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SEPARATING POINTS OF $\beta\mathbb{N}$ BY MINIMAL FLOWS

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ABSTRACT. We consider minimal left ideals L of the universal semigroup compactification βS of a topological semigroup S. We show that the enveloping semigroup of L is homeomorphically isomorphic to βS if and only if given $q \neq r$ in βS, there is some p in the smallest ideal of βS with $qp \neq rp$. We derive several conditions, some involving minimal flows, which are equivalent to the ability to separate q and r in this fashion, and then specialize to the case that $S = \mathbb{N}$, and the compactification is $\beta\mathbb{N}$. Included is the statement that some set A whose characteristic function is uniformly recurrent has $q \in \mathrm{cl}(A)$ and $r \notin \mathrm{cl}(A)$.

Consider a flow with compact phase space X and semigroup (or group) of transformations S. One measure of the complexity of the flow is the complexity of the algebraic and topological structure of the enveloping semigroup of the flow. For example, if all transformations in the flow consist of contractions toward one fixed point of X, then the constant mapping to that point is the only additional transformation in the enveloping semigroup which was not already in S. At the other extreme is the case that the enveloping semigroup is the universal semigroup compactification of S (the enveloping semigroup is always a semigroup compactification of S). Even in the simple case that S is $(\mathbb{N}, +)$, the natural numbers under addition acting as a transformation and its powers, the universal compactification $(\beta\mathbb{N}, +)$ already exhibits an extremely complex structure, both topologically and algebraically.

The major problem that we investigate in this work is the extent to which the full complexity of the enveloping semigroup can be exhibited in a *minimal* flow. In particular, we consider the question of whether $\beta\mathbb{N}$ can be the enveloping semigroup of a minimal flow. We are unable to answer this question, but we do present some interesting alternate formulations involving the cancellative structure of the semigroup $\beta\mathbb{N}$ (or more generally the universal semigroup compactification). Indeed the answer is affirmative if and only if given $q \neq r$, there exists p in the smallest ideal of $\beta\mathbb{N}$ such that $q + p \neq r + p$. More generally, we feel that the investigation of the enveloping semigroup structure of a minimal flow is a natural and worthy one, and one that additionally suggests interesting corresponding problems of a purely semigroup theoretic nature.

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1. **Preliminaries.** Let S be a semigroup equipped with a Hausdorff topology. For $s,t \in S$ with product st, we write

$$\rho_t(s) = st = \lambda_s(t);$$

the functions ρ_t and λ_s on S are called *right* and *left translations* respectively. The multiplication is said to be *right continuous* and the semigroup S is said to be *right topological* if all right translations are continuous (in which case multiplication is continuous in the *left* variable); in this case an important role is played by the subsemigroup $\Lambda(S) := \{s \in S : \lambda_s \text{ is continuous}\}$. The semigroup S is *semitopological* if for all left and right translations are continuous and *topological* if the multiplication function itself is continuous.

Let S be a semitopological semigroup. A compact right topological semigroup T, together with a continuous homomorphism $\varphi\colon S \longrightarrow T$, is called a *semigroup compactification* if $\varphi(S)$ is dense in T and the action of S on T,

$$(s,t) \longmapsto s.t = \varphi(s)t\colon S \times T \longrightarrow T$$

is continuous (Ruppert [12] calls these dynamical compactifications). The semigroup compactification is *universal* if all others are homomorphic images, where the homomorphism makes the appropriate diagram commute. It is shown in [12] or [13] that universal semigroup compactifications exist and are unique up to topological isomorphism.

If S is a discrete semigroup, then one obtains the universal semigroup compactification of S by taking the Stone-Čech compactification βS endowed (uniquely) with a semigroup multiplication extending that of S and such that βS is a right topological semigroup and λ_s is continuous for each $s \in S$ (see *e.g.* [11]). In this case βS may be identified with the set of ultrafilters on S, and S then embeds as a subsemigroup of βS by identifying each point with the principal ultrafilter it determines. The special case that S is the additive semigroup $(\mathbb{N},+)$ and the compactification is $(\beta\mathbb{N},+)$ has been extensively studied and is of particular interest. In the literature, one standardly sees a left-right switch in the type of continuity considered here, but the study of the enveloping semigroup makes the right topological semigroups natural for our considerations.

An *S-flow* is a triple (S,X,π) such that X is a compact Hausdorff space, called the *phase space*, and $\pi\colon S\times X \longrightarrow X$ is a continuous action of S on X; we write $\pi(s,x) = s.x = \pi^s(x)$. A particular important type of flow is the case of a semigroup compactification (T,φ), with the standard action $(s,t) \longmapsto s.t$ of S on T.

The notion of the enveloping semigroup $\epsilon(S,X)$ of a flow was introduced by R. Ellis ([4], or see [5]). It is the closure of the semigroup of mappings $\{\pi^s : s \in S\}$ in the compact product space X^X of all functions from X into X, and is a compact right topological subsemigroup (where X^X is endowed with the semigroup operation of composition) in the relative topology. Furthermore, the mapping $s \longmapsto \pi^s\colon S \longrightarrow \epsilon(S,X)$ is a semigroup compactification of S.

Let $\psi\colon S \longrightarrow \beta S$ be the universal semigroup compactification of a topological semigroup S. In general one does not expect the universal semigroup compactification of S to be the Stone-Čech compactification. However, as we have seen, this is true in the case where S is discrete, which is the case of most interest to us, so we denote the universal semigroup compactification of S by βS. Let (S, X, π) be a flow. Then the mapping $\varphi(s) := \pi^s$ from S to $\epsilon(S, X)$ is a semigroup compactification, and thus there exists a unique continuous semigroup homomorphism $\Phi \colon \beta S \longrightarrow \epsilon(S, X)$ such that $\Phi \circ \psi = \varphi$; the mapping Φ is called the *canonical homomorphism*. Since the image of S in $\epsilon(S, X)$ is dense, the mapping Φ is surjective. The function $(s, x) \longmapsto s.x := \Phi(s)(x)$ is then a right continuous action of βS on X, called the *extended action*. Conversely if one is given the extended action, then the canonical homomorphism Φ may be recovered by $\Phi(s) \colon X \longrightarrow X$ where $\Phi(s)$ is defined by $\Phi(s)(x) = s.x$. Thus the extended action and the canonical homomorphism of βS onto $\epsilon(S, X)$ uniquely determine each other.

