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A REMARK ON A WEIGHTED LANDAU INEQUALITY OF KWONG AND ZETTL

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ABSTRACT. In this note we extend a theorem of Kwong and Zettl concerning the inequality

$$\int_0^\infty t^\beta |u'|^p \leq K\Big(\int_0^\infty t^\gamma |u|^p\Big)^{1/2}\Big(\int_0^\infty t^\alpha |u''|^p\Big)^{1/2}.$$

The Kwong-Zettl result holds for $1 \leq p < \infty$ and real numbers α, β, γ such that the conditions (i) $\beta = (\alpha+\gamma)/2$, (ii) $\beta > -1$, and (iii) $\gamma > -1-p$ hold. Here the inequality is proved with β satisfying (i) for all α, γ except $p-1$, $-1-p$. In this case the inequality is false; however u is shown to satisfy the inequality

$$\int_0^\infty t^{-1}|u'|^p \leq K_1\Big\{\Big(\int_0^\infty t^{-1-p}|u|^p\Big)^{1/2}\Big(\int_0^\infty t^{p-1}|u''|^p\Big)^{1/2} + \int_0^\infty t^{-1-p}|u|^p\Big\}.$$

1. **Notation.** Let $I = (a,b)$, $-\infty \leq a < b \leq \infty$, and "$AC_{\text{loc}}(I)$" denote the class of locally absolutely continuous functions on I. If α, γ are real numbers define

$$\begin{aligned}
\mathcal{D}_{\alpha\gamma}(I) &:= \Big\{u : u' \in AC_{\text{loc}}(I) : \int_I t^\gamma |u|^p, \int_I t^\alpha |u''|^p < \infty\Big\},\\
\mathcal{D}_L^0(I) &:= \{u \in \mathcal{D}_{\alpha\gamma}(I) : \lim_{t\to a^+} u(t) = 0\},\\
\mathcal{D}_L^1(I) &:= \{u \in \mathcal{D}_{\alpha\gamma}(I) : \lim_{t\to a^+} u'(t) = 0\},\\
\mathcal{D}_R^0(I) &:= \{u \in \mathcal{D}_{\alpha\gamma}(I) : \lim_{t\to b^-} u(t) = 0\},\\
\mathcal{D}_R^1(I) &:= \{u \in \mathcal{D}_{\alpha\gamma}(I) : \lim_{t\to b^-} u'(t) = 0\},\\
\mathcal{D}_L(I) &:= \mathcal{D}_L^0(I) \cap \mathcal{D}_L^1(I),\\
\mathcal{D}_R(I) &:= \mathcal{D}_R^0(I) \cap \mathcal{D}_R^1(I).
\end{aligned}$$

Additionally let K denote a constant of interest whose value may change from line to line; if required different constants will be denoted by K_1, K_2, *etc.*

2. **A weighted multiplicative inequality.** In [5, Theorem 9] Kwong and Zettl proved the following result:

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THEOREM 1. *Suppose* $1 \le p < \infty$, β, γ *and* α *are real numbers such that*

$$\beta = \frac{\alpha+\gamma}{2}. \tag{1}$$

Then there is a constant K independent of u such that the inequality

$$\int_0^\infty t^\beta |u'|^p \le K\Big(\int_0^\infty t^\gamma |u|^p\Big)^{1/2}\Big(\int_0^\infty t^\alpha |u''|^p\Big)^{1/2} \tag{2}$$

holds for $u \in \mathcal{D}_{\alpha\gamma}\big((0,\infty)\big)$ *if* $\beta > -1$ *and* $\gamma > -1-p$.

We are going to extend this theorem by proving:

THEOREM 2. *Let* $u \in \mathcal{D}_{\alpha\gamma}\big((0,\infty)\big)$, $1 \le p < \infty$, *then the inequality (2) holds if and only if the following conditions are satisfied:*

(i) $\{\alpha,\beta,\gamma\} \ne \{p-1,-1,-1-p\}$.
(ii) β *satisfies (1).*
(iii) $\lim_{t\to 0^+} u'(t) = 0$ *when* $\beta \le -1$ *and* $\beta > \alpha - p$.
(iv) $\lim_{t\to\infty} u'(t) = 0$ *when* $\beta \ge -1$ *and* $\beta < \alpha - p$.

