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FUNCTION CLASSES RELATED TO RUSCHEWEYH DERIVATIVES

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Abstract

We investigate a family consisting of functions whose convolution with $z/(1-z)^{n+1}$ is starlike of order α, $0 \le \alpha < 1$. We determine extreme points, inclusion relations, and show how this family acts under various linear operators.

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1. Introduction

Let A denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ that are analytic in the unit disk $\Delta = \{z : |z| < 1\}$. Let $S^*(\alpha)$ and $K(\alpha)$ denote the usual classes consisting of functions starlike of order α and convex of order α, respectively. In [4], Ruscheweyh introduced subclasses

$$K_n = \left\{ f \in A : \operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \tfrac{1}{2}, z \in \Delta \right\}$$

of $S^*(1/2)$, where

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \qquad n \in N_0 = \{0, 1, 2, \ldots\}, \tag{1}$$

and the operation $*$ stands for the Hadamard product of power series, that is, if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ then $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Here $K_0 = S^*(1/2)$, $K_1 = K(0)$ and $K_{n+1} \subset K_n$, $n \in N_0$

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(see [4]). Recently, Ahuja [1, 2] has introduced the classes, denoted by $R_n(\alpha)$, of functions f in A which satisfy the condition $\operatorname{Re}\{z(D^n f(z))'/D^n f(z)\} > \alpha$ for some α $(0 \le \alpha < 1)$ and for all $z \in \Delta$. In particular,

$$f \in R_n(\alpha) \text{ if and only if } D^n f \in S^*(\alpha). \tag{2}$$

It is observed [1] that for each $n \ge 0$, $R_n(\alpha) \subset R_n(0)$, and for each $n \ge 1$, $R_n(\alpha) \subset K_n$. The class $R_n \equiv R_n(0)$ was studied by R. Singh and S. Singh [8]. In [2], it was seen that $R_{n+1}(\alpha) \subset R_n(\alpha)$ for each $n \in N_0$ and for all α. These inclusion relations establish that $R_n(\alpha) \subset S^*(\alpha)$ for each $n \ge 0$ and $R_n(\alpha) \subset K(\alpha)$ for each $n \ge 1$. In fact, for α fixed and $n = n(\alpha)$ sufficiently large, we can show that $R_n = R_n(0) \subset K(\alpha)$.

THEOREM 1. *For any α, $0 \le \alpha < 1$, $R_n \subset K(\alpha)$ for $n \ge n_0 = [32/(1-\alpha)]$.*

PROOF. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, a computation applied to (1) shows that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} a_k z^k. \tag{3}$$

If $f \in R_n$, then $D^n f \in S^*(0)$ and we must have $\binom{k+n-1}{n}|a_k| \le k$ or, equivalently,

$$|a_k| \le k\binom{k+n-1}{n}^{-1} \qquad \text{for every } k \ge 2. \tag{4}$$

It is known [6] that $f \in K(\alpha)$ if $\sum_{k=2}^{\infty} k(k-\alpha)|a_k| \le 1-\alpha$. In view of (4) it thus suffices to show that $\sum_{k=2}^{\infty} k^2|a_k| \le \sum_{k=2}^{\infty} k^3\binom{k+n-1}{n}^{-1} \le 1-\alpha$ for $n \ge n_0$. Since $\sum_{k=2}^{\infty}(1/k^2) < 1$, we need only show that $\sum_{k=2}^{\infty} k^3\binom{n+k-1}{n}^{-1} \le (1-\alpha)\sum_{k=2}^{\infty}(1/k^2)$, $n \ge n_0$, which is true if

$$c_k = k^5\binom{n+k-1}{n}^{-1} \le 1-\alpha \qquad (n \ge n_0,\ k \ge 2). \tag{5}$$

Now $\binom{n+k-1}{n}^{-1}$ is a decreasing function of n, so it suffices to prove (5) for $n = n_0$. Inequality (5) follows for $k = 2$ because $c_2 = 32/(n_0+1) \le 1-\alpha$. The proof will be completed by showing that c_k is a decreasing function of $k (\ge 2)$ for $n = n_0$. We have that $c_{k+1}/c_k = (1+1/k)^5(k/(n_0+k)) \le 1$ is equivalent to $g(k) = (n_0-5)k^4 - 10k^3 - 10k^2 - 5k - 1 \ge 0$. But $g(k) \ge 27k^4 - 10k^3 - 10k^2 - 5k - 1 \ge k^4 > 0$ and the proof is complete.