LEMMA 1.1. *The canonical homomorphism $\Phi \colon \beta S \longrightarrow \epsilon(S, X)$ is a topological isomorphism if and only if the extended action is effective, i.e., given $s \neq t \in \beta S$, there exists $x \in X$ such that $s.x \neq t.x$.*

PROOF. Suppose Φ is a topological isomorphism. Then $\Phi(s) \neq \Phi(t)$ for $s \neq t$. Thus there exists $x \in X$ such that $\Phi(s)(x) \neq \Phi(t)(x)$, *i.e.*, $s.x \neq t.x$. Conversely, suppose that the extended action is effective. From the preceding remarks Φ is a continuous surjective homomorphism and is closed by compactness considerations. Thus we need only show that Φ is injective. But the argument just given may be reversed to obtain this. ∎

REMARK. The enveloping semigroup gives some measure of the complexity of the action of S on X. In particular, if the canonical homomorphism is actually an isomorphism, then the enveloping semigroup reflects the full complexity of the universal semigroup compactification. The lemma states that this is equivalent to the extended action being effective. The major problem of interest in this paper is whether this can happen for minimal flows.

2. **Enveloping semigroups of left ideals.** A non-empty subset L of a semigroup S is called a *left ideal* if $SL \subseteq L$. Right ideals are defined dually; a set that is both a left ideal and right ideal is called an *ideal*. The left ideal L is a *minimal left ideal* if it is minimal in the set of left ideals. By elementary semigroup theory, L is a minimal left ideal if and only if $L = Lx = Sx$ for each $x \in L$.

If (T, α) is a semigroup compactification of a topological semigroup S, then any closed left ideal of T determines an S-flow $(s, y) \longmapsto \alpha(s)y \colon S \times L \longrightarrow L$.

REMARK 2.1. In the case that L is a closed left ideal in the universal semigroup compactification βS of S, the extended action of βS on L for the flow (S, L) is just multiplication in βS, *i.e.*, $s.v = sv$ for $s \in \beta S$ and $v \in L$. This may be verified directly from the right continuity of the extended action, the right continuity of multiplication in βS, and the density of the image of S in βS.

The next proposition follows directly from Lemma 1.1.

COROLLARY 2.2. *Let S be a topological semigroup, and let L be a closed left ideal of the universal compactification βS. If there exists a point p of L at which right cancellation holds, then the canonical homomorphism $\Phi\colon \beta S \longrightarrow \epsilon(S,L)$ is a topological isomorphism.* ■

It is known that there are many points $p \in \beta\mathbb{N}$, the universal semigroup compactification of $(\mathbb{N},+)$, at which right cancellation holds [8]. In fact there are points in the closure of the smallest ideal in $\beta\mathbb{N}$ at which right cancellation holds. Then the left ideal $\beta\mathbb{N}+p$ generated by each such point is a copy of $\beta\mathbb{N}$, and by Corollary 2.2 the canonical homomorphism onto the enveloping semigroup is a topological isomorphism. As a consequence of recent results of [14], there is a whole downward chain of type ω_1 of such left ideals contained in the closure of the smallest ideal. On the other hand, if L is a minimal left ideal of an arbitrary semigroup S and L is proper in S, then right cancellation fails at every point of L. (Indeed let $p \in L$ and $s \in S \setminus L$. Then $sp \in L$. By minimality $Lp = L$, so there exists $t \in L$ such that $tp = sp$, *i.e.*, right cancellation fails at p.) Compact right topological semigroups always have a smallest ideal K, and a left ideal is minimal if and only if it is contained in K.

We recall some of the basic structure theory of compact right topological semigroups [3]. As just mentioned, such a semigroup S has a (unique) smallest ideal K which can be represented as a disjoint union of minimal left ideals as well as a disjoint union of minimal right ideals (such ideals are said to be *completely simple* in semigroup parlance). For each minimal left ideal L with $p \in L$, we have $L = Sp = Lp$, and hence L is a closed principal left ideal. The intersection of each minimal left ideal with each minimal right ideal is a (nonempty) group, and K is the disjoint union of these groups.

THEOREM 2.3. *Let L be a minimal left ideal and let K be the smallest ideal in a compact right topological semigroup T. Let $q \neq r \in T$. The following are equivalent:*

(1) There exists $p \in L$ such that $qp \neq rp$.

(2) There is an idempotent $u = u^2 \in L$ with $qu \neq ru$.

(3) There is an idempotent $e \in K$ such that $qe \neq re$.

(4) There exists $p \in K$ such that $qp \neq rp$.

PROOF. The implications (2) implies (3) implies (4) and (2) implies (1) implies (4) are trivial. We thus assume (4) and derive (2). By the preceding comments $p \in R$ for some minimal right ideal R. Then $L \cap R$ is a group and hence contains some idempotent u. Since R is a minimal right ideal, $uR = R$. Then $p = ut$ for some $t \in R$, and $up = uut = ut = p$. If $qu = ru$, then $qp = qup = rup = rp$, a contradiction. So $qu \neq ru$. ■

DEFINITION 2.4. Let T be a semigroup which has a smallest ideal K. The semigroup T is right K-reductive if given $q \neq r \in T$, there exists $p \in K$ such that $qp \neq rp$. Note that Theorem 2.3 gives other equivalences to this definition.

We recall some elementary facts about minimal flows (see, *e.g.*, [11]). The flow (S,X) is minimal if the orbit Sp is dense in X for every $p \in X$. This is equivalent to $\beta S.p = X$ for every $p \in X$ for the extended action. Suppose that (T,α) is a semigroup compactification of the topological semigroup S. Then the flow (S, L) for a closed left ideal L of T is a minimal flow if and only if L is a minimal left ideal of T. In particular, for each minimal left ideal L of the universal semigroup compactification βS, the flow (S, L) is a minimal flow (and is actually a universal minimal flow).

THEOREM 2.5. *Let S be a topological semigroup, and let $(\beta S, \psi)$ be the universal semigroup compactification. The following are equivalent:*

(1) The semigroup βS is right K-reductive.

(2) If L is a minimal left ideal of βS, then the canonical homomorphism for the minimal flow (S, L) is an isomorphism, and hence the enveloping semigroup of this minimal flow is topologically isomorphic to βS.

3. Given $q \neq r \in \beta S$, there exists a minimal flow (S, X) and $x \in X$ such that $q.x \neq r.x$ with respect to the extended action.

