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THE HAUSDORFF DIMENSION DISTRIBUTION OF FINITE MEASURES IN EUCLIDEAN SPACE

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1. Introduction. Let E be a Borel set of $\mathbf{R}^N$. The α-outer Hausdorff measure of E has been defined to be

$$H^\alpha(E) = \lim_{\delta \to 0^+} H^\alpha_\delta(E)$$

where

$$H^\alpha_\delta(E) = \inf_{\substack{\bigcup_i B_i \supseteq E \\ d(B_i) \leqq \delta}} \sum (d(B_i))^\alpha$$

and each B_i is a closed ball. $d(B_i)$ denotes the diameter of B_i.

It is easily seen that the same value $H^\alpha(E)$ is obtained if we consider coverings of E by open balls or by balls which may be either open or closed.

By $\dim(E)$ we will mean the usual Hausdorff-Besicovitch dimension of E, where

$$\dim(E) = \sup\{\alpha | H^\alpha(E) = \infty\} = \inf\{\alpha | H^\alpha(E) = 0\}.$$

The following (see [7]) are well known elementary properties of $\dim(E)$:

(1) $0 \leqq \dim(E) \leqq N$.

(2) if E is countable then $\dim(E) = 0$ while if $\lambda(E) > 0$ then $\dim(E) = N$ (where λ denotes N-dimensional Lebesgue measure, a notation to be maintained throughout this paper).

$$(3) \quad \dim\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup_n \dim(E_n).$$

These properties will be used freely without comment in the following.

Let μ be a probability measure on $\mathbf{R}^N$. In this paper we introduce the notion of the dimension distribution $\hat{\mu}$ of μ. $\hat{\mu}$ is a probability measure on $[0, N]$ and the quantity $\hat{\mu}(E)$ can be interpreted as the proportion of mass of μ supported strictly on sets with dimension lying in E. Associated with μ and $\hat{\mu}$ is a real-valued random variable $\hat{\alpha}$ (which we will call the dimension concentration map determined by μ) and a family $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, of probability measures on $\mathbf{R}^N$ to be referred to as the

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dimension derivative family of μ. This family is used to obtain an integral representation of μ with respect to its dimension distribution (called the dimension disintegration formula) which leads to an intuitive and elegant proof of a dimension decomposition theorem first presented and proved in another manner by Rogers and Taylor, [**8**]. These results of course extend to finite Borel measures by using the appropriate normalizations.

We also develop a characterization of dimension derivative families which turns out to be extremely useful in constructing measures μ having a desired dimension distribution. While it is easy to construct measures having an atomic dimension distribution (for example, an absolutely continuous measure will always have an atomic dimension distribution with all mass concentrated at the point N) the problem of building measures with diffuse dimension distributions is much more difficult. It was partially addressed by Rogers and Taylor in [**8**] who, without the actual concept of dimension distribution, constructed a specific measure which they demonstrated to have a "diffuse dimension spectrum"; that is, it had no mass concentrated on any set of any particular dimension. Their construction is lengthy and nontrivial and in this paper we prove a much stronger result via the simpler techniques of dimension derivatives, namely that to each probability distribution γ on $[0, N]$ there corresponds a probability measure μ on $\mathbf{R}^N$ satisfying $\hat{\mu} = \gamma$. Furthermore the measure μ is constructed explicitly as an integral with respect to γ. Extensions of results due to Billingsley [**1**, **2**, **3**] are employed in the proof.

In the final section of the paper the same techniques of Billingsley are used to develop an alternative characterization of the dimension concentration map $\hat{\alpha}$; this leads to a more tractable definition of $\hat{\mu}$ which we expect will prove useful in the statistical estimation of $\hat{\mu}$ for high-dimensional spatially-distributed data. We connect these results with work of Gács [**5**] who defined a numerical quantity called the Hausdorff dimension of a probability measure and examined its relationship to Renyi dimension and entropy. We see that in fact Gács' number is precisely the mean of the dimension distribution.

By $\mathscr{B}(\mathbf{R}^N)$ and $\mathscr{B}([0, N])$ we will mean the Borel sets of $\mathbf{R}^N$ and $[0, N]$ respectively.

2. Dimension distributions and derivative families. Let μ be a finite Borel measure on $\mathbf{R}^N$. For each $\alpha \in [0, N]$ define the set function μ_α on $\mathscr{B}(\mathbf{R}^N)$ by

$$\mu_\alpha(B) = \sup_{\dim(D) \leqq \alpha} \mu(B \cap D)$$

where D is always assumed to be a Borel set. Clearly

$$\mu_\alpha(B) \leqq \mu_\beta(B) \leqq \mu(B)$$

whenever $\alpha \leqq \beta$ and if $\dim(B) \leqq \alpha$ then $\mu_\alpha(B) = \mu(B)$.

LEMMA 2.1. *μ_α is a measure supported on a set D_α satisfying* $\dim(D_\alpha) \leqq \alpha$ *and can be expressed as*

$$\mu_\alpha(B) = \mu(B \cap D_\alpha).$$

Proof. Countable subadditivity of μ_α is obvious. To verify finite superadditivity let B_1 and B_2 be disjoint Borel sets and let $\epsilon > 0$. Then there exist D_1 and D_2 with $\dim(D_1) \leqq \alpha$, $\dim(D_2) \leqq \alpha$, such that

$$\mu(B_1 \cap D_1) \geqq \mu_\alpha(B) - \epsilon/2 \quad \text{and}$$
$$\mu(B_2 \cap D_2) \geqq \mu_\alpha(B_2) - \epsilon/2.$$

Letting $D^* = D_1 \cup D_2$ then $\dim(D^*) \leqq \alpha$ and we have

$$\begin{aligned}\mu_\alpha(B_1 \cup B_2) &= \sup_{\dim(D)\leqq\alpha} \mu((B_1 \cup B_2) \cap D)\\ &= \sup_{\dim(D)\leqq\alpha} \mu(B_1 \cap D) + \mu(B_2 \cap D)\\ &\geqq \mu(B_1 \cap D^*) + \mu(B_2 \cap D^*)\\ &\geqq \mu_\alpha(B_1) + \mu_\alpha(B_2) - \epsilon.\end{aligned}$$

Thus we conclude μ_α is a measure and since

$$\mu_\alpha(\mathbf{R}^N) = \sup_{\dim(D)\leqq\alpha} \mu(D)$$

there exists a sequence $\{D_n\}_{n=1}^{\infty}$ of sets with $\dim(D_n) \leqq \alpha$ such that

$$\mu_\alpha(\mathbf{R}^N) = \lim_{n\to\infty} \mu(D_n).$$

Without loss of generality we can choose $D_n \subseteq D_{n+1}$. Let

$$D_\alpha = \bigcup_{n=1}^{\infty} D_n.$$

Then

$$\dim(D_\alpha) \leqq \alpha \quad \text{and} \quad \mu_\alpha(\mathbf{R}^N) = \mu(D_\alpha) = \mu_\alpha(D_\alpha).$$

Thus D_α supports μ_α and we obtain

$$\mu_\alpha(B) = \mu_\alpha(B \cap D_\alpha) = \mu(B \cap D_\alpha).$$

The quantity $\mu_\alpha(B)$ represents the amount of μ-mass of B which is concentrated on sets of dimension not exceeding α. The set D_α will be called an *α-support of μ* and is obviously not unique.

A family $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, will be called an *α-support chain of μ* if, for each α, D_α is an α-support of μ and $D_\alpha \subseteq D_\beta$ whenever $\alpha \leqq \beta$.

LEMMA 2.2. *An α-support chain of μ always exists.*

Proof. For each rational $q \in [0, N]$ let E_q be a q-support of μ. Note then that E_q supports μ_α for all $\alpha \leqq q$. Let

$$E_q^* = \bigcup_{q' \leqq q} E_{q'}$$

where q' is rational and define

$$D_\alpha = \bigcap_{q \geqq \alpha} E_q^*.$$

It follows that $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, is an α-support chain of μ.

