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Operators with Closed Range, Pseudo-Inverses, and Perturbation of Frames for a Subspace

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Abstract. Recent work of Ding and Huang shows that if we perturb a bounded operator (between Hilbert spaces) which has closed range, then the perturbed operator again has closed range. We extend this result by introducing a weaker perturbation condition, and our result is then used to prove a theorem about the stability of frames for a subspace.

1 Introduction

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. It is well known that every bounded operator $T\colon \mathcal{K} \to \mathcal{H}$ with closed range has a *generalized inverse*, usually called the *pseudo-inverse*, or the *Moore-Penrose inverse*. In a recent paper Ding and Huang [DH2] find conditions implying that a perturbation of an operator with closed range again has closed range. They connect the results with norm estimates for the corresponding pseudo-inverse operators.

In the first part of the present paper we extend one of Ding and Huang's results such that it fits with our applications in Section 3, which concerns perturbations of frames. A frame is a family $\{f_i\}_{i=1}^{\infty}$ of elements in a Hilbert space with the property that every element in the space can be written as a (infinite) linear combination of the frame elements. The question we consider is whether a family $\{g_i\}_{i=1}^{\infty}$ which is "close" to a frame $\{f_i\}_{i=1}^{\infty}$ is itself a frame. Beginning with work of Heil [H], there has been some interest in this question. Most of the work has been done with the goal to find results which are easy to apply to frames arising in wavelet theory, see *e.g.*, [FZ], [GZ]. Together with Heil and Casazza, the present author has contributed with theoretical results [CC1], [C1], [C2], [CH]. From our point of view the major drawback of the results presented so far is that they can only be applied if $\{g_i\}_{i=1}^{\infty}$ is contained in $\overline{\text{span}}\{f_i\}_{i=1}^{\infty}$. That is, if $\{f_i\}_{i=1}^{\infty}$ is only a frame for $\overline{\text{span}}\{f_i\}_{i=1}^{\infty}$ and not for the underlying Hilbert space, the theory can not be applied without restrictions on $\{g_i\}_{i=1}^{\infty}$. This is a problem, *e.g.*, in sampling theory, where sequences $\{f(\cdot - \lambda_i)\}_{i=1}^{\infty}$ consisting of translates of the single function $f \in L^2(R)$ plays an important role. A sequence $\{f(\cdot - \lambda_i)\}_{i=1}^{\infty}$ can be a frame for $\overline{\text{span}}\{f(\cdot - \lambda_i)\}_{i=1}^{\infty}$ (see the papers [BL], [CCK] for sufficient conditions), but as shown in [CDH] it can not be a frame for $L^2(R)$.

Easy examples suggest that such a case would be outside the scope of the theory. It is therefore a surprise for the author that a different way of proof (involving the above mentioned results on pseudo-inverse operators) leads to a perturbation theorem which is very similar to previous results, but which covers the case of a frame for a subspace.

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The results presented here lead immediately to an extension of the celebrated Kadec 1/4 Theorem.

2 Operators with Closed Range

Let $\mathcal{K}$, $\mathcal{H}$ denote Hilbert spaces and $B(\mathcal{K},\mathcal{H})$ the set of bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. The range and the kernel of $T \in B(\mathcal{K},\mathcal{H})$ will be denoted by R_T and N_T, respectively.

Suppose that the operator $T \in B(\mathcal{K},\mathcal{H})$ has closed range. The *pseudo-inverse* of T is, by definition, the uniquely determined operator $T^\dagger \colon \mathcal{H} \to \mathcal{K}$ satisfying:

$$T^\dagger T x = x, \quad \forall x \in N_T^\perp \quad \text{and} \quad T^\dagger y = 0, \quad \forall y \in {R_T}^\perp.$$

It is well known [K, p. 231] that the operator $T \in B(\mathcal{K},\mathcal{H})$ has closed range if and only if

$$\gamma(T) := \inf_{x \in N_T^\perp, \|x\|=1} \|Tx\| > 0.$$

It can be shown [DH1] that if R_T is closed, then

$$\gamma(T) = \frac{1}{\|T^\dagger\|}.$$

Let V, W be subspaces of the same Hilbert space. If $V \neq 0$, the *gap* between V and W is defined by:

$$\delta(V,W) := \sup_{x \in V, \|x\|=1} \operatorname{dist}(x,W) = \sup_{x \in V, \|x\|=1} \inf_{y \in W} \|x - y\|.$$