PROOF. (1) $\Rightarrow$ (2): By Lemma 1.1 we need only show that the extended action of βS on L is effective. But this follows directly from hypothesis and Theorem 2.3, in light of Remark 2.1.

(2) $\Rightarrow$ (3): Take for the minimal flow (S, L) and apply Lemma 1.1.

(3) $\Rightarrow$ (1): Let $q \neq r \in \beta S$, and let (S, X) be a minimal flow such that $q.x \neq r.x$ for some $x \in X$. Let L be a minimal left ideal in βS. Then $L.x$ is closed and invariant under S, and hence must be all of X since the flow is minimal. Thus $v.x = x$ for some $v \in L$. Then $qv = rv$ would imply $q.x = qv.x = rv.x = r.x$, a contradiction. So $qv \neq rv$, and thus βS is right K-reductive. ∎

The proof of Theorem 2.5 carries over to prove a corresponding local version, which is useful to have on record.

THEOREM 2.6. *Let S be a topological semigroup, and let $(\beta S, \psi)$ be the universal semigroup compactification. Let $q \neq r \in \beta S$. The following are equivalent:*

(1) There exists $p \in K$, the smallest ideal of βS, such that $qp \neq rp$.

(2) If L is a minimal left ideal of βS, then q and r have distinct images under the canonical homomorphism for the minimal flow (S, L).

(3) There exists a minimal flow (S, X) and $x \in X$ such that $q.x \neq r.x$ with respect to the extended action. ∎

3. **Separating points of βS in K.** Let S be a topological semigroup with universal semigroup compactification βS. In light of Theorems 2.5 and 2.6, we want to know, given $q \neq r \in \beta S$, whether there is some $p \in K(\beta S)$, the smallest ideal of βS, such that $qp \neq rp$.

We recall the definition of an almost periodic point (see [7] or [11]).

DEFINITION 3.1. Let (S, X) be a flow. A point $p \in X$ is an almost periodic point (resp. uniformly recurrent point) if given any neighborhood U of p, there exists a compact (resp. finite) subset K of S such that given $s \in S$, there exists $k \in K$ with $ksp \in U$.

REMARK. The uniformly recurrent points are precisely those points that are almost periodic for the flow (S, X) when S is endowed with the discrete topology.

We recall the following result from [11].

THEOREM 3.2. *Let (S,X) be a flow, $x \in X$, and L a minimal left ideal in the universal semigroup compactification βS. The following are equivalent:*

(1) The orbit closure $\overline{Sx}$ contains x and is a minimal S-flow.

(2) For the extended action of βS on X, there exists $t \in K$, the smallest ideal in βS, and $y \in X$ such that $x = t.y$.

(3) $x \in L.x$.

(4) There exists an idempotent $e \in L$ such that $e.x = x$.

(5) $\overline{Sx} = L.x$.

(6) The point x is almost periodic.

(7) The point x is uniformly recurrent.

PROOF. The equivalence of the first five items is just Theorem 3.5 of [11] and equivalence (6) is Theorem 4.2 in the same reference. The equivalence of (7) follows by converting to the discrete flow and noting that condition (1) is independent of the topology on S.

We add to the string of equivalences of Theorem 2.6. (Observe that condition (1) of Theorem 3.3 and condition (1) of Theorem 2.6 are equivalent by Theorem 2.3.)

THEOREM 3.3. *Let S be a topological semigroup with universal compactification βS, and let $q \neq r \in \beta S$. Let L be a minimal left ideal of βS. The following are equivalent:*

(1) There exists $p \in L$ such that $qp \neq rp$.

(2) There exists a flow (S,X) and an almost periodic (resp. uniformly recurrent) point $x \in X$ such that $q.x \neq r.x$ for the extended action.

(3) Let (S,Y) be any flow for which the canonical homomorphism from βS onto $\epsilon(S,Y)$ is an isomorphism. Then there exists an almost periodic (resp. uniformly recurrent) point $y \in Y$ such that $q.y \neq r.y$.

PROOF. (1) $\Rightarrow$ (2): Consider the minimal flow (S,L). Then by hypothesis $qp \neq rp$. Since L is minimal, $Lp = L$, so $p \in Lp$, and by Theorem 3.2, p is almost periodic and uniformly recurrent.

(2) $\Rightarrow$ (1): Let (S,X) be a flow, and let x be an almost periodic (or uniformly recurrent) point in X such that $q.x \neq r.x$. By Theorem 3.2, $x \in L.x$, so there exists $p \in L$ such that $x = p.x$. Then $qp.x = q.p.x = q.x \neq r.x = r.p.x = rp.x$, so $qp \neq rp$.

(1) $\Rightarrow$ (3): Let (S,Y) be a flow for which the canonical homomorphism is an isomorphism. By Lemma 1.1, there exists $y \in Y$ such that $q.p.y = qp.y \neq rp.y = r.p.y$, and by Theorem 3.2 the point $p.y$ is almost periodic and recurrent (since condition (2) holds.)

(3) $\Rightarrow$ (1): Consider the flow $(S,\beta S^1)$ of the universal semigroup compactification $(\beta S,\psi)$, where 1 is a discrete point added to βS, and $s.1 = \psi(s)$ for all $s \in S$. The multiplication of βS extends to βS^1 by making 1 act as an identity of βS, and βS^1 remains a right topological semigroup. Then for $q \neq r \in \beta S$, we have $q.1 = q \neq r = r.1$, so by Lemma 1.1, the canonical homomorphism is an isomorphism. By hypothesis, there exists an almost periodic point p in βS^1 such that $qp \neq rp$. By Theorem 3.2, $p \in Lp$. Thus there exists $u \in L$ such that $p = up$. Then $qup = qp \neq rp = rup$, and so $qu \neq ru$. ∎

4. **The case of βN.** We specialize now to the case that the semigroup under consideration is the positive integers $(\mathbb{N}, +)$ under addition with universal semigroup compactification $\beta\mathbb{N}$, the space of ultrafilters on $\mathbb{N}$. See [10] for an elementary description of the semigroup $(\beta\mathbb{N}, +)$.

For our purposes, it will be convenient to identify the semigroup and its compactification with a specific flow.