If $\{D_\alpha\}_\alpha$ is an α-support chain of μ we will let

$$D_\alpha^- = \bigcup_{\beta<\alpha} D_\beta \quad \text{and} \quad D_\alpha^+ = \bigcap_{\beta>\alpha} D_\beta.$$

The chain will be said to be *right-continuous* if

$$D_\alpha = D_\alpha^+ \quad \text{for each } \alpha.$$

Right-continuous versions exist as it is easily seen that $\{D_\alpha^+\}_\alpha$ is always a right-continuous α-support chain.

LEMMA 2.3. *$\mu_\alpha(\mathbf{R}^N)$ is an increasing right-continuous function of α.*

Proof. Let $\{D_\alpha\}_\alpha$ be an α-support chain of μ. Then

$$\lim_{\alpha \to \alpha_0^+} \mu_\alpha(\mathbf{R}^N) = \lim_{\alpha \to \alpha_0^+} \mu(D_\alpha) = \mu(D_{\alpha_0}^+) = \mu_{\alpha_0}(\mathbf{R}^N).$$

Thus $\mu_\alpha(\mathbf{R}^N)$ is the distribution function of a finite measure $\hat{\mu}$ defined on the Borel sets of $[0, N]$ via the relation

$$\hat{\mu}([0, \alpha]) = \mu_\alpha(\mathbf{R}^N).$$

Note the total mass of $\hat{\mu}$ is

$$\hat{\mu}([0, N]) = \mu_N(\mathbf{R}^N) = \mu(\mathbf{R}^N).$$

We will refer to $\hat{\mu}$ as the *dimension measure of* μ and to the normalized quantity

$$\hat{\hat{\mu}} = \hat{\mu}/\mu(\mathbf{R}^N)$$

as the *dimension distribution of* μ.

Remark. Lemma 2.3 also holds true when $\mathbf{R}^N$ is replaced by an arbitrary Borel set B, enabling us to define $\hat{\mu}_B$, the *dimension measure of* μ *at* B, by

$$\hat{\mu}_B([0, \alpha]) = \mu_\alpha(B).$$

We will see that the measures $\hat{\mu}_B$ also arise naturally in yet another way.

For the remainder of this section we will for convenience assume that μ

is a probability measure. Let $\{D_\alpha\}_\alpha$ be some α-support chain of μ and let

$$D^0_\alpha = D^+_\alpha \backslash D^-_\alpha .$$

Note the family $\{D^0_\alpha\}_\alpha$ partitions D_N. Define the map

$$\hat{\alpha}: D_N \to [0, N]$$

by

$$\hat{\alpha}(x) = \inf\{\alpha | x \in D_\alpha\}.$$

It is easily seen that

$$\{x | \hat{\alpha}(x) \leqq \alpha\} = D^+_\alpha$$

while

$$\{x | \hat{\alpha}(x) = \alpha\} = D^0_\alpha.$$

As $\mu(D_N) = 1$ it follows that $\hat{\alpha}$ is μ-a.e. defined on $\mathbf{R}^N$ and thus may be regarded as a random variable from the probability space $(\mathbf{R}^N, \mathscr{B}(\mathbf{R}^N), \mu)$ into $[0, N]$.

Part (ii) of the following theorem shows that, up to a set of μ-measure zero, the definition of $\hat{\alpha}$ does not depend on the choice of α-support chain for μ. Thus $\hat{\alpha}$ is μ-a.e. uniquely defined and will be called the *dimension concentration map* determined by μ. This enables us to associate a dimensional number $\hat{\alpha}(x)$ with each point x. In Section 5 we develop an alternate characterization of $\hat{\alpha}(x)$ which clarifies the meaning of this dimensional number in terms of the behaviour of μ in neighbourhoods of x.

THEOREM 2.1. (i) *The random variable $\hat{\alpha}$ has distribution $\hat{\mu}$ on* $[0, N]$. *That is,* $\hat{\mu} = \mu\hat{\alpha}^{-1}$.

(ii) *If $\{D_\alpha\}_\alpha$ and $\{E_\alpha\}_\alpha$ are two α-support chains of μ with*

$$\hat{\alpha}_D(x) = \inf\{\alpha | x \in D_\alpha\} \quad \textit{and}$$

$$\hat{\alpha}_E(x) = \inf\{\alpha | x \in E_\alpha\}$$

then $\hat{\alpha}_D = \hat{\alpha}_E$ μ-a.e. *Equivalently,*

$$\mu\left(\bigcup_{0 \leqq \alpha \leqq N} (D^0_\alpha \Delta E^0_\alpha)\right) = 0.$$

Proof. (i) This is trivial as

$$\mu\hat{\alpha}^{-1}([0, \alpha]) = \mu(\{x | \hat{\alpha}(x) \leqq \alpha\})$$
$$= \mu(D^+_\alpha) = \hat{\mu}([0, \alpha]).$$

(ii) To prove (ii) we first note that $\{E_\alpha \cap D_\alpha\}_\alpha$ is also an α-support chain of μ. Hence

$$\begin{aligned}\mu(\{x|\hat{\alpha}_E(x) \leqq \alpha \text{ and } \hat{\alpha}_D(x) \leqq \alpha\}) &= \mu(E_\alpha^+ \cap D_\alpha^+) \\ &= \hat{\mu}([0, \alpha]) \\ &= \mu(\{x|\hat{\alpha}_E(x) \leqq \alpha\}).\end{aligned}$$

Thus, for each α,

$$\mu(\{x|\hat{\alpha}_E(x) \leqq \alpha \text{ and } \hat{\alpha}_D(x) > \alpha\}) = 0.$$

As

$$\{x|\hat{\alpha}_D(x) > \hat{\alpha}_E(x)\} = \bigcup_{q \in [0,N]} \{x|\hat{\alpha}_E(x) \leqq q \text{ and } \hat{\alpha}_D(x) > q\},$$

where q is rational, it then follows from symmetry that

$$\mu(\{x|\hat{\alpha}_D(x) \neq \hat{\alpha}_E(x)\}) = 0.$$

Noting that

$$\bigcup_{0 \leqq \alpha \leqq N} (D_\alpha^0 \Delta E_\alpha^0) = \{x|\hat{\alpha}_D(x) \neq \hat{\alpha}_E(x)\}$$

the proof is complete.

We now wish to develop the notion of the dimension derivative family of μ. For convenience let $\hat{x}$ denote the identity map on $\mathbf{R}^N$. Thus, considered as a random vector on $(\mathbf{R}^N, \mathscr{B}(\mathbf{R}^N), \mu)$, $\hat{x}$ has distribution μ. Let $\tilde{\mu}$ denote the joint distribution of $(\hat{x}, \hat{\alpha})$ on the product space $\mathbf{R}^N \times [0, N]$. That is,

$$\tilde{\mu}(G) = \mu(\{x| (x, \hat{\alpha}(x)) \in G\}) \text{ for each } G \in \mathscr{B}(\mathbf{R}^N \times [0, N]).$$

For product sets $B \times E$ with $B \in \mathscr{B}(\mathbf{R}^N)$ and $E \in \mathscr{B}([0, N])$ we note that

$$\tilde{\mu}(B \times E) = \mu(B \cap \hat{\alpha}^{-1}(E)).$$

In particular,

$$\begin{aligned}\tilde{\mu}(B \times [0, \alpha]) &= \mu(B \cap \hat{\alpha}^{-1}([0, \alpha])) \\ &= \mu(B \cap D_\alpha^+) = \mu_\alpha(B) = \hat{\mu}_B([0, \alpha])\end{aligned}$$

where $\hat{\mu}_B$ is defined as in the remark following Lemma 2.3. Thus

$$\tilde{\mu}(B \times E) = \mu_B(\hat{E})$$

and so the measures $\hat{\mu}_B$ occur naturally as the partial second marginals of $\tilde{\mu}$. Clearly μ and $\hat{\mu}$ are, respectively, the first and second marginals of $\tilde{\mu}$.

The conditional distributions of $\hat{x}$ given $\hat{\alpha} = \alpha$ consist of a family of probability measures on $\mathbf{R}^N$ which we will call the *dimension derivative family of* μ. Of less interest are the conditional distributions of $\hat{\alpha}$ given $\hat{x} = x$; since $\hat{x}$ uniquely determines $\hat{\alpha}$ these are simply point masses which we will refer to as the *dimension concentration family of* μ. These definitions are included in Theorem 2.2.