As a convention we use $\delta(0,W) = 0$. Usually δ is most conveniently found using the orthogonal projection P of $\mathcal{H}$ onto $\overline{W}$:

$$\delta(V,W) = \delta(V,\overline{W}) = \sup_{v \in V, \|x\|=1} \|x - Px\|.$$

Given operators $T, U \in B(\mathcal{K},\mathcal{H})$ we let

$$\delta_N := \delta(N_T, N_U).$$

Using this notation Ding and Huang [DH2, Theorem 3.1] have proven a stability result for the closedness of the range of an operator:

Theorem 2.1 *Let $T, U \in B(\mathcal{K},\mathcal{H})$ and suppose that T has closed range. If ${\delta_N}^2 + \|T - U\|^2 \cdot \|T^\dagger\|^2 < 1$, then R_U is closed, and*

$$\|U^\dagger\| \leq \frac{\|T^\dagger\|}{(1 - {\delta_N}^2)^{1/2} - \|T - U\| \cdot \|T^\dagger\|}.$$

Theorem 2.1 can be reformulated by saying that if an operator T has closed range and U is a small perturbation of T (in the sense that $\|T - U\| < \frac{(1-{\delta_N}^2)^{1/2}}{\|T^\dagger\|}$), then also U has closed

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range. We need a more general stability result which will be the key to our most important frame result in the next section.

Theorem 2.2 *Let $T, U \in B(\mathcal{K}, \mathcal{H})$. Suppose that $\delta_N < 1$ and that there exist numbers $\lambda_1 \in [0; 1[$, $\lambda_2 \in]-1; \infty[$ and $\mu \geq 0$ such that*

$$(1) \qquad \|Tx - Ux\| \leq \lambda_1 \|Tx\| + \lambda_2 \|Ux\| + \mu \|x\|, \quad \forall x \in \mathcal{K}.$$

Then

(i) $\gamma(U) \geq \frac{(1-\lambda_1)\gamma(T)(1-\delta_N{}^2)^{1/2}-\mu}{1+\lambda_2}$
(ii) If R_T is closed and $\lambda_1 + \frac{\mu}{\gamma(T)(1-\delta_N{}^2)^{1/2}} < 1$, then R_U is closed and

$$\|U^\dagger\| \leq \frac{(1+\lambda_2)\|T^\dagger\|}{(1-\lambda_1)(1-\delta_N{}^2)^{1/2} - \mu \|T^\dagger\|}.$$

Proof (i) For $x \in \mathcal{K}$,

$$\|Ux\| \geq \|Tx\| - \|Tx - Ux\| \geq (1-\lambda_1)\|Tx\| - \lambda_2 \|Ux\| - \mu \|x\|,$$

so

$$\|Ux\| \geq \frac{(1-\lambda_1)\|Tx\| - \mu \|x\|}{1+\lambda_2},$$

and

$$\gamma(U) = \inf\{\|Ux\| \mid x \in N_U^\perp, \|x\| = 1\} \geq \frac{(1-\lambda_1)\inf\{\|Tx\| \mid x \in N_U^\perp, \|x\| = 1\} - \mu}{1+\lambda_2}.$$

Now (i) follows by the calculation in [DH2, Lemma 3.4], where it is shown that $\inf\{\|Tx\| \mid x \in N_U^\perp, \|x\| = 1\} \geq \gamma(T)(1-\delta_N{}^2)^{1/2}$.

(ii) R_U is closed if $\gamma(U) > 0$, and by (i) this is satisfied if $\lambda_1 + \frac{\mu}{\gamma(T)(1-\delta_N{}^2)^{1/2}} < 1$. Also, in this case $U^\dagger$ exists, and

$$\|U^\dagger\| = \frac{1}{\gamma(U)} \leq \frac{1+\lambda_2}{(1-\lambda_1)\frac{1}{\|T^\dagger\|}(1-\delta_N{}^2)^{1/2} - \mu} = \frac{(1+\lambda_2)\|T^\dagger\|}{(1-\lambda_1)(1-\delta_N{}^2)^{1/2} - \mu\|T^\dagger\|}. \qquad \blacksquare$$

Remark From the point of view that U is considered as a perturbation of T it might be surprising that λ_2 can be arbitrarily large. But in a special case it is necessary to restrict λ_2, namely if U is not *a priori* known to be bounded. In this case the boundedness of U follows from the inequality (1) if we assume that $\lambda_2 < 1$. We need this observation in Theorem 3.2.