Also in the exceptional case $\{\alpha,\gamma\} = \{p-1,-1-p\}$ *the inequality*

$$\int_0^\infty t^{-1}|u'|^p \le K_1\Big\{\Big(\int_0^\infty t^{-1-p}|u|^p\Big)^{1/2}\Big(\int_0^\infty t^{p-1}|u''|\Big)^{1/2} + \int_0^\infty t^{-1-p}|u|^p\Big\} \tag{3}$$

is valid.

PROOF. That (2) implies (1) is a statement of "dimensional balance" and follows if we introduce the change of variables $t = \lambda s$ in (2). Next suppose that $\beta \ne \alpha - p$. If $\beta > \alpha - p$ we get that $(\alpha-\gamma)/2p < 1$, and if $\beta < \alpha - p$ then $(\alpha-\gamma)/2p > 1$; thus in either case $(\alpha-\gamma)/2p \ne 1$.

CASE (i). Let $\beta \le -1$ and $\beta > \alpha - p$. Assume that (2) holds. Now

$$u'(t) = u'(s) - \int_t^s u''.$$

Since $\alpha < p-1$, $\alpha p'/p < 1$, so that $t^{-\alpha p'/p}$ is integrable if $p > 1$ on right neighborhoods of 0; also $t^{|\alpha|}$ is bounded near 0 if $p = 1$. These facts and Hölder's inequality imply that $\lim_{t\to 0^+}\int_t^s u''$ is finite; consequently $u'(0^+)$ exists. Since $t^\beta|u'|^p$ is integrable (by (2)) while t^β is not integrable on right neighborhoods of 0, $u'(0^+) = 0$. On the other hand, suppose $u'(0^+) = 0$. Because $\beta > \alpha - p$ a form of Hardy's inequality for $\mathcal{D}_L^1\big((0,1]\big)$ (see [6, Example 6.8(i)]) gives

$$\int_0^1 t^\beta|u'|^p \le K\int_0^1 t^{\beta+p}|u''|^p < K\int_0^1 t^\alpha|u''|^p$$

when $\beta < -1$, and

$$\int_0^1 t^\beta|u'|^p \le K\int_0^1 t^{\alpha-p}|u'|^p \le K\int_0^1 t^\alpha|u''|^p$$

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when $\beta = -1$. The sum inequality on $\mathcal{D}^1_{\alpha\gamma}\big((0,1)\big)$

$$\int_0^1 t^\beta |u'|^p \le K\Big\{\int_0^1 t^\gamma |u|^p + \int_0^1 t^\alpha |u''|^p\Big\} \tag{4}$$

follows trivially. By existing theory (take $I = [1,\infty)$, $\delta := (\alpha-\gamma)/2p < 1$, and $\epsilon = 1$ in [1, Example 1]) we obtain the sum inequality

$$\int_1^\infty t^\beta |u'|^p \le K\Big\{\int_1^\infty t^\gamma |u|^p + \int_1^\infty t^\alpha |u''|^p\Big\} \tag{5}$$

on $\mathcal{D}_{\gamma\alpha}\big([1,\infty)\big)$. Addition of (4) and (5) gives the sum inequality on the entire interval. Set $t = \lambda s$. Then $u_\lambda := u(\lambda s)$ is in $\mathcal{D}_{\alpha\gamma}\big((0,\infty)\big)$ so that

$$\int_0^\infty s^\beta |u_\lambda'(s)|^p\, ds \le K\Big\{\int_0^\infty s^\gamma |u_\lambda(s)|^p\, ds + \int_0^\infty s^\alpha |u_\lambda''(s)|^p\, ds\Big\},$$

which is equivalent to the inequality

$$\int_0^\infty t^\beta |u'(t)|^p\, dt \le K\Big\{\lambda^\phi \int_0^\infty t^\gamma |u(t)|^p\, dt + \lambda^{-\phi}\int_0^\infty t^\alpha |u''(t)|^p\, dt\Big\} \tag{6}$$

where $\phi = (\alpha-\gamma)/2 - p$. (2) follows by minimizing the right side of (6) with respect to λ (the minimization is possible since $(\alpha-\gamma)/2p \ne 1$).