Given a point x, by δ_x we will mean the unit mass (or Dirac measure) at x. That is,

$$\delta_x(B) = \begin{cases} 1 & x \in B \\ 0 & x \notin B. \end{cases}$$

This notation will be maintained throughout the paper.

THEOREM 2.2. *Let μ be a probability measure on the Borel sets of* $\mathbf{R}^N$.

(i) *There exists a family* $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, *of probability measures on* $\mathscr{B}(\mathbf{R}^N)$ *such that*

$$\tilde{\mu}(B \times E) = \int_E \psi(B, \alpha)\hat{\mu}(d\alpha)$$

for each $B \in \mathscr{B}(\mathbf{R}^N)$, $E \in \mathscr{B}([0, N])$.

The family $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, *is unique up to an α-set of $\hat{\mu}$-measure zero and will be called the dimension derivative family of* μ.

(ii) *There exists a family* $\{\eta(\cdot, x)\}$, $x \in \mathbf{R}^N$, *of probability measures on* $\mathscr{B}([0, N])$ *such that*

$$\tilde{\mu}(B \times E) = \int_B \eta(E, x)\mu(dx)$$

for each $B \in \mathscr{B}(\mathbf{R}^N)$, $E \in \mathscr{B}([0, N])$.

The family $\{\eta(\cdot, x)\}$, $x \in \mathbf{R}^N$, *is unique up to an x-set of μ-measure zero and will be called the dimension concentration family of* μ. *Furthermore*

$$\eta(\cdot, x) = \delta_{\hat{\alpha}(x)}(\cdot)\ \mu\text{-a.e.}$$

Proof. It is well known that a.e.-unique regular conditional distributions exist for random vectors defined on Euclidean space. (See, for example, [**6**], (CD_1) and (CD_2), p. 30.) Thus we set

$$\psi(\cdot, \alpha) = \tilde{\mu}(\cdot|\hat{\alpha} = \alpha)$$

the conditional distribution of $\hat{x}$ given $\hat{\alpha} = \alpha$, and

$$\eta(\cdot, x) = \tilde{\mu}(\cdot|\hat{x} = x)$$

the conditional distribution of $\hat{\alpha}$ given $\hat{x} = x$. Noting that

$$\tilde{\mu}(B \times E) = \mu(B \cap \hat{\alpha}^{-1}(E)) = \int_B \delta_{\hat{\alpha}(x)}(E)\mu(dx)$$

we conclude from uniqueness that

$$\eta(\cdot, x) = \delta_{\hat{\alpha}(x)}(\cdot)\ \mu\text{-a.e.}$$

The following theorem lists the basic properties of dimension derivative families. Note that part (iii), the dimension disintegration formula, provides a representation of μ as an integral with respect to its dimension measure $\hat{\mu}$. This representation will be used in obtaining the dimension decomposition theorem of Section 3.

THEOREM 2.3. *Let* $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, *be the dimension derivative family of a probability measure* μ *on* $\mathbf{R}^N$. *Let* $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, *denote any* α-*support chain of* μ.

(i) *For every* $B \in \mathscr{B}(\mathbf{R}^N)$,

$$\psi(B, \cdot)|_{(\dim(B),N]} = 0 \ \hat{\mu}\text{-a.e.},$$

where $\psi(B, \cdot)|_{(\dim(B),N]}$ *denotes the restriction of the function* $\psi(B, \cdot)$ *to the interval* $(\dim(B), N]$.

(ii) *For each* α

$$\psi(D_\alpha, \cdot)|_{[0,\alpha]} = 1 \ \hat{\mu}\text{-a.e.}$$

and in fact the stronger result

$$\hat{\mu}(\{\alpha|\psi(D_\alpha^0, \alpha) \neq 1\}) = 0$$

holds. That is, for $\hat{\mu}$-*almost all* α, $\psi(\cdot, \alpha)$ *is supported on* D_α^0.

(iii) (*Dimension Disintegration Formula*): *For each* $0 \leqq \alpha \leqq N$ *we have*

$$\mu_\alpha(\cdot) = \int_{[0,\alpha]} \psi(\cdot, \beta)\hat{\mu}(d\beta)$$

and in particular

$$\mu(\cdot) = \int_{[0,N]} \psi(\cdot, \beta)\hat{\mu}(d\beta).$$

Proof. (i) Noting that

$$\begin{aligned}\tilde{\mu}(B \times [0, \dim(B)]) &= \mu(B \cap D^+_{\dim(B)}) \\ &= \mu(B) = \tilde{\mu}(B \times [0, N])\end{aligned}$$

we obtain

$$\int_{[0,\dim(B)]} \psi(B, \alpha)\hat{\mu}(d\alpha) = \int_{[0,N]} \psi(B, \alpha)\hat{\mu}(d\alpha)$$

and thus

$$\psi(B, \cdot)|_{(\dim(B),N]} = 0 \ \hat{\mu}\text{-a.e.}$$

(ii) From (i) it follows that for every rational q

$$\psi(D_q, \cdot)|_{(q,N]} = 0 \ \hat{\mu}\text{-a.e.}$$

Letting

$$A = \{\alpha|\psi(D_q, \alpha) = 0 \text{ for all } q < \alpha\},$$

then

$$\hat{\mu}([0, N]\backslash A) = 0$$

and for each $\alpha \in A$ we have

$$\psi(D_\alpha^-, \alpha) = \lim_{q \to \alpha^-} \psi(D_q, \alpha) = 0.$$

Thus $\psi(D_\alpha^-, \alpha) = 0$ for $\hat{\mu}$-almost all α.

To show $\psi(D_\alpha^+, \alpha) = 1$ for $\hat{\mu}$-almost all α choose $\alpha_0 \in [0, N]$ arbitrarily. Then

$$\begin{aligned}\hat{\mu}([0, \beta]) &= \tilde{\mu}(\mathbf{R}^N \times [0, \beta]) = \tilde{\mu}(D_\beta \times [0, \beta]) \\ &\leqq \tilde{\mu}(D_{\alpha_0} \times [0, \beta]) \text{ whenever } \beta \leqq \alpha_0 \\ &\leqq \hat{\mu}([0, \beta]).\end{aligned}$$

Hence

$$\hat{\mu}([0, \beta]) = \tilde{\mu}(D_{\alpha_0} \times [0, \beta]) = \int_{[0,\beta]} \psi(D_{\alpha_0}, \alpha)\hat{\mu}(d\alpha)$$

for every $\beta \leqq \alpha_0$ from which we conclude

$$\psi(D_{\alpha_0}, \cdot)\,|_{[0,\alpha_0]} = 1 \ \hat{\mu}\text{-a.e.}$$

Letting

$$A^* = \{\alpha | \psi(D_q, \alpha) = 1 \text{ for all } q > \alpha\}$$

where q is rational, it follows that

$$\hat{\mu}([0, N]\backslash A^*) = 0.$$

Furthermore for each $\alpha \in A^*$ we have

$$\psi(D_\alpha^+, \alpha) = \lim_{q \to \alpha^+} \psi(D_q, \alpha) = 1.$$

Thus $\psi(D_\alpha^+, \alpha) = 1$ for $\hat{\mu}$-almost all α and hence

$$D_\alpha^0 = D_\alpha^+ \backslash D_\alpha^-$$

supports $\psi(\cdot, \alpha)$ for $\hat{\mu}$-almost all α.

(iii) For each $\alpha \in [0, N]$ and $B \in \mathscr{B}(\mathbf{R}^N)$ we have

$$\mu_\alpha(B) = \tilde{\mu}(B \times [0, \alpha]) = \int_{[0,\alpha]} \psi(B, \beta)\hat{\mu}(d\beta).$$

Since $\mu_N = \mu$ we obtain

$$\mu(B) = \int_{[0,N]} \psi(B, \beta)\hat{\mu}(d\beta).$$

The following is a useful characterization of dimension derivative families which provides a method of constructing a measure having a desired dimension distribution. This technique will be exploited in the proof of the existence theorem of Section 4.