EXAMPLE 4.1. Let $X := \{0, 1\}^{\mathbb{N}}$ be the countable product of the two-point discrete space, and define the shift operator T on X by $T(x)(n) = x(n+1)$ for $x \in X$. If members of X are viewed as infinite tuples, then T shifts a tuple one place to the left, discarding the first entry. We consider the semigroup of continuous functions $\{T^n : n \in \mathbb{N}\}$, and let $E := \epsilon(\{T^n : n \in \mathbb{N}\}, X)$ denote the enveloping semigroup. Define $\varphi \colon \mathbb{N} \longrightarrow E$ by $\varphi(n) = T^n$, and also denote by φ its extension, the canonical homomorphism from $\beta\mathbb{N} \longrightarrow E$.

The following result is a slight modification of a result of Ruppert [13].

THEOREM 4.2. *The canonical homomorphism $\varphi \colon \beta\mathbb{N} \longrightarrow E$ of Example 4.1 is a topological isomorphism.*

PROOF. We apply Lemma 1.1. Let p and q be distinct ultrafilters in $\beta\mathbb{N}$. Since φ clearly separates points of $\mathbb{N}$, we may assume without loss of generality that p is not a principal ultrafilter. Hence there exists an infinite set $A \subseteq \mathbb{N}$ such that A is in the ultrafilter p, and its complement is in q. Set $B = A + 1$, and let $x \in X$ be the characteristic function χ_B of B.

For each $n \in A$, $\varphi(n)(x) = T^n(x)$ and $T^n(x)(1) = x(n+1) = 1$ since $n + 1 \in B$ and x is the characteristic function of B. Since p is in the closure of A, we conclude that $p.x(1) = \big(\varphi(p)(x)\big)(1) = 1$. Similarly if $n \notin A$, then $\varphi(n)(x)(1) = 0$, and so $q.x(1) = 0$. Thus $p.x \neq q.x$, and so φ is topological isomorphism by Lemma 1.1. ∎

REMARK 4.3. In light of Theorems 3.3 and 4.2, the problem of K-separating two distinct points q and r in $\beta\mathbb{N}$ reduces to taking their distinct images in the enveloping semigroup E of $\{T^n : n \in \mathbb{N}\}$ and finding a uniformly recurrent point in X at which they do not agree. (The referee observed: Uniformly recurrent points can be characterized also as those characteristic functions χ_A on $\mathbb{N}$ which are almost periodic functions. If χ_A is almost periodic then A is usually also called an almost periodic subset of $\mathbb{N}$.)

We recall from Definition 3.1 with $S = \mathbb{N}$ that a point x in X is uniformly recurrent with respect to T if and only if given any neighborhood U of x there is some $b \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ some $i \in \{1, 2, 3, \ldots, b\}$ satisfies $T^{n+1}(x) \in U$. We also recall that a string x of 0's and 1's which is a uniformly recurrent point in X can be alternately described as one with the property that given any positive integer k, there exists another positive integer $n(k)$ such that any substring of x of length $n(k)$ contains within a string of length k which agrees with the string of the first k-entries of x (see *e.g.* [6]).

LEMMA 4.4. *Given $p \in \beta\mathbb{N}$, $x \in X$, $n \in \mathbb{N}$, then $\varphi(p)(x)(n) = 1$ if and only if $x^{-1}(\{1\}) - n \in p$.*

PROOF. Let $p \in \beta\mathbb{N}$, $x \in X$ and $n \in \mathbb{N}$ be given. Let $A = x^{-1}[\{1\}]$. Let $i = \varphi(p)(x)(n)$ and let $U = \{y \in X : y(n) = i\}$. Then U is a neighborhood of $\varphi(p)(x)$ so that $V = \{f \in E : f(x) \in U\}$ is a neighborhood of $\varphi(p)$. Pick $B \in p$ such that $\varphi[\bar{B}] \subseteq V$.

We show that if $i = 0$ then $B \cap (A - n) = \emptyset$ and if $i = 1$ then $B \subseteq A - n$. Let $m \in B$ be given. Then $T^m = \varphi(m) \in V$ so that $T^m(x) \in U$ and $T^m(x)(n) = i$. That is $x(m + n) = i$. If $i = 0$ this says $m + n \notin A$. If $i = 1$ it says $m + n \in A$. ∎

Given $A \subseteq \mathbb{N}$ we denote by χ_A the characteristic function of A.

DEFINITION 4.5. Let $A \subseteq \mathbb{N}$.
(a) $\gamma(A)$ is the statement that χ_A is uniformly recurrent on $\{0,1\}^{\mathbb{N}}$.
(b) $\psi(A)$ is the statement that there exist a sequence $\{B_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{N}$ and a sequence $\{k(n)\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that
 (1) $(\bigcup_{n \in A} B_n + n) \cap (\bigcup_{n \in \mathbb{N}\setminus A} B_n + n) = \emptyset$
 (2) for all $n \in \mathbb{N}$, $\mathbb{N} = \bigcup_{t=1}^{k(n)} (B_n - t)$, and
 (3) for all $n \in \mathbb{N}$, $B_{n+1} \subseteq B_n$.

We shall see that the statements $\psi(A)$ and $\gamma(A)$ both serve to characterize our problem. We do not know whether they are equivalent. We do have one implication however.

LEMMA 4.6. *Let $A \subseteq \mathbb{N}$. If $\psi(A)$, then $\gamma(A)$.*

PROOF. Pick $\langle B_n\rangle_{n=1}^{\infty}$ and $\langle k(n)\rangle_{n=1}^{\infty}$ as guaranteed by the definition of $\psi(A)$. Let $x = \chi_A$ and let U be a neighborhood of x in X. We may presume that we have some $d \in \mathbb{N}$ such that $U = \{y \in X : \text{for each } i \in \{1,2,3,\ldots,d\},\ y(i) = x(i)\}$. We show that for all $z \in \mathbb{N}$ there exists $t \in \{1,2,\ldots,k(d)\}$ with $T^{z+t}(x) \in U$. To this end, let $z \in \mathbb{N}$ be given and let $m = z + k(d) + d$. Pick some $w \in B_m$. Now $w + z \in \bigcup_{t=1}^{k(d)} B_d - t$ so pick $t \in \{1,2,\ldots,k(d)\}$ with $w + z + t \in B_d$. Suppose $T^{z+t}(x) \notin U$ and pick $i \in \{1,2,\ldots,d\}$ such that $T^{z+t}(x)(i) \neq x(i)$. Let $b = z + t + i$. Then $x(b) \neq x(i)$ so either $b \in A$ and $i \notin A$ or $b \notin A$ and $i \in A$. In any event $(B_b + b) \cap (B_i + i) = \emptyset$. But $w \in B_m \subseteq B_b$ (since $b \leq m$) so $w + b \in B_b + b$. Also $w + b - i = w + z + t \in B_d \subseteq B_i$ so $w + b \in B_i + i$, a contradiction. ∎

LEMMA 4.7. *Let $A \subseteq N$. If $\gamma(A)$, then $\gamma(A+1)$ or $\gamma(A+1)$ or $\gamma\big((A+1) \cup \{1\}\big)$.*

PROOF. Let $x = \chi_A$. For each $d \in \mathbb{N}$ let $U_d = \{y \in X : \text{for all } i \in \{1,2,\ldots,d\},\ y(i) = x(i)\}$. Now $\{n \in \mathbb{N} : T^n(x) \in U_d\}$ is non empty. Thus we may pick $\ell(d) \in \mathbb{N}$ such that $T^{\ell(d)}(x) \in U_d$.