In concrete cases it can be difficult to estimate δ_N. Therefore it is important to notice that this number can be avoided in some special cases. Recall that an operator $T \in B(\mathcal{K}, \mathcal{H})$ is said to have an *index* if $\dim(N_T) < \infty$ or if $\dim(\mathcal{H}/R_T) < \infty$. In this case the index is defined as

$$\operatorname{ind}(T) = \dim(N_T) - \dim(\mathcal{H}/R_T).$$

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We shall only work with operators T which have closed range, in which case $\dim(\mathcal{H}/R_T) = \operatorname{codim}(R_T)$. In our terminology [G, Theorem V.3.6] now reads:

Theorem 2.3 *Let $T, U \in B(\mathcal{K}, \mathcal{H})$. Suppose that T has closed range and that T has an index. If there exist $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\gamma(T)} < 1$ and*

$$\|Tx - Ux\| \leq \lambda\|Tx\| + \mu\|x\|, \quad \forall x \in \mathcal{K},$$

then U has closed range. Furthermore $\dim(N_U) \leq \dim(N_T)$, $\operatorname{codim}(R_U) \leq \operatorname{codim}(R_T)$, *and* $\operatorname{ind}(U) = \operatorname{ind}(T)$.

3 Applications to Frames

In this section $\mathcal{H}$ denotes a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. We begin with some definitions.

A family $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *frame (for $\mathcal{H}$)* if

$$\exists A, B > 0: \quad A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

A and B are called *frame bounds*.

$\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *frame sequence* if $\{f_i\}_{i=1}^{\infty}$ is a frame for $\overline{\operatorname{span}}\{f_i\}_{i=1}^{\infty}$.

$\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *Bessel sequence* if at least the upper frame bound B exists. In this case one can define a bounded operator by

$$T: \ell^2(N) \to \mathcal{H}, \quad T\{c_i\}_{i=1}^{\infty} := \sum_{i=1}^{\infty} c_i f_i.$$

T is usually called the *pre-frame operator*. Composing T with its adjoint operator T^* gives the *frame operator*

$$S: \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i.$$

If both of the frame conditions are satisfied, then S is invertible and self-adjoint, which immediately leads to the *frame decomposition*

$$f = SS^{-1}f = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i\rangle f_i, \quad \forall f \in \mathcal{H}.$$

The possibility of making such a decomposition of every $f \in \mathcal{H}$ is the reason for the importance of frames. For a more detailed discussion we refer to [HW].

Our goal here is to arrive at a perturbation theorem for frame sequences. Before we state our main result we need a lemma, which is proven in [C3]:

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Lemma 3.1

(i) *A Bessel sequence* $\{f_i\}_{i=1}^{\infty}$ *is a frame sequence if and only if* R_T *is closed.*
(ii) *A Bessel sequence* $\{f_i\}_{i=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *if and only if* $R_T = \mathcal{H}$.
(iii) *If* $\{f_i\}_{i=1}^{\infty}$ *is a frame, then the optimal bounds (i.e., maximal lower bound, minimal upper bound) are* $A = \frac{1}{\|T^{\dagger}\|^2}$, $B = \|T\|^2$.

To Bessel sequences $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ we associate the pre-frame operators T, U. Corresponding to those operators we use the notation δ_N from Section 2. The use of pseudo-inverse operators leads to a surprisingly simple argument for the lower frame bound in our main theorem:

Theorem 3.2 *Let* $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *be a frame sequence with bounds* A, B. *Let* $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ *and suppose that there exists numbers* $\lambda_2, \in [0;1[$ *and* $\lambda_1, \mu \geq 0$ *such that*

$$\Big\|\sum_{i=1}^{n} c_i(f_i - g_i)\Big\| \leq \lambda_1\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \lambda_2\Big\|\sum_{i=1}^{n} c_i g_i\Big\| + \mu\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{1/2} \tag{2}$$

for all scalars $c_1, \dots, c_n$ $(n \in N)$. *Then* $\{g_i\}_{i=1}^{\infty}$ *is a Bessel sequence with upper bound* $B\big(1 + \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\big)^2$. *If furthermore* $\delta_N < 1$ *and* $\lambda_1 + \frac{\mu}{\sqrt{A}(1-\delta_N{}^2)^{1/2}} < 1$, *then* $\{g_i\}_{i=1}^{\infty}$ *is a frame sequence with lower bound* $A(1-\delta_N{}^2)\big(1 - \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}(1-\delta_N{}^2)^{1/2}}}{1+\lambda_2}\big)^2$.