THEOREM 2.4. *A family* $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, *of probability measures defined over the Borel sets of* $\mathbf{R}^N$ *is the dimension derivative family of some probability measure on* $\mathbf{R}^N$ *if and only if the following conditions hold*:

1. $\psi(B, \cdot)$ *is a Borel measurable function of* α *for each* $B \in \mathscr{B}(\mathbf{R}^N)$.

2. *There exists a probability measure* γ *on the Borel sets of* $[0, N]$ *such that*

(i) $\psi(B, \cdot)|_{(\dim(B),N]} = 0$ γ-a.e.

for each $B \in \mathscr{B}(\mathbf{R}^N)$

(ii) *for each* $\alpha \in [0, N]$ *there exists a set* D_α *satisfying* $\dim(D_\alpha) \leqq \alpha$ *such that*

$$\psi(D_\alpha, \cdot)|_{[0,\alpha]} = 1 \ \gamma\text{-a.e.}$$

If these conditions are met then $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, *is the dimension derivative family of the probability measure* μ *defined by*

$$\mu(B) = \int_{[0,N]} \psi(B, \alpha)\gamma(d\alpha)$$

and furthermore $\hat{\mu} = \gamma$.

Proof. Necessity of these conditions is immediate by applying Theorem 2.3 with $\gamma = \hat{\mu}$ and $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, any α-support chain of μ.

To show sufficiency let $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, be a family of probability measures satisfying the conditions of the theorem. Let γ and $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, be as in condition 2. Define μ by

$$\mu(B) = \int_{[0,N]} \psi(B, \alpha)\gamma(d\alpha).$$

Clearly μ is a probability measure. Now for each $B \in \mathscr{B}(\mathbf{R}^N)$ and $\alpha \in [0, N]$ we must have, by the definition of μ and conditions 2 (i) and 2 (ii):

$$\begin{aligned}\mu(B \cap D_\alpha) &= \int_{[0,N]} \psi(B \cap D_\alpha, \beta)\gamma(d\beta) \\ &= \int_{[0,\alpha]} \psi(B \cap D_\alpha, \beta)\gamma(d\beta) = \int_{[0,\alpha]} \psi(B, \beta)\gamma(d\beta).\end{aligned}$$

But if $\dim(B) \leqq \alpha$ condition 2 (i) also implies

$$\mu(B) = \int_{[0,\alpha]} \psi(B, \beta)\gamma(d\beta).$$

Hence $\mu(B) = \mu(B \cap D_\alpha)$ whenever $\dim(B) \leqq \alpha$ which clearly implies D_α is in fact an α-support of μ. Thus we obtain for each $B \in \mathscr{B}(\mathbf{R}^N)$ and $\alpha \in [0, N]$:

$$\tilde{\mu}(B \times [0, \alpha]) = \mu(B \cap D_\alpha)$$

by definition of $\tilde{\mu}$

$$= \int_{[0,\alpha]} \psi(B, \beta)\gamma(d\beta)$$

while

$$\hat{\mu}([0, \alpha]) = \tilde{\mu}(\mathbf{R}^N \times [0, \alpha]) = \int_{[0,\alpha]} \psi(\mathbf{R}^N, \beta)\gamma(d\beta) = \gamma([0, \alpha])$$

and so $\hat{\mu} = \gamma$. Hence

$$\tilde{\mu}(B \times [0, \alpha]) = \int_{[0,\alpha]} \psi(B, \beta)\hat{\mu}(d\beta)$$

and by uniqueness we conclude $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, is the dimension derivative family of μ.

Remark. If we additionally assume the family $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, in 2 (ii) is nested it is tempting to try to replace 2 (i) by

2 (i)′ $\psi(D_\alpha, \cdot)|_{(\alpha,N]} = 0$ γ-a.e.

In fact it is easy to construct counterexamples showing that 2 (i)′ is not sufficient.

Note that if $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, and γ satisfy the conditions of Theorem 2.4 and ν is some probability measure on $[0, N]$ such that $\nu \ll \gamma$ then the conditions of the theorem are also met with ν in place of γ. Hence $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, is also the dimension derivative family of the measure μ^* defined by

$$\mu^*(B) = \int_{[0,N]} \psi(B, \alpha)\nu(d\alpha).$$

Thus, while a given measure μ determines its dimension derivative family $\hat{\mu}$-a.e. uniquely, any particular dimension derivative family gives rise to an equivalence class of measures sharing that derivative family but possessing distinct dimension distributions.

3. The dimension decomposition of measures. In the previous section we restricted μ to be a probability measure only for the convenience of using the joint and conditional distributions of certain random variables. It is clear that the dimension concentration map $\hat{\alpha}$ is μ-a.e. uniquely defined even when $\mu(\mathbf{R}^N) \neq 1$. The measure $\hat{\mu}$ is simply defined via the relation

$$\tilde{\mu}(B \times E) = \mu(B \cap \hat{\alpha}^{-1}(E)).$$

As a result we have the following linearity lemma (the proof is elementary and therefore omitted).

LEMMA 3.1. (i) *Let μ be a finite Borel measure on $\mathbf{R}^N$ and let $c > 0$. Then μ and $c\mu$ determine the same dimension concentration map μ-a.e. Furthermore $\widetilde{c\mu} = c\tilde{\mu}$ and as a consequence*

$$(\widehat{c\mu})_B = c\hat{\mu}_B \text{ for each } B \in \mathscr{B}(\mathbf{R}^N).$$

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(ii) *Let* $\mu, \mu_1, \mu_2, \ldots$ *be finite Borel measures on* $\mathbf{R}^N$ *such that*

$$\mu = \sum_n \mu_n.$$

Then

$$\widetilde{\mu} = \sum_n \widetilde{\mu}_n$$

and as a consequence

$$\hat{\mu}_B = \sum_n (\hat{\mu}_n)_B.$$

We now note that Theorem 2.2 holds true for any finite Borel measure μ. This can be seen by first applying Theorem 2.2 to the normalized measure $\nu = \mu/c$ (where $c = \mu(\mathbf{R}^N)$) and then noting that $\widetilde{\mu} = c\widetilde{\nu}$ and $\hat{\mu} = c\hat{\nu}$ (from Lemma 3.1). Thus to each finite Borel measure μ we can associate a dimension derivative family $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, satisfying (i) of Theorem 2.2 and (i), (ii), (iii) of Theorem 2.3. If $\mu(\mathbf{R}^N) \neq 1$ it is perhaps more aesthetic to consider $\widetilde{\mu}$ (and μ in the dimension disintegration formula) as an integral with respect to the dimension distribution $\hat{\hat{\mu}}$ rather than the nonnormalized quantity $\hat{\mu}$. This is easily accomplished by modifying the total mass of the dimension derivative family, setting

$$\psi^*(\cdot, \alpha) = c\psi(\cdot, \alpha)$$

where $c = \mu(\mathbf{R}^N)$. We will continue to present $\widetilde{\mu}$ as an integral with respect to $\hat{\mu}$, however, as the linearity properties of $\hat{\mu}$ simplify proofs and discussions.

We now note that if μ is a measure such that $\hat{\hat{\mu}} = \delta_\alpha$ then this is equivalent to saying that μ can be supported on a set of dimension α but has no mass on any set of smaller dimension; in the terminology of Rogers and Taylor μ is of *exact dimension* α. Any measure with an atomic dimension distribution is simply a sum of such exact dimensional measures. In the following theorem we formalize the fact that any measure without a diffuse singular component necessarily has an atomic dimension distribution. The Cantor measure ν defined over $[0, 1]$ is an easy example of a diffuse singular measure with an atomic dimension distribution; in fact

$$\hat{\nu} = \delta_{\frac{\log 2}{\log 3}}.$$

THEOREM 3.1. *Let* μ *be a finite Borel measure on* $\mathbf{R}^N$ *having no diffuse singular component. Then*

$$\hat{\mu} = \mu_a(\mathbf{R}^N)\delta_0 + \mu_{ac}(\mathbf{R}^N)\delta_N$$

where μ_a *and* μ_{ac} *are respectively the atomic and absolutely continuous components of* μ.