The extreme points of the closed convex hull of $S^*(\alpha)$ and $K(\alpha)$ were determined by Brickman, Hallenbeck, MacGregor, and Wilken in [3]. We denote the closed convex hull of a family F by $\overline{\text{cl}}\ F$ and make use of some results in [3] to determine the extreme points of $\overline{\text{cl}}\ R_n(\alpha)$.

2. Extreme points

Theorem 2. *The extreme points of* $\overline{\text{cl}}\ R_n(\alpha)$, $0 \leq \alpha < 1$, *are given by the functions*

$$f_x(z) = z + \sum_{k=2}^{\infty} \frac{(2-2\alpha)_{k-1} n! x^{k-1} z^k}{(k+n-1)!}, \tag{6}$$

$|x| = 1$, $z \in \Delta$, *where* $(a)_k = a(a+1)\cdots(a+k-1)$.

Proof. In [3] it is shown that the extreme point of $S^*(\alpha)$ are

$$\left\{ \frac{z}{(1-xz)^{2(1-\alpha)}} = z + \sum_{k=2}^{\infty} \frac{(2-2\alpha)_{k-1}}{(k-1)!} x^{k-1} z^k, |x| = 1 \right\}.$$

Since $D^n: f \to D^n f$ is an isomorphism from $R_n(\alpha)$ to $S^*(\alpha)$, and consequently preserves extreme points, we see from (3) that the extreme points of $\overline{\text{cl}}\ R_n(\alpha)$ are given by

$$z + \sum_{k=2}^{\infty} \binom{k+n-1}{n}^{-1} \frac{(2-2\alpha)_{k-1}}{(k-1)!} x^{k-1} z^k, \qquad |x| = 1,$$

which simplifies to $f_x(z)$ defined in (6).

Remark. The special cases $n = 0$ and $n = 1$ in Theorem 2 reduce to the extreme points of $\overline{\text{cl}}\ S^*(\alpha)$ and $\overline{\text{cl}}\ K(\alpha)$, respectively, found in [3].

Theorem 2 enables us to solve some extremal problems in $R_n(\alpha)$; for example, we have

Corollary 1. *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in R_n(\alpha)$, *then*

$$|a_k| \leq \frac{(2-2\alpha)_{k-1} n!}{(k+n-1)!}, \qquad k \geq 2,$$

with equality for

$$f_x(z) = z + \sum_{k=2}^{\infty} \frac{(2-2\alpha)_{k-1} n!}{(k+n-1)!} x^{k-1} z^k, \qquad |x| = 1.$$

COROLLARY 2. *If $f \in R_n(\alpha)$, then*

$$|f(z)| \leq r + \sum_{k=2}^{\infty} \frac{(2-2\alpha)_{k-1}}{(k+n-1)!} n! r^k \qquad (|z| = r),$$

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} \frac{(2-2\alpha)_{k-1}}{(k+n-1)!} n! k r^{k-1} \qquad (|z| = r),$$

with equality for $f_x(z)$ at $z = \bar{x}r$.

REMARK. It would be of interest to get $f_x(z)$ in (6) into closed form to obtain additional information and solutions to extremal problems. For example, we believe that the lower bounds for $|f(z)|$ and $|f'(z)|$ when $f \in R_n(\alpha)$ occur for $f_x(z)$ at $z = -\bar{x}r$. This is true for $n = 0$ and $n = 1$ (see [3]).

The determination of the extreme points of $\overline{\text{cl}}\ K_n$ is an immediate consequence of inclusion relations for K_n.