The other possibilities concerning β follow a similar logic.

CASE (ii). Assume $\beta > \max\{-1, \alpha - p\}$. Then Hardy's inequality for $\mathcal{D}^1_R\big((0,1]\big)$ (see [6, Example 6.8(ii)]), Minkowski's inequality, and the integrability of t^β on $(0,1]$ gives

$$\int_0^1 t^\beta |u'|^p \le K\Big\{\int_0^1 t^\alpha |u''|^p + |u'(1)|^p\Big\}. \tag{7}$$

Since [1, Lemma 2.1]

$$|u'(1)| \le K\Big\{\int_1^2 |u| + \int_1^2 |u''|\Big\}, \tag{8}$$

a standard Hölder's inequality argument applied to (8) in conjunction with (7) yields that

$$\int_0^1 t^\beta |u'|^p \le K\Big\{\int_0^\infty t^\gamma |u|^p + \int_0^\infty t^\alpha |u''|^p\Big\}. \tag{9}$$

Since (5) remains valid, addition of (5) and (6) gives the sum inequality on $(0,\infty)$ and the the same scaling argument as in the previous case may be applied.

CASE (iii). If $\beta < \alpha - p$ and $\beta < -1$, Hardy's inequality for $\mathcal{D}^1_L\big([1,\infty)\big)$ (see [6, Example 6.9(i)]), Minkowski's inequality, Lemma 2.1 of [1], *etc.*, give as in Case (ii) the sum inequality (5). On the other hand since $(\alpha-\gamma)/2p > 1$, existing theory (see [1, Example 2]) gives

$$\int_0^1 t^\beta |u'|^p \le K\Big\{\int_0^1 t^\gamma |u|^p + \int_0^1 t^\alpha |u''|^p\Big\}; \tag{10}$$

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we then add and scale as before.

CASE (iv). If $\beta < \alpha - p$ and $\beta \geq -1$, we can show that $\lim_{t\to\infty} u'(t) = 0$ by an argument similar to Case (i). Hardy's inequality for $\mathcal{D}^1_R\big([1,\infty)\big)$ (see [6, Example 6.9(ii)]) then leads trivially to (5). Adding this to (10) (the argument of Case (iii) continues to apply) and finishing the argument as before completes the proof.

Now suppose that $\beta = \alpha - p$. Let $N = t^\beta$, $W = t^\gamma$, $P = t^\alpha$. Let $f(t) = t^\delta$ where $\delta := (\alpha - \gamma)/2p = 1$. Then combining Examples 1 and 2 of [1] gives the sum inequality

$$\int_0^\infty N|u'|^p \leq K(\epsilon)\Big\{\epsilon^{-p}\int_0^\infty W|u|^p + \epsilon^p \int_0^\infty P|u''|^p\Big\}. \tag{11}$$

In particular this implies that $t^\beta|u'|^p$ is integrable on $(0,\infty)$. (This fact is needed in the argument below.)