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Proof. Since μ_a is atomic it can be supported on a countable set and thus

$$\hat{\mu}_a = \mu_a(\mathbf{R}^N)\delta_0.$$

If $\mu_{ac}(E) > 0$ then $\lambda(E) > 0$ and hence μ_{ac} has no mass on any set of dimension less than N. Therefore

$$\hat{\mu}_{ac} = \mu_{ac}(\mathbf{R}^N)\delta_N.$$

Applying (ii) of Lemma 3.1 we conclude

$$\hat{\mu} = \mu_a(\mathbf{R}^N)\delta_0 + \mu_{ac}(\mathbf{R}^N)\delta_N.$$

We now present our version of Rogers and Taylor's dimension decomposition theorem for measures.

THEOREM 3.2. *Let μ be a finite Borel measure on $\mathbf{R}^N$. Then there exists a unique countable set $A \subseteq [0, N]$ and unique finite measures μ^α, $\alpha \in$ A, and μ^d such that μ^α is of exact dimension α, μ^d has a diffuse dimension measure, and*

$$\mu = \sum_{\alpha \in A} \mu^\alpha + \mu^d.$$

Furthermore A is precisely the set of atoms of $\hat{\mu}$ and

$$\mu^\alpha(\cdot) = \hat{\mu}(\{\alpha\})\psi(\cdot, \alpha)$$

where $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, is the dimension derivative family of μ.

Proof. Let

$$\mu^\alpha(\cdot) = \hat{\mu}(\{\alpha\})\psi(\cdot, \alpha)$$

for each atom α of $\hat{\mu}$. As $\hat{\mu}(\{\alpha\}) > 0$ it follows from (i) of Theorem 2.3 that $\psi(B, \alpha) = 0$ whenever $\dim(B) < \alpha$ while (ii) of Theorem 2.3 implies $\psi(\cdot, \alpha)$ is supported on D_α^0 (where D_α is any α-support of μ). Hence $\hat{\psi}(\cdot, \alpha) = \delta_\alpha$ and so μ^α is of exact dimension α. Let A denote the set of atoms of $\hat{\mu}$. By the dimension disintegration formula we have

$$\begin{aligned}
\mu(\cdot) &= \int_{[0,N]} \psi(\cdot, \beta)\hat{\mu}(d\beta) \\
&= \int_A \psi(\cdot, \beta)\hat{\mu}(d\beta) + \int_{[0,N]\setminus A} \psi(\cdot, \beta)\hat{\mu}(d\beta) \\
&= \sum_{\alpha \in A} \hat{\mu}(\{\alpha\})\psi(\cdot, \alpha) + \int_{[0,N]\setminus A} \psi(\cdot, \beta)\hat{\mu}(d\beta) \\
&= \sum_{\alpha \in A} \mu^\alpha(\cdot) + \int_{[0,N]\setminus A} \psi(\cdot, \beta)\hat{\mu}(d\beta).
\end{aligned}$$

Setting

$$\mu^d(\cdot) = \int_{[0,N]\setminus A} \psi(\cdot, \beta)\hat{\mu}(d\beta)$$

we thus have

$$\mu = \sum_{\alpha \in A} \mu^\alpha + \mu^d.$$

To show $\hat{\mu}^d$ is diffuse first note that

$$\hat{\mu}^\alpha = \hat{\mu}(\{\alpha\})\delta_\alpha.$$

Applying Lemma 3.1 we obtain

$$\hat{\mu} = \sum_{\alpha \in A} \hat{\mu}^\alpha + \hat{\mu}^d$$

$$= \sum_{\alpha \in A} \hat{\mu}(\{\alpha\})\delta_\alpha + \hat{\mu}^d.$$

As the atomic component of $\hat{\mu}$ must be

$$\sum_{\alpha \in A} \hat{\mu}(\{\alpha\})\delta_\alpha$$

it follows that $\hat{\mu}^d$ is diffuse. Thus we have the desired decomposition. We need only verify uniqueness. Suppose

$$\mu = \sum_{\alpha \in A'} \gamma^\alpha + \gamma^d$$

is another such decomposition for μ. Then for each $B \in \mathscr{B}(\mathbf{R}^N)$ we have

$$\hat{\mu}_B = \sum_{\alpha \in A'} \gamma^\alpha(B)\delta_\alpha + \hat{\gamma}_B^d = \sum_{\alpha \in A} \mu^\alpha(B)\delta_\alpha + \hat{\mu}_B^d.$$

Taking $B = \mathbf{R}^N$ and equating atomic and diffuse components we immediately obtain $A' = A$. Then noting that we must have $\gamma^\alpha(B) = \mu^\alpha(B)$ for each B we conclude $\gamma^\alpha = \mu^\alpha$ for each $\alpha \in A$.

Remark. A version of Theorem 3.2 can be extended to finite signed measures (Rogers and Taylor originally stated the decomposition theorem for this case). If ν is a finite signed Borel measure on $\mathbf{R}^N$ let $|\nu|$ denote the total variation of ν. It can be shown that there exists a family $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, of finite signed measures on $\mathbf{R}^N$ such that

$$\nu(B) = \int_{[0,N]} \psi(B, \alpha)\, |\hat{\nu}|\, (d\alpha)$$

where $|\hat{\nu}|$ denotes the dimension measure of $|\nu|$ and $\{\, |\psi|\, (\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, is the dimension derivative family of $|\nu|$. We will call $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, the dimension derivative family of ν. Applying this result and Theorem 3.2 we can obtain a unique decomposition

$$\nu = \sum_{\alpha \in A} \nu^{\alpha} + \nu^{d}$$

where $|\nu^{\alpha}|$ has exact dimension α, $|\nu^{d}|$ has a diffuse dimension measure, and A is the set of atoms of $|\hat{\nu}|$. Furthermore the dimension decomposition of $|\nu|$ is related to that of ν by

$$|\nu| = \sum_{\alpha \in A} |\nu^{\alpha}| + |\nu^{d}|$$

(that is, $|\nu|^{\alpha} = |\nu^{\alpha}|$ and $|\nu|^{d} = |\nu^{d}|$).

4. The existence theorem. The main goal of this section is to prove that every probability distribution γ on $[0, N]$ is the dimension distribution of some measure on $\mathbf{R}^N$. This will clearly be accomplished if we exhibit the existence of a complete dimension derivative family, where we define a collection $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, of probability measures on $\mathbf{R}^N$ to be a complete dimension derivative family if, for every probability distribution γ on $[0, N]$, $\{\psi(\cdot, \alpha)\}_{\alpha}$ is the dimension derivative family of some measure μ satisfying $\hat{\mu} = \gamma$. This implies for every choice of γ, the measure

$$\mu(\cdot) = \int_{[0,N]} \psi(\cdot, \alpha)\gamma(d\alpha)$$

satisfies $\hat{\mu} = \gamma$.

The following result describes the structure of such a family. It is not known whether condition 3 is necessary.

THEOREM 4.1. *A collection* $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, *of probability measures on* $\mathbf{R}^N$ *will be a complete dimension derivative family if the following conditions hold*:

1. $\psi(B, \cdot)$ *is a Borel measurable function of* α *for each* $B \in \mathscr{B}(\mathbf{R}^N)$.

2. $\hat{\psi}(\cdot, \alpha) = \delta_{\alpha}$ *for each* α.

3. *There exists a chain* $\{D_{\alpha}\}$, $0 \leqq \alpha \leqq N$, *of Borel sets satisfying* $\dim(D_{\alpha}) = \alpha$ *and* $D_{\alpha} \subseteq D_{\beta}$ *whenever* $\alpha < \beta$ *such that* D_{α} *supports* $\psi(\cdot, \alpha)$ *for each* α.

Furthermore conditions 1 *and* 2 *are also necessary.*

Proof. Sufficiency is immediate by noting that the conditions of Theorem 2.4 are met for any probability distribution γ on $[0, N]$. Condition 1 is obviously necessary. To show the necessity of 2 fix $\alpha \in [0, N]$ and take $\gamma = \delta_{\alpha}$. If $\{\psi(\cdot, \beta)\}_{\beta}$ is complete then the measure

$$\mu(\cdot) = \int_{[0,N]} \psi(\cdot, \beta)\delta_{\alpha}(d\beta) = \psi(\cdot, \alpha)$$

satisfies $\hat{\mu} = \delta_{\alpha}$ and hence $\hat{\psi}(\cdot, \alpha) = \delta_{\alpha}$ as claimed.