THEOREM 3. *The extreme points of $\overline{\text{cl}}\ K_n$ are $\{z/(1-xz); |x| = 1, n \in N_0\}$.*

PROOF. Note first from (1) that $D^n(z/(1-xz)) = z/(1-xz)^{n+1}$ so the family of functions $\{z/(1-xz)\}$ is contained in K_n for every n. Thus we have the double inclusion

$$\{z/(1-xz)\} \subset K_n \subset K_0 = S^*(1/2) \qquad (n = 0, 1, 2, \dots).$$

Since the extreme points of $\overline{\text{cl}}\ S^*(1/2)$ are $\{z/(1-xz): |x| = 1\}$ (see [3]) the result follows.

3. Convolution invariance

For

$$(7) \qquad h_n(z) = \frac{z}{(1-z)^{n+1}} = z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} z^k$$

we may express $D^n f$ as $D^n f = h_n * f$. We also denote by $h_n^{-1}(z)$ the function normalized by $h_n^{-1}(0) = 0$ with $(h_n^{-1} * h_n)(z) = z/(1-z)$. Then $h_n^{-1}(z) = z + \sum_{k=2}^{\infty} \binom{k+n-1}{n}^{-1} z^k$. With this notation, we may rewrite (2) as $f \in R_n(\alpha)$ if and only if $h_n * f \in S^*(\alpha)$ or, equivalently, $g \in S^*(\alpha)$ if and only if $h_n^{-1} * g \in R_n(\alpha)$.

The work of Ruscheweyh and Sheil-Small in [5] shows that the convolution of a convex function with a function in $S^*(\alpha)$ yields a functions in $S^*(\alpha)$. We make use of this result in establishing convolution properties for $R_n(\alpha)$.

THEOREM 4. *If* $f, g \in R_n(\alpha)$, $n \geq 1$, *then* $(f * g)(z) \in R_n(\alpha)$.

PROOF. With h_n defined by (7) we must show that if $f * h_n \in S^*(\alpha)$ and $g * h_n \in S^*(\alpha)$, then $(f * g) * h_n \in S^*(\alpha)$. Since $f \in R_n(\alpha) \subset R_1(\alpha) = K(\alpha)$, we have $(f * g) * h_n = f * (g * h_n)$ is the convolution of a convex function with a function in $S^*(\alpha)$ and must therefore be in $S^*(\alpha)$. Hence $(f * g)(z) \in R_n(\alpha)$, and the proof is complete.

Theorem 4 may be put in an equivalent form.

THEOREM 4a. *If* $z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} a_k z^k$ *and* $z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} b_k z^k$ *are both in* $S^*(\alpha)$, $n \geq 1$, *then so is* $z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} a_k b_k z^k$.

Compare this with the following remarkable result of Suffridge.

THEOREM A [9]. *Define* $\gamma(\alpha, k)$, $\alpha \leq 1$, *by*

$$\frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{k=2}^{\infty} \gamma(\alpha, k) z^k.$$

If $z + \sum_{k=2}^{\infty} \gamma(\alpha, k) a_k z^k$ *and* $z + \sum_{k=2}^{\infty} \gamma(\alpha, k) b_k z^k$ *are both in* $S^*(\alpha)$, *then so is* $z + \sum_{k=2}^{\infty} \gamma(\alpha, k) a_k b_k z^k$.

Another equivalent form to Theorem 4a is

THEOREM 4b. *If* $z + \sum_{k=2}^{\infty} a_k z^k$ *and* $z + \sum_{k=2}^{\infty} b_k z^k$ *are both in* $S^*(\alpha)$, *then so is* $z + \sum_{k=2}^{\infty} \binom{k+n-1}{n}^{-1} a_k b_k z^k$ *for* $n \geq 1$.

Setting $b_k = a_k$ and $n = 2$ in Theorem 4b, we obtain the following

COROLLARY. *If* $z + \sum_{k=2}^{\infty} a_k z^k \in S^*(\alpha)$, *then* $z + \sum_{k=2}^{\infty} \frac{2a_k^2}{k(k+1)} z^k \in S^*(\alpha)$.

REMARK. Theorem 4 cannot be extended to include the case $n = 0$. The Koebe function $k(z) = z/(1-z)^2$ is in $R_0 = S^*(0)$ but $(k^* k)(z) = z + \sum_{m=2}^{\infty} m^2 z^m$ is not even univalent in Δ.