To obtain (2) we modify an argument previously given in the proof of [2, Theorem 2.1]: Define a bi-infinite partition $\{t_i\}_{-\infty}^\infty$ by letting $t_0 = 1$ and $t_i = 2^i$. Let ϕ be a C_0^∞ function with support on $[-3/4, 1]$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $[-1/2, 0]$. For $m \in \mathbb{Z}$ and $u \in \mathcal{D}_{\alpha\gamma}\big((0,\infty)\big)$ set

$$y_m(t) = u(t)\phi\big((t - t_m)/t_m\big) \tag{12}$$

where t_m is a point of the partition defined above. It follows that y_m has support on $[t_{m-2}, t_{m+1}]$ and $y_m = u$ on $[t_{m-1}, t_m]$. It is not difficult to show applying Leibniz's rule of differentiation that there is a constant C independent of u and m such that

$$|y''_m(t)| \leq C\sum_{i=0}^{2} |u^{(i)}|/t_m^{2-i}, \text{ a.e.} \tag{13}$$

Next we recall that if $\alpha = \beta = \gamma = 0$, then (2) is a special case of a far more general and well known Gabushin inequality (*cf.* [3]). (Also note that if $p > 1$ the unweighted inequality follows from Case (ii) above.) Substituting (12) into this inequality and using (13) gives

$$\begin{aligned}\Big(\int_{t_{m-1}}^{t_m} |u'|^p\Big) &\leq \Big(\int_{t_{m-2}}^{t_{m+1}} |y'_m|^p\Big)\\ &\leq K\Big(\int_{t_{m-2}}^{t_{m+1}} |y_m|^p\Big)^{1/2}\Big(\int_{t_{m-2}}^{t_{m+1}} |y''_m|^p\Big)^{1/2}\\ &\leq KC^{1/2}\Big(\int_{t_{m-2}}^{t_{m+1}} |u|^p\Big)^{1/2}\Big(\int_{t_{m-2}}^{t_{m+1}} \sum_{i=0}^{2} |u^{(i)}|^p/(t_m^{2-i})^p\Big)^{1/2}.\end{aligned} \tag{14}$$

We multiply the last line of (14) by t_m^β, noting both that β satisfies (1) and that if $t \in [t_{m-2}, t_{m+1}]$, then $1/4 \leq t/t_m \leq 2$ because of the nature of the partition. This gives

$$\int_{t_{m-1}}^{t_m} t^\beta|u'|^p \leq K_1\Big(\int_{t_{m-2}}^{t_{m+1}} t^\gamma|u|^p\Big)^{1/2}\Big(\int_{t_{m-2}}^{t_{m+1}} \sum_{i=0}^{2} t^{\alpha-(2-i)p}|u^{(i)}|^p\Big)^{1/2} \tag{15}$$

for a constant K_1 independent of u. Summing (15) over m and using the discrete sum form of the Cauchy-Schwartz inequality yields that

$$\int_0^\infty t^\beta |u'|^p \le K_1 \Big(\sum_{m=-\infty}^{\infty} \int_{t_{m-2}}^{t_{m+1}} t^\gamma |u|^p \Big)^{1/2} \Big(\sum_{m=-\infty}^{\infty} \int_{t_{m-2}}^{t_{m+1}} \Big(\sum_{i=0}^{2} t^{\alpha-(2-i)p} |u^{(i)}|^p \Big) \Big)^{1/2}. \tag{16}$$

Because each t belongs in at most three intervals $[t_{m-2}, t_{m+1}]$ and by Minkowski's inequality applied to the last integral in (16), it follows that

$$\int_0^\infty t^\beta |u'|^p \le K_2 \Big(\int_0^\infty t^\gamma |u|^p \Big)^{1/2} \Big[\sum_{i=0}^{2} \Big(\int_0^\infty t^{\alpha-(2-i)p} |u^{(i)}|^p \Big) \Big]^{1/2}. \tag{17}$$

Assume now that $\beta < -1$ so that $\gamma < -1 - p$. Because $\alpha < p - 1$ and $\beta < -1$ an argument given in Case (i) using the integrability of $t^\beta |u'|$ (established in equation (11) above) shows that $\lim_{t\to 0^+} u'(t) = 0$. Similarly the fact that

$$\frac{-\beta p'}{p} > \frac{1}{p-1} > 0$$

together with Hölder's inequality applied to the integral in the identity

$$u(t) = u(s) - \int_t^s u'$$

demonstrates that $\lim_{t\to 0^+} u(t)$ exists. Since $\gamma < -1$ the limit is 0. This shows that $\mathcal{D}_{\alpha\gamma}\big((0,\infty)\big) = \mathcal{D}_L\big((0,\infty)\big)$. Since $\alpha - p < -1$, the Hardy inequalities (see [6, Example 6.7]