Thus we wish to construct a family of measures satisfying the conditions of Theorem 4.1. In order to do this we need to establish Theorems 4.2 and 4.3 which are a generalization to N dimensions of the one-dimensional results given by Billingsley in Theorem 14.1 of [**3**]. We note that while it is possible to arrive at Theorems 4.2 and 4.3 by reformulating the problem in terms of a finite state space stochastic process and applying Theorems 2.2 and 2.4 of Billingsley [**2**] (see also [**1**]) it is simpler and more coherent to develop the results directly using the methods of Billingsley in [**3**].

If $r \geqq 2$ is a fixed positive integer then by an *r-adic interval* in [0, 1] we will mean an interval I_n of the form

$$\begin{aligned} I_n &= I_n(x_1, \dots, x_n) \\ &= \left\{ x \in [0, 1] \,\middle|\, \sum_{j=1}^{n} x_j r^{-j} \leqq x < \sum_{j=1}^{n} x_j r^{-j} + r^{-n} \right\} \end{aligned}$$

where n is a positive integer and the possible values of x_j are $0, 1, \dots, r - 1$. Equivalently $x \in I_n$ if and only if the first n terms of the terminating (if one exists) base r expansion of x coincide with $x_1, \dots, x_n$.

An *r-adic cube* C in the unit cube $[0, 1]^N$ will be a product of N r-adic intervals of equal length, so there will exist n such that

$$C = I_n^1 \times \dots \times I_n^N.$$

If μ is a diffuse probability measure on the Borel sets of $[0, 1]^N$ we define the set function L_μ^α on the subsets of $[0, 1]^N$ by

$$L_\mu^\alpha(E) = \lim_{\delta \to 0^+} L_{\mu,\delta}^\alpha(E)$$

where

$$L_{\mu,\delta}^\alpha(E) = \inf_{\substack{\bigcup_k C_k \supseteq E \\ \mu(C_k) \leqq \delta}} \sum (\mu(C_k))^\alpha$$

and each C_k is an r-adic cube.

By a $\mu - \delta$ covering of E we will mean a countable covering of E by Borel sets $\{S_k\}_k$ satisfying $\mu(S_k) \leqq \delta$ for each k. Since μ is diffuse it follows that $\mu - \delta$ coverings of E by r-adic cubes exist for each $\delta > 0$.

If $\alpha = 1$ then

$$L_\mu^1(E) = \lim_{\delta \to 0^+} \inf_{\substack{\bigcup_k C_k \supseteq E \\ \mu(C_k) \leqq \delta}} \sum \mu(C_k) = \mu^*(E)$$

where

$$\mu^*(E) = \inf_{\bigcup_k C_k \supseteq E} \sum \mu(C_k)$$

is the outer measure of μ constructed using the algebra of finite unions of

r-adic cubes. (Equality of L^1_μ and μ^* follows from the fact that any r-adic cube is the finite disjoint union of smaller cubes each having μ-mass not exceeding δ.) As a consequence L^1_μ is finite. In a proof analogous to that for Hausdorff α-outer measures it can be shown that for each set E there exists a unique value α_0 where $0 \leqq \alpha_0 \leqq 1$ such that $L^\alpha_\mu(E) = \infty$ if $\alpha < \alpha_0$ and $L^\alpha_\mu(E) = 0$ if $\alpha > \alpha_0$. We define

$$\dim_\mu (E) = \alpha_0.$$

If E is a Borel set then

$$L^1_\mu(E) = \mu^*(E) = \mu(E)$$

and so $\mu(E) > 0$ implies

$$\dim_\mu (E) = 1.$$

It is also easily established that

$$\dim_\mu (\bigcup_n E_n) = \sup_n \dim_\mu (E_n).$$

(The one-dimensional analogue of this notion is developed in [**3**]; a somewhat different presentation in terms of stochastic processes is given in [**2**].)

In order that the ratio of logarithms used below in Theorem 4.2 is always defined we adopt the following conventions.

If $0 < a < 1$ and $0 < b < 1$, then

$$\frac{\log a}{\log 0} = \frac{\log 1}{\log b} = \frac{\log 1}{\log 0} = 0,$$

$$\frac{\log 0}{\log b} = \frac{\log a}{\log 1} = \frac{\log 0}{\log 1} = \infty,$$

$$\frac{\log 0}{\log 0} = \frac{\log 1}{\log 1} = 1.$$

THEOREM 4.2. *Let $E \subseteq [0, 1]^N$. Let μ and ν be any two diffuse probability measures on the Borel sets of $[0, 1]^N$. For each $x \in [0, 1]^N$ let $C_n(x)$ denote that unique r-adic cube of volume r^{-nN} which contains x.*

If

$$E \subseteq \left\{ x \middle| \liminf_{n\to\infty} \frac{\log \nu(C_n(x))}{\log \mu(C_n(x))} \geqq \eta \right\}$$

then

$$\dim_\mu (E) \geqq \eta \dim_\nu (E).$$

Proof. It can be shown that Billingsley's proof for the one-dimensional case serves equally well in N dimensions. We present here a modified proof which is somewhat more compact.

If $\eta = 0$ the theorem is trivially true. Thus let $\eta > 0$. It is sufficient to show

$$\dim_\nu (E) \leqq \xi\alpha \text{ whenever } \alpha > \dim_\mu (E) \text{ and } \xi > \frac{1}{\eta}.$$

If $x \in E$ then there exists $N(x)$ such that $n \geqq N(x)$ implies

$$\log \nu(C_n(x)) \leqq \frac{1}{\xi} \log \mu(C_n(x)).$$

Hence for $n \geqq N(x)$ we have

$$\nu(C_n(x))^\xi \leqq \mu(C_n(x))$$

and so

$$\nu(C_n(x))^{\xi\alpha} \leqq \mu(C_n(x))^\alpha.$$

Therefore

$$E = \bigcup_{m=1}^{\infty} E_m$$

where

$$E_m = \bigcap_{n=m}^{\infty} \{x \in E | \nu(C_n(x))^{\xi\alpha} \leqq \mu(C_n(x))^\alpha\}.$$

Since

$$\dim_\nu (E) = \sup_m \dim_\nu (E_m)$$

it is sufficient to prove

$$\dim_\nu (E_m) \leqq \xi\alpha.$$

Let $\delta > 0$ and let $\{C_k\}_k$ be a $\mu - \delta$ covering of E_m by r-adic cubes. Without loss of generality we can assume C_k meets E_m for each k and hence $C_k = C_n(x)$ for some n and some $x \in E_m$. If $n \geqq m$ then let $C'_k = C_k$. If $n < m$ then C_k is the finite disjoint union of no more than r^{mN} subcubes of volume r^{-mN}. We will let $\{C_{kj}\}_j$ denote the collection of those subcubes which also meet E_m. (Hence each $C_{kj} = C_m(x)$ for some $x \in E_m$.) Then the collection $\{D_i\}_i$ consisting of all C'_k and C_{kj} is a $\mu - \delta$ covering of E_m and

$$(\nu(D_i))^{\xi\alpha} \leqq (\mu(D_i))^\alpha \text{ for each } i.$$

Thus $\{D_i\}_i$ is also a $\nu - \delta^{1/\xi}$ covering of E_m and

$$\begin{aligned} \sum (\nu(D_i))^{\xi\alpha} &\leqq \sum (\mu(D_i))^\alpha \\ &= \sum (\mu(C'_k))^\alpha + \sum (\mu(C_{kj}))^\alpha \\ &\leqq r^{mN} \sum_k (\mu(C_k))^\alpha. \end{aligned}$$

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Hence

$$L^{\xi\alpha}_{\nu,\delta^{1/\xi}}(E_m) \leqq r^{mN} L^{\alpha}_{\mu,\delta}(E_m).$$

Letting $\delta \to 0^+$ we conclude

$$L^{\xi\alpha}_{\nu}(E_m) \leqq r^{mN} L^{\alpha}_{\mu}(E_m).$$

But

$$\alpha > \dim_{\mu}(E) \geqq \dim_{\mu}(E_m)$$

so $L^{\alpha}_{\mu}(E_m) = 0$ and hence $L^{\xi\alpha}_{\nu}(E_m) = 0$. Thus

$$\dim_{\nu}(E_m) \leqq \xi\alpha.$$

THEOREM 4.3. *If μ and ν are diffuse probability measures on* $[0, 1]^N$ *and*

$$E \subseteq \left\{ x \,\middle|\, \lim_{n\to\infty} \frac{\log \nu(C_n(x))}{\log \mu(C_n(x))} = \eta \right\}$$

then $\dim_{\mu}(E) = \eta \dim_{\nu}(E)$.