CASE 1. $\{d \in \mathbb{N} : \ell(d) \in A\}$ is infinite. Let $B = (A+1) \cup \{1\}$. We show that $\gamma(B)$. Let $y = \chi_B$, let $a \in \mathbb{N}$, and let $V_d = \big\{z \in X : \text{for all } i \in \{1,2,\ldots,d\},\ z(i) = y(i)\big\}$. Pick $d' > d$ such that $\ell(d') \in A$. Pick $b \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ some $t \in \{1,2,\ldots,b\}$ satisfies $T^{n+t}(x) \in U_{\ell(d')+d}$. We show that for all $n \in \mathbb{N}$, some $t \in \{1,2,3,\ldots,b+\ell(d')\}$ has $T^{n+t}(y) \in V_d$. Let $n \in \mathbb{N}$ be given. Pick $i \in \{1,2,\ldots,b\}$ such that $T^{n+i}(x) \in U_{\ell(d')+d}$. Let $t = i + \ell(d')$. Then $t \in \{1,2,\ldots,b+\ell(d)\}$. We claim that $T^{n+t}(y) \in V_d$. To this end, let $j \in \{1,2,\ldots,d\}$ be given. Assume first $j = 1$. Then $y(j) = 1$ and $T^{n+t}(y)(j) = y(n+t+1) = x(n+t) = x\big(n+i+\ell(d')\big)$. Now $T^{(n+i)}(x) \in U_{\ell(d')+d}$ and so $T^{(n+i)}(x)\big(\ell(d')\big) = x\big(\ell(d')\big) = 1$. That is $T^{n+t}(y)(j) = y(j)$. Now assume $j \in \{2,3,\ldots,d\}$.

Then $y(j) = x(j-1)$ and $T^{n+t}(y)(j) = y(n+t+j) = x(n+t+j-1) = x\big(n+i+\ell(d')+j-1\big)$. Again $T^{n+i}(x) \in U_{\ell(d')+d}$ so $T^{n+i}(x)\big(\ell(d')+j-1\big) = x\big(\ell(d')+j-1\big)$. Since $T^{\ell(d')}(x) \in U_{d'}$ we have $x\big(\ell(d')+j-1\big) = T^{\ell(d')}(x)(j-1) = x(j-1)$. Thus $T^{n+t}(y) = y(j)$ as required.

CASE 2. $\{d \in \mathbb{N} : \ell(d) \in A\}$ is finite. Then $\{d \in \mathbb{N} : \ell(d) \notin A\}$ is infinite. So we let $B = A + 1$ and proceed as in Case 1. ∎

We are now ready to exhibit our characterizations of when distinct points can be separated by points in the smallest ideal. (Recall from Lemma 1.1 and Theorem 2.3 that the ability to always separate distinct points in $\beta\mathbb{N}$ by points in the smallest ideal is precisely what is needed for the enveloping semigroup of a minimal flow to be a topological and algebraic copy of $\beta\mathbb{N}$).

THEOREM 4.8. *Let q and r be distinct members of $\beta\mathbb{N}$. The following statements are equivalent:*

(1) There exists $p \in K(\beta\mathbb{N})$ such that $q+p \neq r+p$.
(2) There exists A in $q \setminus r$ such that $\psi(A)$.
(3) There exists A in $q \setminus r$ such that $\gamma(A)$.
(4) There is a uniformly recurrent point y in $\{0,1\}^{\mathbb{N}}$ such that $\varphi(q)(y) \neq \varphi(r)(y)$.

PROOF. To see that (1) implies (2) pick $p \in K(\beta\mathbb{N})$ with $q+p \neq r+p$ and pick $D \in (q+p) \setminus (r+p)$. (Since $q+p$ and $r+p$ are maximal filters, neither is contained in the other.) Let $A = \{x \in \mathbb{N} : D - x \in p\}$. Then $A \in q$. Since $\mathbb{N} \setminus A = \{x \in \mathbb{N} : (\mathbb{N} \setminus D) - x \in p\}$ we also have $A \notin r$. For each $n \in \mathbb{N}$, let $C_n = \bigcap\{D - x : x \in A$ and $x \leq n\} \cap \bigcap\{(\mathbb{N} \setminus D) - x : x \in \mathbb{N} \setminus A$ and $x \leq n\}$, and let $E_n = \{x \in \mathbb{N} : C_n - x \in p\}$. Now each $C_n \in p$ and $p \in K(\beta\mathbb{N})$ so by [9, Theorem 7.23], given $n \in \mathbb{N}$ we may pick $k(n) \in \mathbb{N}$ such that $\mathbb{N} = \bigcup_{j=1}^{k(n)} E_n - j$. We assume $k(n) \geq n$. For each $i \in \mathbb{N}$ pick $x(i,n) \in \{1,2,\ldots,k(n)\}$ such that $i + x(i,n) \in E_n$ (so that $C_n - i - x(i,n) \in p$). For each $m \in \mathbb{N}$, choose $t_m \in \bigcap_{n=1}^{m} \bigcap_{i=1}^{m} \big(C_n - i - x(i,n)\big)$. (This intersection is in p and is thus nonempty).