We next show how to move to different classes of $R_n(\alpha)$ through convolution with hypergeometric functions. Recall the generalized hypergeometric function

$$\begin{aligned} &{}_mF_n(\alpha_1, \alpha_2, \ldots, \alpha_m; \beta_1, \beta_2, \ldots, \beta_n; z) \\ &\quad = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_m)_k}{(\beta_1)_k (\beta_2)_k \cdots (\beta_n)_k k!} z^k, \end{aligned}$$

where $(a)_0 = 1$ and $(a)_k = a(a+1) \cdots (a+k-1)$ for $k \geq 1$. We will apply this operator after establishing the following lemma.

Lemma. *Let $J: A \to A$ be defined by $J(f) = \frac{n+1}{z^n}\int_0^z t^{n-1}f(t)\,dt$. Then $f \in R_n(\alpha)$ if and only if $J(f) \in R_{n+1}(\alpha)$.*

Proof. We need to show that $D^n f \in S^*(\alpha)$ if and only if $D^{n+1}J(f) \in S^*(\alpha)$. In fact, we will show that $D^n f = D^{n+1}J(f)$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ we have $J(f) = z + \sum_{k=2}^{\infty} \frac{n+1}{n+k} a_k z^k$. Hence

$$D^{n+1}J(f) = z + \sum_{k=2}^{\infty} \left(\frac{k+n}{n+1}\right)\left(\frac{n+1}{n+k}\right) a_k z^k$$
$$= z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} a_k z^k = D^n f.$$

Theorem 5. *Let*

$$H(z) = {}_{m+1}F_m(n+1, n+1, \ldots, n+1, 1; n+2, n+2, \ldots, n+2; z)$$

be a hypergeometric function. Then $f \in R_n(\alpha)$ if and only if $f * zH(z)$ belongs to the class $R_{n+m}(\alpha)$ for any $m = 1, 2, \ldots$.

Proof. For $f(z) = \sum_{k=1}^{\infty} a_k z^k \in A$, $a_1 = 1$, and J defined in the lemma, we have that

$$J(f) = \sum_{k=1}^{\infty} \frac{n+1}{n+k} a_k z^k = \left(z\sum_{k=0}^{\infty} \frac{n+1}{n+k+1}\right) z^k * \left(\sum_{k=1}^{\infty} a_k z^k\right)$$
$$= \left(z\sum_{k=0}^{\infty} \frac{(n+1)_k(1)_k}{(n+2)_k k!} z^k\right) * f(z) = [z\,{}_2F_1(n+1, 1; n+2; z)] * f(z)$$

belongs to $R_{n+1}(\alpha)$. By repeated use of the lemma, the result follows.

Finally, we give a necessary and sufficient convolution condition for a function to be in $R_n(\alpha)$. In [7] it was shown that $f \in S^*(\alpha)$ if and only if

$$(8) \qquad f * \frac{z + \left(\frac{x+2\alpha-1}{2-2\alpha}\right)z^2}{(1-z)^2} \neq 0 \qquad (0 < |z| < 1,\ |x| = 1).$$

We use this result to prove

Theorem 6. *The function f is in $R_n(\alpha)$ if and only if*

$$f * \frac{z + \left(\frac{x(1+n)+n-1+2\alpha}{2-2\alpha}\right)z^2}{(1-z)^{n+2}} \neq 0 \qquad (0 < |z| < 1, |x| = 1).$$

Proof. An application of (2) to (8) shows that $f \in R_n(\alpha)$ if and only if

$$(9) \qquad f * \left(\frac{z}{(1-z)^{n+1}} * \frac{z + \left(\frac{x+2\alpha-1}{2-2\alpha}\right)z^2}{(1-z)^2}\right) \neq 0 \qquad (0 < |z| < 1, |x| = 1).$$

Since $g(z) * (\frac{z}{(1-z)^2} + \frac{Bz^2}{(1-z)^2}) = zg' + B(zg' - g)$, the result follows from (9) upon setting $g(z) = z/(1-z)^{n+1}$ and $B = (x + 2\alpha - 1)/(2 - 2\alpha)$, and then simplifying.

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