Proof. The result follows by applying Theorem 4.2 then interchanging ν and μ and applying Theorem 4.2 again with $1/\eta$ in place of η.

In order to apply Theorem 4.2 and 4.3 for our purpose we need to note the relationship between $\dim_{\lambda}(E)$ and $\dim(E)$. In fact

$$\dim_{\lambda}(E) = \dim(E)|N,$$

simply a change of scale due to the fact that L^{α}_{λ} is calculated in terms of the Lebesgue measures of the members of the coverings while H^{α} is calculated in terms of the diameters. It can be shown that using coverings by r-adic cubes or by balls does not affect the value obtained for the dimension.

THEOREM 4.4. *A complete dimension derivative family exists on* $\mathbf{R}^N$.

Proof. We will construct a family $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, on the unit cube $[0, 1]^N$. To each real number $x \in [0, 1]$ associate its terminating (if one exists) dyadic expansion

$$x = \sum_{j=1}^{\infty} x(j) 2^{-j}$$

where $x(j) = 0$ or 1. Let

$$s_n(x) = \sum_{j=1}^{n} x(j).$$

For each $0 \leqq p \leqq \frac{1}{2}$ define the set $M(p)$ in $[0, 1]^N$ by

$$M(p) = \left\{ (x_1, \ldots, x_n) \in [0, 1]^N \,\middle|\, \lim_{n\to\infty} \frac{s_n(x_k)}{n} = p \text{ for each } k = 1, \ldots, N \right\}.$$

Let

$$M^*(p) = \bigcup_{0 \leqq p' \leqq p} M(p').$$

The collection $\{M^*(p)\}$, $0 \leqq p \leqq \frac{1}{2}$, is a nested increasing family of sets; we will show shortly that

$$\dim(M^*(p)) = Nd(p)$$

where

$$d(p) = \frac{p \log\left(\frac{1}{p}\right) + (1 - p) \log\left(\frac{1}{1 - p}\right)}{\log 2}.$$

Let

$$C_n = I_n^1 \times \ldots \times I_n^N$$

be a dyadic cube of volume 2^{-nN}. By definition

$$I_n^k = I_n^k(x, \ldots, x_n)$$

for some choice of $x_1, \ldots, x_n$ (which depends on k). We will let

$$s_n^k = s_n^k(I_n^k) = \sum_{j=1}^{n} x_j.$$

For each $0 \leqq p \leqq \frac{1}{2}$ let μ^p denote the N-dimensional product (Bernoulli) measure on $[0, 1]^N$ whose value over dyadic cubes is given by

$$\begin{aligned}
\mu^p(C_n) &= \mu^p(I_n^1 \times \ldots \times I_n^N) \\
&= (p^{s_n^1}(1 - p)^{n - s_n^1}) \ldots (p^{s_n^N}(1 - p)^{n - s_n^N}) \\
&= p^{\sum_{k=1}^{N} s_n^k} (1 - p)^{nN - \sum_{k=1}^{N} s_n^k}.
\end{aligned}$$

It is well-known (and easily proved by applying the strong law of large numbers) that μ^p is supported on $M(p)$ and hence also on $M^*(p)$. μ^p is a diffuse singular measure for $0 < p < \frac{1}{2}$ while $\mu^0 = \delta_0$ and $\mu^{1/2} = \lambda$.

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Note that if $\underline{x} \in [0, 1]^N$ and $C_n(\underline{x})$ is that unique dyadic cube of volume 2^{-nN} which contains $\underline{x}$ then

$$\frac{\log \lambda(C_n(\underline{x}))}{\log \mu^p(C_n(\underline{x}))}$$

$$= \frac{nN \log\left(\frac{1}{2}\right)}{\sum_{k=1}^{N} s_n(x_k) \log p + \left(nN - \sum_{k=1}^{N} s_n(x_k)\right) \log(1 - p)}$$

$$= \frac{N \log 2}{\sum_{k=1}^{N} \frac{s_n(x_k)}{n} \log\left(\frac{1}{p}\right) + \sum_{k=1}^{N} \left[1 - \frac{s_n(x_k)}{n}\right] \log\left(\frac{1}{1-p}\right)}.$$

If $\underline{x} \in M(p)$ then

$$\lim_{n\to\infty} \frac{s_n(x_k)}{n} = p$$

for each k and so

$$\lim_{n\to\infty} \frac{\log \lambda(C_n(\underline{x}))}{\log \mu^p(C_n(\underline{x}))} = \frac{N \log 2}{Np \log\left(\frac{1}{p}\right) + N(1 - p) \log\left(\frac{1}{1-p}\right)}$$

$$= 1/d(p).$$

Thus

$$M(p) \subseteq \left\{\underline{x} \in [0, 1]^N \,\middle|\, \lim_{n\to\infty} \frac{\log \lambda(C_n(\underline{x}))}{\log \mu^p(C_n(\underline{x}))} = 1/d(p)\right\}$$

and applying Theorem 4.3 we conclude

$$\dim_\lambda (M(p)) = d(p) \dim_{\mu^p} (M(p)).$$

As $\mu^p(M(p)) > 0$ we have

$$\dim_{\mu^p} (M(p)) = 1$$

and thus

$$\dim(M(p)) = N \dim_\lambda (M(p)) = Nd(p).$$

It is easily seen that if $0 \leqq p' \leqq p \leqq \frac{1}{2}$ then

$$d(p) \geqq \frac{p' \log\left(\frac{1}{p}\right) + (1 - p') \log\left(\frac{1}{1-p}\right)}{\log 2}.$$

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Thus if $\underline{x} \in M^*(p)$ (and hence there exists $p' \leqq p$ such that

$$\lim_{n\to\infty} \frac{s_n(x_k)}{n} = p'$$

for all k) we obtain

$$\lim_{n\to\infty} \frac{\log \lambda(C_n(\underline{x}))}{\log \mu^p(C_n(\underline{x}))} = \frac{\log 2}{p' \log\left(\frac{1}{p}\right) + (1 - p') \log\left(\frac{1}{1-p}\right)}$$
$$\geqq 1/d(p).$$

Hence

$$M^*(p) \subseteq \left\{\underline{x} \in [0, 1]^N \middle| \lim_{n\to\infty} \frac{\log \lambda(C_n(\underline{x}))}{\log \mu^p(C_n(\underline{x}))} \geqq 1/d(p)\right\}$$

and from Theorem 4.2 we conclude

$$\dim(M^*(p)) = N \dim_\lambda (M^*(p))$$
$$\leqq Nd(p) \dim_{\mu^p} (M^*(p)) = Nd(p).$$

But also

$$\dim(M^*(p)) \geqq Nd(p)$$

since $M^*(p) \supseteq M(p)$. Thus

$$\dim(M^*(p)) = Nd(p).$$

Furthermore if $\mu^p(A) > 0$ then

$$\mu^p(A \cap M(p)) > 0$$

and since

$$A \cap M(p) \subseteq \left\{\underline{x} \in [0, 1]^N \middle| \lim_{n\to\infty} \frac{\log \lambda(C_n(\underline{x}))}{\log \mu^p(C_n(\underline{x}))} = 1/d(p)\right\}$$

we conclude

$$\dim(A) \geqq \dim(A \cap M(p)) = Nd(p).$$

Thus μ^p has no mass on any set of dimension less than $Nd(p)$ yet is supported on $M^*(p)$. Hence μ^p is of exact dimension $Nd(p)$; that is, letting $d^*(p) = Nd(p)$ we have

$$\hat{\mu}^p = \delta_{d^*(p)}.$$

Noting that d^* maps the interval $\left[0, \frac{1}{2}\right]$ onto $[0, N]$ in a one-one strictly increasing continuous fashion we define the probability measures

$\psi(\cdot, \alpha)$ on $[0, 1]^N$ for $0 \leqq \alpha \leqq N$ by

$$\psi(\cdot, \alpha) = \mu^{(d^*)^{-1}(\alpha)}.$$

Note that $\hat{\psi}(\cdot, \alpha) = \delta_\alpha$ and $\psi(\cdot, \alpha)$ is supported on

$$D_\alpha = M^*((d^*)^{-1}(\alpha));$$

the chain $\{D_\alpha\}$, $0 \leqq \alpha \leqq N$, satisfies 3 of Theorem 4.1. It is also easily verified that $\psi(B, \cdot)$ is a Borel measurable function of α by noting that $\psi(C_n, \cdot)$ is in fact continuous for any dyadic cube C_n.