Let $H_0 = \mathbb{N}$ and pick an infinite subset H_1 of H_0 such that for all $m, k \in H_1$ one has $t_m + 1 \in C_1$ if and only if $t_k + 1 \in C_1$. Inductively, given infinite $H_{\ell-1}$, pick an infinite subset H_ℓ of $H_{\ell-1}$ such that for all $m, k \in H_\ell$ and all $n, i \in \{1,2,\ldots,\ell\}$ one has $t_m + i \in C_n$ if and only if $t_k + i \in C_n$. (We are simply applying the pigeon hole principle). Having chosen $\langle H_n \rangle_{n=1}^{\infty}$, pick for each $n \in \mathbb{N}$ some $m(n) \in H_n$ with $m(n) \geq n$ and let $B_n = \{i \in \mathbb{N} : i \leq n$ and $t_{m(n)} + i \in C_n\} \cup \{i \in \mathbb{N} : i > n$ and $t_{m(i)} + i \in C_n\}$.

We claim that: $(*)$ for all i, n and ℓ in $\mathbb{N}$ with $\ell \geq \max\{i,n\}$ one has $t_{m(\ell)} + i \in C_n$ if and only if $i \in B_n$.

To see this, let i, n and ℓ be given. Assume first $i \leq n$. Then $H_\ell \subseteq H_n$ so $m(n)$ and $m(\ell)$ are in H_n so

$$t_{m(\ell)} + i \in C_n \Longleftrightarrow t_{m(n)} + i \in C_n \Longleftrightarrow i \in B_n.$$

Now assume $i > n$. Then $H_\ell \subseteq H_i$ and, $m(i)$, $m(\ell)$ are in H_i so

$$t_{m(\ell)} + i \in C_n \Longleftrightarrow t_{m(i)} + i \in C_n \Longleftrightarrow i \in B_n.$$

Thus $(*)$ is established.

To see that $\left(\bigcup_{n\in A}(B_n+n)\right)\cap\left(\bigcup_{n\in\mathbb{N}\setminus A}(B_n+n)\right)=\emptyset$, let $n\in A$, $k\in\mathbb{N}\setminus A$, and suppose we have some $z\in(B_n+n)\cap(B_k+k)$. Let $\ell=\max\{n,k,z-n,z-k\}$. Now $z-n\in B_n$ and $z-k\in B_k$ so by $(*)$ we have $t_{m(\ell)}+(z-n)\in C_n$ and $t_{m(\ell)}+(z-k)\in C_k$. Since $n\in A$, we have $C_n\subseteq D-n$. So $t_{m(\ell)}+z\in D$. Also $k\in\mathbb{N}\setminus A$ and so $C_k\subseteq(\mathbb{N}\setminus D)-k$ and hence $t_{(m)\ell}+z\in\mathbb{N}\setminus D$, a contradiction.

To see that $\mathbb{N}=\bigcup_{t=1}^{k(n)}(B_n-t)$ for all $n\in\mathbb{N}$, let n and i be in $\mathbb{N}$ and let $\ell=k(n)+i$. Now $n\leq k(n)<\ell\leq m(\ell)$ and $i<\ell\leq m(\ell)$ so $t_{m(\ell)}\in C_n-i-x(i,n)$. That is, $t_{m(\ell)}+i+x(i,n)\in C_n$. Also $m(\ell)\geq\ell\geq i+x(i,n)$ and $m(\ell)\geq\ell>k(n)\geq n$ so by $(*)$, $i+x(i,n)\in B_n$. Thus $i\in B_n-x(i,n)\subseteq\bigcup_{t=1}^{k(n)}B_n-t$.

Finally, given $n\in\mathbb{N}$ and $i\in B_{n+1}$, let $\ell=\max\{n+1,i\}$. Invoking $(*)$ twice and the obvious fact that $C_{n+1}\subseteq C_n$, we conclude that $i\in B_n$.

That (2) implies (3) follows from Lemma 4.6.

To see that (3) implies (4), pick A in $q\setminus r$ such that $\gamma(A)$. Pick by Lemma 4.7 $B\subseteq\mathbb{N}$ such that $B-1=A$ and $\gamma(B)$. Let $y=\chi_B$. Then y is uniformly recurrent. Since $B-1\in q$ and $B-1\notin r$ we have by Lemma 4.4 that $\varphi(q)(y)(1)=1$ and $\varphi(r)(y)(1)=0$ so that $\varphi(q)(y)\neq\varphi(r)(y)$.

That (4) implies (1) follows from Remark 4.3. ∎

We close this section with a question. Observe that by Lemma 4.6 we know that for each $A\subseteq\mathbb{N}$, $\psi(A)$ implies $\gamma(A)$. By Theorem 4.8 we know that if ultrafilters q and r can be separated by a set satisfying $\gamma(A)$, then they can be separated by a set satisfying $\psi(A)$. But we don't know the answer to the following.

QUESTION 4.8. Does there exist $A\subseteq\mathbb{N}$ satisfying $\gamma(A)$ but not satisfying $\psi(A)$?

5. **Some sufficient conditions.** We investigate here some algebraic conditions which guarantee that we can separate at least certain pairs in $\beta\mathbb{N}$ by points in $K(^*\beta\mathbb{N})$.

In [2] it was shown that there are many idempotents e and f in $K(\beta\mathbb{N})$ such that $e+f$ is not an idempotent. It was asked there whether there were any idempotents e and f in $K(\beta\mathbb{N})$ such that $e+f$ is an idempotent and $e+f\neq e$ and $e+f\neq f$. We establish now an alternate formulation of this question.

LEMMA 5.1. *The following statements are equivalent.*

(1) Whenever e and f are idempotents in $K(\beta\mathbb{N})$, if $e+f$ is an idempotent then $e+f=e$ or $e+f=f$.

(2) Whenever e and g are distinct idempotents in the same minimal right ideal of $\beta\mathbb{N}$ and f is any idempotent in $\beta\mathbb{N}+g$ with $f\neq g$ one has $e+f\neq g$.

PROOF. To see that (1) implies (2), let e and g be distinct idempotents in the same minimal right ideal R and let f be an idempotent in $\beta\mathbb{N}+g$ with $f\neq g$. Suppose that $e+f=g$. Then $e+f$ is an idempotent and $e+f\neq e$ and $e+f\neq f$, a contradiction.

To see that (2) implies (1), let e and f be distinct idempotents in $K(\beta\mathbb{N})$ and let $g=e+f$. Assume g is an idempotent and $g\neq e$. Then $g\in e+\beta\mathbb{N}$, which is a minimal right ideal. (See [3] for this and any other unfamiliar points about the structure of $K(\beta\mathbb{N})$.) Thus g

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and e are distinct idempotents in the same minimal right ideal of $\beta\mathbb{N}$. Also $e+f=g$ so $g \in \beta\mathbb{N}+f$, a minimal left ideal, so $f \in \beta\mathbb{N}+g$. Since $e+f=g$ one must have $f=g$, *i.e.* $e+f=f$. ∎

We are prepared to make a conjecture which is (on its face) significantly weaker than the statement of Lemma 5.1. (There are, after all, 2^{c} idempotents in $\beta\mathbb{N}+g$ if $g \in K(\beta\mathbb{N})$ [1].)

