuous and $\Phi(t)$ lies in the face of a sphere of X^{**} of radius $\rho(t)$. Now $G \cap G_1$ is residual in A. Consider a non-empty open subset W of A and $t_0 \in G \cap G_1 \cap W$. There exists some $\widehat{x}_0 \in \Phi(t_0) \cap \widehat{X}$. But x_0 is a β denting point of $\rho(t_0)B(X)$. So there exists a $g \in S(X^*)$ and a $\delta > 0$ such that $x_0 \in S(\rho(t_0)B(X), g, \delta)$ and $\beta(S(\rho(t_0)B(X), g, \delta)) < \varepsilon/2$. We can choose $1 < \lambda < 2$ such that $x_0 \in S(\lambda\rho(t_0)B(X), g, \lambda\delta) = \lambda S(\rho(t_0)B(X), g, \delta)$ and then by property (2) of Proposition 2.2, $\beta(\lambda S(\rho(t_0)B(X), g, \delta)) < \varepsilon$. Since ρ is continuous at t_0 there exists a non-empty open subset V' of W containing t_0 such that $\Phi(t) \cap \lambda\rho(t_0)B(X^{**}) \neq \emptyset$ for all $t \in V'$. So by Proposition 3.2, $\Phi(V') \subseteq \lambda\rho(t_0)B(X^{**})$. Since $\Phi(V') \not\subseteq \{F \in X^{**} : F(g) \leqslant \lambda\rho(t_0) - \lambda\delta\}$ then again by Proposition 3.2, there exists a non-empty open subset V of V' and so of W such that $\Phi(V) \subseteq S(\lambda\rho(t_0)B(X^{**}), \widehat{g}, \lambda\delta)$. Now, since $\beta(\lambda S(\rho(t_0)B(X), g, \delta)) < \varepsilon$, there exists a sequence $\{x_n\}_{n=1}^\infty$ contained in X such that $\lambda S(\rho(t_0)B(X), g, \delta) \subseteq \left(\bigcup_{n=1}^\infty \{x_n\} + \varepsilon B(X)\right)$.

We now prove that there exists a non-empty open subset U of V such that $\operatorname{diam}\Phi(U) \leqslant 2\varepsilon$. We start by noting that V with the induced topology is a Baire space. Now, if $\Phi(U') \subseteq (\widehat{x}_1 + \varepsilon B(X^{**}))$ for some non-empty open set U' contained in V, write

$U = U'$, but if not, we have by Lemma 3.3 that there exists a dense open set $O_1 \subseteq V$ such that $\Phi(O_1) \cap (\widehat{x}_1 + \varepsilon B(X^{**})) = \emptyset$. Now if $\Phi(U') \subseteq (\widehat{x}_2 + \varepsilon B(X^{**}))$ for some non-empty open set U' contained in V write $U = U'$, but if not, we have by Lemma 3.3 that there exists a dense open set $O_2 \subseteq V$ such that $\Phi(O_2) \cap (\widehat{x}_2 + \varepsilon B(X^{**})) = \emptyset$. Continue in this way. We will have defined U at some stage, because if not, we have a dense G_δ subset $O_\infty \subseteq V$ where $O_\infty = \bigcap_{n=1}^{\infty} O_n$ and $\Phi(O_\infty) \cap (\bigcup_{n=1}^{\infty} \widehat{x}_n + \varepsilon B(X^{**}))$ is empty. However for any $t \in O_\infty \cap (G \cap V)$ which is dense in V, $\Phi(t) \cap \left(\bigcup_{n=1}^{\infty} \widehat{x}_n + \varepsilon B\left(\widehat{X}\right)\right) \neq \emptyset$. So V contains a non-empty open set U with $\operatorname{diam} \Phi(U) \leqslant 2\varepsilon$. Therefore $\emptyset \neq U \subseteq O_\varepsilon \cap W$, and so O_ε is dense. We conclude that Φ is single-valued and norm upper semi-continuous on the dense G_δ subset $\bigcap_{n=1}^{\infty} O_{1/n}$ of A. □

The question now arises as to whether the class of Banach spaces we have been considering in this theorem is larger than the class in the original theorem we have generalised. It is an open question whether spaces of our class can be equivalently renormed to have the more restricted condition.

References

[1] M.M. Day, *Normed linear spaces*, Third edition (Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[2] J.R. Giles and W.B. Moors, 'A continuity property related to Kuratowski's index of non-compactness, its relevance to the drop property and its implications for differentiability theory', (preprint).

[3] P.S. Kenderov and J.R. Giles, 'On the structure of Banach spaces with Mazur's intersection property', *Math. Annal.* (to appear).

[4] C. Kuratowski, 'Sur les espaces complets', *Fund. Math.* **15** (1930), 301–309.

[5] I. Namioka and R.R. Phelps, 'Banach spaces which are Asplund spaces', *Duke Math. J.* **42** (1975), 735–749.

[6] R.R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics, 1364 (Springer-Verlag, Berlin, Heidelberg, New York, 1989).

[7] R.R. Phelps, 'Weak* support points of convex sets in E^*', *Israel J. Math.* **2** (1964), 177–182.

Department of Mathematics
The University of Newcastle
Newcastle NSW 2308
Australia