From Theorem 4.1 we conclude $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, is a complete dimension derivative family.

COROLLARY 4.4.1. (The Existence Theorem). *Let γ be any probability distribution on the Borel sets of* $[0, N]$. *Then there exists a probability distribution μ defined over the Borel sets of the unit cube* $[0, 1]^N$ *such that $\hat{\mu} = \gamma$.*

Proof. Let $\{\psi(\cdot, \alpha)\}$, $0 \leqq \alpha \leqq N$, be the complete dimension derivative family provided by Theorem 4.4. Define

$$\mu(\cdot) = \int_{[0,N]} \psi(\cdot, \alpha)\gamma(d\alpha).$$

5. A characterization of the dimension concentration map. In this section we develop an alternative form of the dimension concentration map which clarifies the meaning of the dimensional number $\hat{\alpha}(x)$ and the structure of α-supports. We use the following result, a version of Theorem 2.1 in [**2**].

THEOREM 5.1. *Let μ and ν be two probability measures on* $[0, 1]^N$ *such that ν is diffuse. Then*

$$\dim_\nu\left(\left\{x \,\middle|\, \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \nu(C_n(x))} \leqq \delta\right\}\right) \leqq \delta,$$

where $C_n(x)$ is the r-adic cube of volume r^{-nN} containing x.

Proof. Let

$$E = \left\{x \,\middle|\, \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \nu(C_n(x))} \leqq \delta\right\}.$$

Let $\delta' > \delta$ and n_0 any positive integer. Let $\mathscr{C}$ be the collection of r-adic cubes $\{C_k\}_k$ for which $C_k = C_n(x)$ for some $n \geqq n_0$ and such that

$$\frac{\log \mu(C_k)}{\log \nu(C_k)} < \delta'.$$

As $x \in E$ implies

$$\inf_{n\geqq n_0} \frac{\log \mu(C_n(x))}{\log \nu(C_n(x))} < \delta'$$

it follows that $\mathscr{C}$ is a covering of E. Furthermore as $\mathscr{C}$ consists of r-adic cubes we can choose a subcollection $\mathscr{C}'$ of $\mathscr{C}$ consisting of disjoint cubes such that $\mathscr{C}'$ covers E. We obtain

$$\sum_{\mathscr{C}'} \nu(C_k)^{\delta'} < \sum_{\mathscr{C}'} \mu(C_k) = \mu(\bigcup_{\mathscr{C}'} C_k) \leqq 1.$$

It follows that, setting $\epsilon_{n_0} = \sup\{\nu(C) \,|\, C = C_n \text{ for some } n \geqq n_0\}$

$$L^{\delta'}_{\nu,\epsilon_{n_0}}(E) \leqq 1$$

and as n_0 was arbitrary we conclude

$$L^{\delta'}_{\nu}(E) \leqq 1.$$

Hence

$$\dim_{\nu}(E) \leqq \delta'.$$

Since this holds for each $\delta' > \delta$ we have

$$\dim_{\nu}(E) \leqq \delta.$$

THEOREM 5.2. *Let μ be a probability measure on the Borel sets of* $[0, 1]^N$ *and let $\hat{\alpha}$ denote the dimension concentration map determined by μ. Then*

$$\hat{\alpha}(x) = N \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \lambda(C_n(x))} \quad \mu\text{-a.e.}$$

Proof. Let

$$M_{\alpha} = \left\{ x \,\middle|\, N \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \lambda(C_n(x))} \leqq \alpha \right\}.$$

We will show that $\{M_{\alpha}\}$, $0 \leqq \alpha \leqq N$, is a right-continuous α-support chain of μ. From the definition of $\hat{\alpha}$ and (ii) of Theorem 2.1 the result will then follow.

Clearly $\{M_{\alpha}\}_{\alpha}$ is a nested chain and $M_{\alpha} = M_{\alpha}^{+}$. From Theorem 5.1 we also have

$$\dim_{\lambda}(M_{\alpha}) \leqq \alpha/N$$

and hence

$$\dim(M_{\alpha}) \leqq \alpha.$$

Thus we need only show that M_{α} supports μ_{α}. If $\mu_{\alpha} = \underline{0}$, the zero measure, there is nothing to prove. Therefore assume

$$\mu_{\alpha}([0, 1]^N) > 0$$

and let

$$M^{\beta} = \left\{ x \,\middle|\, N \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \lambda(C_n(x))} \geqq \beta \right\}.$$

As

$$M^\beta \uparrow [0, 1]^N \backslash M_\alpha \text{ as } \beta \to \alpha^+$$

we will show that

$$\mu_\alpha([0, 1]^N \backslash M_\alpha) = 0$$

by proving $\mu_\alpha(M^\beta) = 0$ for $\beta > \alpha$ and hence M_α will be a support of μ_α.

Let D_α be any α-support of μ. If $\mu_\alpha(M^\beta) > 0$ for some $\beta > \alpha$ then

$$\mu_\alpha(M^\beta) = \mu(M^\beta \cap D_\alpha) > 0.$$

Thus

$$\dim_\mu (M^\beta \cap D_\alpha) = 1.$$

But

$$M^\beta \cap D_\alpha \subseteq \left\{ x \,\middle|\, \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \lambda(C_n(x))} \geqq \beta/N \right\}$$

and so Theorem 4.2 implies

$$\dim(M^\beta \cap D_\alpha) = N \dim_\lambda (M^\beta \cap D_\alpha) \geqq \beta \dim_\mu (M^\beta \cap D_\alpha) = \beta$$

which contradicts the fact that

$$\dim(M^\beta \cap D_\alpha) \leqq \alpha.$$

Thus we must have

$$\mu_\alpha(M^\beta) = 0 \text{ for every } \beta > \alpha.$$

Remark. In **[5]** Gács considered the function

$$f(x) = \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log |C_n(x)|}$$

where $C_n(x)$ is a dyadic cube and $|C_n(x)| = N2^{-n}$ is the sum of edges of $C_n(x)$. Given a probability measure μ he defined a numerical quantity (which we will denote by $d(\mu)$) called the Hausdorff dimension of μ; he demonstrated that

$$d(\mu) = \int_{\mathbf{R}^N} f(x)\mu(dx)$$

and for this reason called $f(x)$ the dimension density of μ. In fact it is easy to see that $f(x) = \hat{\alpha}(x)$ and since $\hat{\mu} = \mu\hat{\alpha}^{-1}$ we obtain

$$d(\mu) = \int_{\mathbf{R}^N} \hat{\alpha}(x)\mu(dx) = \int_{[0,N]} \alpha\hat{\mu}(d\alpha)$$

which is simply the mean of $\hat{\mu}$. We prefer the term dimension concentration map rather than density as in this context the latter term conveys the erroneous impression that $\hat{\alpha}$ is the Radon-Nikodym derivative of $\hat{\mu}$ with respect to Lebesgue measure.

Note that we can now express

$$\hat{\mu}([0, \alpha]) = \mu\left(\left\{x \,\middle|\, N \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \lambda(C_n(x))} \leqq \alpha\right\}\right)$$

which is somewhat more tractable than the original supremum definition. In [**4**] Cutler has discussed the measurability of the map $\mu \to \hat{\mu}$ considered as a function on the space of measures under the topology of weak convergence; some applications to measure-valued stochastic processes are given.

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