CONJECTURE 5.2. Whenever e and g are distinct idempotents in the same minimal right ideal of $\beta\mathbb{N}$ there is some idempotent f in $\beta\mathbb{N}+g$ such that $e+f \neq g$.

One would expect that points in $K(\beta\mathbb{N})$ would be at least as hard to separate as any. We see that the validity of the conjecture would at least allow us to do that.

THEOREM 5.3. *Assume the validity of Conjecture 5.2. If q and r are distinct points $K(\beta\mathbb{N})$, then there is some $p \in K(\beta\mathbb{N})$ such that $q+p \neq r+p$.*

PROOF. As points of $K(\beta\mathbb{N})$, q and r each lie in some minimal right ideals of $\beta\mathbb{N}$. If these minimal right ideals are distinct (hence disjoint) one has that, for all $p \in \beta\mathbb{N}$, $q+p \neq r+p$. Consequently we may assume q and r lie in the same minimal right ideal R. Let $L=\beta\mathbb{N}+q$, the minimal left ideal in which q lies. Now $R \cap L$ is a group (see [3]). Let g be the identity of $R \cap L$. Then g is a right identity for L. If $r \in L$ we have $r+g=r \neq q=q+g$. Thus we may assume $r \notin L$. Let $L'=\beta\mathbb{N}+r$ and let e be the identity of $L' \cap R$. By the assumed validity of Conjecture 5.2, pick an idempotent f in L such that $e+f \neq g$.

We claim that either $q+g \neq r+g$ or $q+f \neq r+f$. Suppose instead we have $q+g=r+g$ and $q+f=r+f$. Now $q+f=q=q+g$ since $q \in L=\beta\mathbb{N}+f=\beta\mathbb{N}+g$. Also

$$
\begin{aligned}
q+g &= q+f \\
&= r+f \\
&= (r+e)+f \quad (r \in L'=\beta\mathbb{N}+e) \\
&= r+(e+f) \\
&= r+\big(g+(e+f)\big) \quad (e+f \in R=g+\beta\mathbb{N}) \\
&= (r+g)+(e+f) \\
&= (q+g)+(e+f) \\
&= q+\big(g+(e+f)\big) \\
&= q+(e+f).
\end{aligned}
$$

Since $q+g=q+(e+f)$ and q, g, and $e+f$ are all in the group $R \cap L$, we conclude that $g=e+f$, a contradiction. ∎

In attempting to establish Conjecture 5.2 in the fashion of [2], we came up with a condition in the style of statements (2) and (3) of Theorem 4.8 which is sufficient to separate points q and r in $\beta\mathbb{N}$. The kinds of sets involved are easy to describe as long as one is comfortable with the ternary representation of numbers. By way of contrast, it can be very difficult to check whether a given set A satisfies $\gamma(A)$, *i.e.* whether the characteristic function of A is uniformly recurrent in $\{0,1\}^{\mathbb{N}}$.

DEFINITION 5.4. Given $x \in \mathbb{N}$, let $F(x)$ be the finite non empty subset of $\mathbb{N} \cup \{0\}$ and let g_x be the function from $F(x)$ to $\{1,2\}$ such that $x = \sum_{n\in F(x)} g_x(n)\cdot 3^n$. Let $m(x) = \max F(x)$. For $A \subseteq \mathbb{N}\cup\{0\}$, let

$$B(A) = \Big\{x \in \mathbb{N} : m(x) \in A \text{ and } g_x\big(m(x)\big) = 1\Big\} \cup \Big\{x \in \mathbb{N} : m(x) \notin A \text{ and } g_x\big(m(x)\big) = 2\Big\}.$$

Thus for example, $B(\mathbb{N}\cup\{0\})$ is the set of all x in $\mathbb{N}$ whose leftmost nonzero ternary digit is 1.

We work with the semigroup $S = \{e, f, ef, fe, efe, fef, efef, fefe\}$ from [2]. The operation on S is determined by the fact that all listed elements are distinct and that $e^2 = efefe = e$ and $f^2 = fefef = f$.

DEFINITION 5.5. Given $A \subseteq \mathbb{N}\cup\{0\}$, we define $\varphi_A \colon \mathbb{N} \longrightarrow S$ as follows.

(1) If $n \in A$, $\varphi_A(3^n) = e$ and $\varphi_A(2\cdot 3^n) = f$.

(2) If $n \notin A$, $\varphi_A(3^n) = f$ and $\varphi_A(2\cdot 3^n) = e$.

(3) For $x \in \mathbb{N}$, $\varphi_A(x) = \prod_{n\in F(x)} \varphi_A\big(g_x(n)\cdot 3^n\big)$ where the product is taken in increasing order of indices.

(4) Denote also by φ_A, the continuous extension of φ_A to $\beta\mathbb{N}$.

Thus, for example, if $F(x) = \{1,6,10\}$ then $\varphi_A(x) = \varphi_A\big(g_x(1)\cdot 3\big)\cdot \varphi_A\big(g_x(6)\cdot 3^6\big)\cdot \varphi_A\big(g_x(10)\cdot 3^{10}\big)$. If $A = \mathbb{N}\cup\{0\}$ and, in ternary, if $x = 20112101000$, then $\varphi_A(x) = eefeef = efef$. (We need to reverse the order as we do because of our choice of right continuity for $\beta\mathbb{N}$).

LEMMA 5.6. *Let $A \subseteq \mathbb{N}\cup\{0\}$. Let $x, y \in \mathbb{N}$. If there is some $n \in \mathbb{N}$ such that $x < 3^n$ and $3^n | y$, then $\varphi_A(x+y) = \varphi_A(x)\cdot\varphi_A(y)$.*

PROOF. One has $F(x+y) = F(x)\cup F(y)$ and $g_{x+y} = g_x \cup g_y$, and $\max F(x) < \min F(y)$ so this is immediate. ∎

DEFINITION 5.7. $T = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta\mathbb{N}}(\mathbb{N}3^n)$.