$$\int_0^\infty t^{\alpha-2p} |u|^p \le K_3 \int_0^\infty t^{\alpha-p} |u'|^p, \tag{18}$$

$$\int_0^\infty t^{\alpha-p} |u'|^p \le K_4 \int_0^\infty t^{\alpha} |u''|^p \tag{19}$$

hold on $\mathcal{D}_{\alpha\gamma}\big((0,\infty)\big)$. Iterating (18) and (19) yields the second order Hardy inequality

$$\int_0^\infty t^{\alpha-2p} |u|^p \le K_5 \int_0^\infty t^{\alpha} |u''|^p. \tag{20}$$

Substitution of (19) and (20) into (17) yields (2). The case $\beta > -1, \gamma > -1 - p$ is covered by Theorem 1. Summarizing, (2) holds for all choices of α, β, and γ satisfying (1) except possibly for

$$\begin{aligned} \alpha &= p - 1, \\ \beta &= -1, \\ \gamma &= -1 - p, \end{aligned} \tag{21}$$

which was to be proved.

We next show by a counterexample that (2) cannot hold in the exceptional case (21), Let

$$u_\delta(t) := \begin{cases} u_{1,\delta}(t) = t^{1+\delta} & \text{for } t \in [0,1] \\ u_{2,\delta} = \big((1+\delta)t^{1-\delta} - 2\delta\big)/(1-\delta) & \text{for } t \in (1,\infty) \end{cases}$$

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where $\delta > 0$ is a parameter. Since u_δ and u'_δ are continuous at 1, $u_\delta \in \mathcal{D}_{\alpha\gamma}\big((0,\infty)\big)$. To prove that (2) cannot hold for this family of functions it is sufficient to show that if

$$Q(u_\delta) := \frac{(\int_0^\infty t^{-1}|u'_\delta|^p)^2}{(\int_0^\infty t^{-1-p}|u_\delta|^p)(\int_0^\infty t^{p-1}|u''_\delta|)},$$

then

$$\lim_{\delta\to 0} Q(u_\delta) = \infty. \tag{22}$$

A calculation yields that

$$\int_0^\infty t^{-1}|u'_\delta|^p = \frac{2(1+\delta)^p}{p\delta} \tag{23}$$

$$\int_0^\infty t^{p-1}|u''_\delta|^p = \frac{2\delta^p(\delta+1)^p}{p\delta}. \tag{24}$$

Moreover

$$\int_1^\infty t^{-1-p}|u_{2,\delta}|^p < \int_1^\infty t^{-1-p}\Big(\frac{1+\delta}{1-\delta}t^{1-\delta}\Big)^p = \frac{(1+\delta)^p}{p\delta(1-\delta)^p}, \tag{25}$$

so that

$$\int_0^\infty t^{-1-p}|u_\delta|^p = \frac{1}{p\delta} + \int_1^\infty t^{-1-p}|u_{2,\delta}|^p < \frac{1}{p\delta} + \frac{(1+\delta)^p}{p\delta(1-\delta)^p}. \tag{26}$$

Combining (25) and (26) and substituting them together with the estimates (23) and (24) into (22) gives

$$\begin{aligned}\lim_{\delta\to 0} Q(u_\delta) &\geq \lim_{\delta\to 0}\frac{2(1+\delta)^p}{\delta^p\Big(1+\frac{(1+\delta)^p}{(1-\delta)^p}\Big)}\\ &= \infty.\end{aligned}$$

It remains to prove (3) in the exceptional case: Let $f(t) = t$, $N = t^{-1}$, $W = t^{-1-p}$, $P = t^{p-1}$, and $J_{t,\epsilon} = [t, t(1+\epsilon)]$ in condition (C_3) of [1]. Then a calculation (see [1, (2.13)] shows that

$$\int_{J_{t,\epsilon}} N|u'|^p \leq K\Big\{\epsilon^{-p}S_2\int_{J_{t,\epsilon}} W|u|^p + \epsilon^p S_1\int_{J_{t,\epsilon}} P|u''|^p\Big\} \tag{27}$$

where

$$S_1 := t^p\Big((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{-1}\,ds\Big)\Big((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{-1}\,ds\Big)^{p-1} \leq 1 \tag{28}$$

$$S_2 := t^{-p}\Big((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{-1}\,ds\Big)\Big((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{(p+1)/(p-1)}\,ds\Big)^{p-1} \leq (1+\epsilon)^{p+1}. \tag{29}$$

Let $\mathcal{J}_\epsilon$ denote the collection of all $J_{t,\epsilon}$, $t \in (0,\infty)$. Let n be a positive integer. Since every $s \in (0,n)$ is the center of some $J_{t,\epsilon} \in \mathcal{J}_\epsilon$ (take $t = 2s/(2+\epsilon)$) we may appeal to the Besicovitch covering theorem (*cf.* [4, Theorem 1.1, p. 2]) to extract finitely many families $\Gamma_1, \ldots, \Gamma_l$ of disjoint intervals in $\mathcal{J}_\epsilon$ where l is independent of n to cover $(0, n)$. From (27), (28), and (29) it follows that

$$\begin{aligned}\int_{\tilde{\Gamma}_i} t^{-1}|u'|^p &\leq K\Big\{\epsilon^{-p}(1+\epsilon)^{p+1}\int_{\tilde{\Gamma}_i} t^{-1-p}|u|^p + \epsilon^p \int_{\tilde{\Gamma}_i} t^{p-1}|u''|^p\Big\}\\ &\leq K\Big\{\epsilon^{-p}(1+\epsilon)^{p+1}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p \int_0^\infty t^{p-1}|u''|^p\Big\}\end{aligned}$$

where $\tilde{\Gamma}_i := \cup\{J_{t,\epsilon} : J_{t,\epsilon} \in \Gamma_i\}$. Hence

$$\int_0^n t^{-1}|u'|^p \leq Kl\Big\{\epsilon^{-p}(1+\epsilon)^{p+1}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p \int_0^\infty t^{p-1}|u''|^p\Big\}.$$

Since n is arbitrary, we finally obtain the inequality

$$\int_0^\infty t^{-1}|u'|^p \leq Kl\Big\{\epsilon^{-p}(1+\epsilon)^{p+1}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p \int_0^\infty t^{p-1}|u''|^p\Big\}.$$

We substitute the elementary inequality

$$(1+\epsilon)^{p+1} \leq 2^p(1+\epsilon^{p+1})$$

into (30). This gives an inequality of the form

$$\int_0^\infty t^{-1}|u'|^p \leq 2^p Kl\{(\epsilon^{-p}+\epsilon)I_1 + \epsilon^p I_2\}$$

where

$$\begin{aligned}I_1 &:= \int_0^\infty t^{-1-p}|u|^p\\ I_2 &:= \int_0^\infty t^{p-1}|u''|^p.\end{aligned}$$

If $I_2 \geq I_1$, set $\epsilon = (I_1/I_2)^{1/2p} \leq 1$. Then

$$\begin{aligned}(\epsilon^{-p}+\epsilon)I_1 + \epsilon^p I_2 &\leq (\epsilon^{-p}+1)I_1 + \epsilon^p I_2\\ &\leq 2(I_1 I_2)^{1/2} + I_1.\end{aligned}$$

If $I_2 < I_1$, set $\epsilon = 1$. This gives the upper bound

$$\begin{aligned}(\epsilon^{-p}+\epsilon)I_1 + \epsilon^p I_2 &\leq 2I_1 + I_2\\ &\leq 2I_1 + (I_1 I_2)^{1/2}.\end{aligned}$$

In either case (3) follows with $K_1 = 2^{p+1}Kl$. The proof is complete.

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