LEMMA 5.8. *Let $A \subseteq \mathbb{N}\cup\{0\}$, let $q \in \beta\mathbb{N}$ and let $p \in T$. Then $\varphi_A(q+p) = \varphi_A(q)\cdot\varphi_A(p)$.*

PROOF. Let $B = \{x \in \mathbb{N} : \varphi_A(x) = \varphi_A(q+p)\}$, $C = \{x \in \mathbb{N} : \varphi_A(x) = \varphi_A(q)\}$, $D = \{x \in \mathbb{N} : \varphi_A(x) = \varphi_A(p)\}$. Since S is discrete we have $B \in q+p$, $C \in q$, and $D \in p$. Let $H = \{x \in \mathbb{N} : B - x \in p\}$. Then $H \in q$. Pick $x \in C\cap H$ and pick $n \in \mathbb{N}$ such that $x < 3^n$. Then $\mathbb{N}3^n \in p$, so pick $y \in \mathbb{N}3^n \cap D \cap (B-x)$. By Lemma 5.6, $\varphi_A(y+x) = \varphi_A(x)\cdot\varphi_A(y)$. Since $y+x \in B$, $x \in C$, and $y \in D$ we thus have $\varphi_A(q+p) = \varphi_A(y+x) = \varphi_A(x)\cdot\varphi_A(y) = \varphi_A(q)\cdot\varphi_A(p)$. ∎

LEMMA 5.9. *Let $A \subseteq \mathbb{N}\cup\{0\}$. Then $\varphi_A[K(\beta\mathbb{N})\cap T] = S$.*

PROOF. By [2, Lemma 2.5] $K(\beta\mathbb{N})\cap T = K(T)$. By Lemma 5.8 the restriction of φ_A to T is a homomorphism which is easily seen to be onto S. Consequently $\varphi_A[K(T)] = K(S)$. One can then easily verify that $K(S) = S$. ∎

THEOREM 5.10. *Let q and r be distinct members of $\beta\mathbb{N}$. If there is some $A \subseteq \mathbb{N} \cup \{0\}$ such that $B(A) \in q \setminus r$, then there exists $p \in K(\beta\mathbb{N})$ such that $q + p \neq r + p$.*

PROOF. Pick A such that $B(A) \in q \setminus r$. We show first that $\varphi_A[B(A)] \subseteq \{e, fe, efe, fefe\}$. Indeed, given $x \in B(A)$ one has $\varphi_A\Big(g_x\big(m(x)\big) \cdot 3^{m(x)}\Big) = e$ so that if $F(x) = \{m(x)\}$, $\varphi_A(x) = e$ and otherwise $\varphi_A(x) = \Big(\prod_{n \in F(x) \setminus \{m(x)\}} \varphi_A\big(g_x(n) \cdot 3^n\big)\Big) \cdot e \in \{e, fe, efe, fefe\}$. Similarly $\varphi_A[\mathbb{N} \setminus B(A)] \subseteq \{f, ef, fef, efef\}$. Thus we have $\varphi_A(q) \in \{e, fe, efe, fefe\}$ and $\varphi_A(r) \in \{f, ef, fef, efef\}$. In particular $\varphi_A(q) \neq \varphi_A(r)$.

A quick look at the multiplication table for S shows that if $a, b \in S$ and $a \neq b$ then either $ae \neq be$ or $af \neq bf$. Essentially without loss of generality we assume $\varphi_A(q)e \neq \varphi_A(r)e$. Pick by Lemma 5.9 some $p \in (\beta\mathbb{N}) \cap T$ such that $\varphi_A(p) = e$. Then by Lemma 5.8,

$$\varphi_A(q + p) = \varphi_A(q)e \neq \varphi_A(r)e = \varphi_A(r + p)$$

so $q + p \neq r + p$. ∎

Observe that $\gamma\big(B(A)\big)$ cannot hold for any $A \subseteq \mathbb{N} \cup \{0\}$, since $B(A)$ and $\mathbb{N} \setminus B(A)$ will always contain arbitrarily long blocks of integers. In fact we see that the sets $B(A)$ do not come close to distinguishing among all pairs of distinct idempotents.

THEOREM 5.11. *Let $q \in \beta\mathbb{N} \setminus \mathbb{N}$. There exists $r \neq q$ in $\beta\mathbb{N} \setminus \mathbb{N}$ such that $\big\{B(A) : A \subseteq \mathbb{N} \cup \{0\}$ and $B(A) \in q\big\} = \big\{B(A) : A \subseteq \mathbb{N} \cup \{0\}$ and $B(A) \in r\big\}$.*

PROOF. Let $C = \{3^n : n \in \mathbb{N}\} \cup \{2 \cdot 3^n : n \in \mathbb{N}\}$ and let $D = \{3^n + 1 : n \in \mathbb{N}\} \cup \{2 \cdot 3^n + 1 : n \in \mathbb{N}\}$. Then $\{C\} \cup \big\{B(A) : A \subseteq \mathbb{N} \cup \{0\}$ and $B(A) \in q\big\}$ and $\{D\} \cup \big\{B(A) : A \subseteq \mathbb{N} \cup \{0\}$ and $B(A) \in q\big\}$ both have the finite intersection property. To see this let $A_1, A_2, \ldots, A_t$ be given with each $B(A_i)$ in q. Pick $x \in \bigcap_{i=1}^t B(A_i)$, with $x \geq 3$. If $g_x\big(m(x)\big) = 1$, then $m(x) \in \bigcap_{i=1}^t A_i$ so that $\{3^{m(x)}, 3^{m(x)} + 1\} \subseteq \bigcap_{i=1}^t B(A_i)$. If $g_x\big(m(x)\big) = 2$, then $m(x) \notin \bigcup_{i=1}^t A_i$ so $\{2 \cdot 3^{m(x)}, 2 \cdot 3^{m(x)} + 1\} \subseteq \bigcap_{i=1}^t B(A_i)$.

Essentially without loss of generality assume $C \notin q$ and pick $r \in \beta\mathbb{N} \setminus \mathbb{N}$ with $\{C\} \cup \big\{B(A) : A \subseteq \mathbb{N} \cup \{0\}$ and $B(A) \in q\big\} \subseteq r$. Also note that for $A \subseteq \mathbb{N} \cup \{0\}$ we always have $\mathbb{N} \setminus B(A) = B(\mathbb{N} \setminus A)$. Thus if $B(A) \in r$ then $B(\mathbb{N} \setminus A) \notin q$ which in turn implies $B(A) \in q$. ∎

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