Proof $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, so we can define a bounded linear operator

$$T\colon \ell^2(N) \to \mathcal{H}, \quad T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i.$$

Furthermore $\|T\| \leq \sqrt{B}$ and R_T is closed. The condition (2) implies that

$$\begin{aligned}\Big\|\sum_{i=1}^{n} c_i g_i\Big\| &\leq \Big\|\sum_{i=1}^{n} c_i(f_i - g_i)\Big\| + \Big\|\sum_{i=1}^{n} c_i f_i\Big\| \\ &\leq (1+\lambda_1)\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \lambda_2\Big\|\sum_{i=1}^{n} c_i g_i\Big\| + \mu\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{1/2}, \quad \forall\{c_i\}_{i=1}^{n},\end{aligned}$$

so

$$\Big\|\sum_{i=1}^{n} c_i g_i\Big\| \leq \frac{1+\lambda_1}{1-\lambda_2}\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \frac{\mu}{1-\lambda_2}\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{1/2}, \quad \forall\{c_i\}_{i=1}^{n}.$$

A Cauchy sequence argument now shows that $\sum_{i=1}^{\infty} c_i g_i$ actually converges for all $\{c_i\}_{i=1}^{\infty} \in \ell^2(N)$, and in (2) and the above estimates the finite sequences $\{c_i\}_{i=1}^{n}$ can be replaced by $\{c_i\}_{i=1}^{\infty} \in \ell^2(N)$. If we define an operator

$$U\colon \ell^2(N) \to \mathcal{H}, \quad U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i g_i,$$

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we have

$$\|T\{c_i\}_{i=1}^\infty - U\{c_i\}_{i=1}^\infty\| \le \lambda_1\|T\{c_i\}_{i=1}^\infty\| + \lambda_2\|U\{c_i\}_{i=1}^\infty\| + \mu\|\{c_i\}_{i=1}^\infty\|, \quad \forall \{c_i\}_{i=1}^\infty \in \ell^2(N)$$

and

$$\begin{aligned}\|U\{c_i\}_{i=1}^\infty\| &\le \frac{1+\lambda_1}{1-\lambda_2}\|T\{c_i\}_{i=1}^\infty\| + \frac{\mu}{1-\lambda_2}\|\{c_i\}_{i=1}^\infty\| \\ &\le \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\|\{c_i\}_{i=1}^\infty\|, \quad \forall \{c_i\}_{i=1}^\infty \in \ell^2(N).\end{aligned}$$

This estimate shows that $\{g_i\}_{i=1}^\infty$ is a Bessel sequence with the upper bound

$$\left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)^2 = B\left(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2.$$

Now we assume that $\delta_N < 1$ and $\lambda_1 + \frac{\mu}{\sqrt{A}(1-\delta_N{}^2)^{1/2}} < 1$. Since $\gamma(T) = \frac{1}{\|T^\dagger\|} \ge \sqrt{A}$ we have that $\lambda_1 + \frac{\mu}{\gamma(T)(1-\delta_N{}^2)^{1/2}} < 1$, so by Theorem 2.2, R_U is closed. Therefore $\{g_i\}_{i=1}^\infty$ is a frame sequence by Lemma 3.1. The optimal lower bound is $\frac{1}{\|U^\dagger\|^2} = \gamma(U)^2$, and

$$\begin{aligned}\gamma(U)^2 &\ge \left(\frac{(1-\lambda_1)\sqrt{A}(1-\delta_N{}^2)^{1/2}-\mu}{1+\lambda_2}\right)^2 \\ &= A(1-\delta_N{}^2)\left(1-\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}(1-\delta_N{}^2)^{1/2}}}{1+\lambda_2}\right)^2. \qquad \blacksquare\end{aligned}$$

Remarks 1) The condition $\lambda_2 < 1$ is only used in the proof of the existence of the upper bound, so if we know that $\{g_i\}_{i=1}^\infty$ is a Bessel sequence we can remove this assumption (but our estimate for the upper bound is no more valid then).

2) It is possible to show a similar result with δ_N replaced by the gap between the ranges of the operators U, T. However, the proof is more involved, *cf.* [CFL].

Easy examples show the optimality of the bounds on $\lambda_1, \lambda_2, \mu$ in Theorem 3.2: the conclusion fails if (2) is only satisfied with $\lambda_1 = 1$ (or $\lambda_2 = 1$ or $\mu = \sqrt{A}, \delta_N = 0$). If $\{f_i\}_{i=1}^\infty$ is a frame for $\mathcal{H}$, it is known that Theorem 3.2 holds with δ_N replaced by 0 and that $\{g_i\}_{i=1}^\infty$ is a frame for $\mathcal{H}$, *cf.* [CC1]. This leads trivially to an extension of the celebrated Kadec's 1/4 Theorem; observe that we denote the index by n and that i denotes the complex unit number in the following result:

Proposition 3.3 *Let* $\{\lambda_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty \subseteq R$. *Suppose that* $\{e^{i\lambda_n x}\}_{n=1}^\infty$ *is a frame for* $L^2(-\pi,\pi)$ *with bounds* A, B. *If there exists a constant* $L < 1/4$ *such that*

$$|\mu_n - \lambda_n| \le L \quad \textit{and} \quad 1-\cos\pi L + \sin\pi L < \sqrt{\frac{A}{B}},$$

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then $\{e^{i\mu_n x}\}_{n=1}^{\infty}$ *is a frame for* $L^2(-\pi,\pi)$ *with bounds*

$$A\left(1-\sqrt{\frac{B}{A}}(1-\cos\pi L+\sin\pi L)\right)^2, B(2-\cos\pi L+\sin\pi L)^2.$$

Proof This is just a trivial adjustment of the standard proof of Kadec's 1/4 Theorem. Observing that $\|\sum c_n e^{i\lambda_n(\cdot)}\|^2 \leq B\sum |c_n|^2$ for all finite sequences $\{c_n\}$, the estimates from [Y, p. 42] gives that

$$\left\|\sum c_n(e^{i\lambda_n(\cdot)}-e^{i\mu_n(\cdot)})\right\| \leq \sqrt{B}(1-\cos\pi L+\sin\pi L)\Big(\sum |c_n|^2\Big)^{1/2} \leq \sqrt{A}\Big(\sum |c_n|^2\Big)^{1/2}. \blacksquare$$

A similar version could have been written down using Theorem 3.2, assuming only that $\{e^{i\lambda_n x}\}_{n=1}^{\infty}$ is a frame sequence. The version presented here is closely related to [DS, Lemma 3], which gives the same conclusion if

$$L < \frac{\ln\left(\frac{A}{4B(e-1)}+1\right)^{1/2}}{\pi},$$

however without estimates of the frame bounds. For $A = B$ this condition is $L < 0.1173\cdots$, where we have the usual Kadec condition $L < 1/4$.

Proposition 3.3 was independently and simultaneously observed by Balan [B].

In the general situation described in Theorem 3.2, the assumption $\delta_N < 1$ is needed, as demonstrated by the following example:

Example 3.4 Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then $\{f_i\}_{i=1}^{\infty} := \{e_1, e_2, 0, 0, 0, \dots\}$ is a frame sequence. Given $\epsilon > 0$, let

$$\{g_i\}_{i=1}^{\infty} = \left\{e_1, e_2, \frac{\epsilon}{3}e_3, \frac{\epsilon}{4}e_4, \dots, \frac{\epsilon}{n}e_n, \dots\right\}.$$

An easy calculation shows that $\delta_N = 1$. By choosing ϵ small enough we can make $\{g_i\}_{i=1}^{\infty}$ as close to $\{f_i\}_{i=1}^{\infty}$ as we want, in the sense that

$$\left\|\sum_{i=1}^{n} c_i(f_i-g_i)\right\| \leq \frac{\epsilon}{3}\Big(\sum_{i=1}^{n}|c_i|^2\Big)^{1/2}, \quad \forall\{c_i\}_{i=1}^{n},$$

but $\{g_i\}_{i=1}^{\infty}$ is not a frame sequence.

As observed in Section 2, the introduction of δ_N can be avoided if the operator T has an index. This condition has a nice interpretation in terms of the underlying frame sequence $\{f_i\}_{i=1}^{\infty}$: $\dim(\mathcal{H}/R_T) < \infty$ means that $\{f_i\}_{i=1}^{\infty}$ is a frame sequence for a space of finite codimension in $\mathcal{H}$, and the case $\dim(N_T) < \infty$ corresponds to what Holub [Ho] calls a *near-Riesz basis* for $\overline{\text{span}}\{f_i\}_{i=1}^{\infty}$, meaning that $\{f_i\}_{i=1}^{\infty}$ consists of a Riesz basis for this space plus finitely many elements. In the later case, Holub also shows that the *excess*, *i.e.*, the number of elements which need to be deleted in order to obtain a Riesz basis for the space, is equal to $\dim(N_T)$.

Theorem 3.5 *Let $\{f_i\}_{i=1}^{\infty}$ be a frame sequence and suppose that the corresponding pre-frame operator T has an index. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and suppose that there exist numbers $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\gamma(T)} < 1$ and*

$$\Big\|\sum_{i=1}^{n} c_i(f_i - g_i)\Big\| \leq \lambda\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \mu\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{1/2}$$

for all scalars $c_1, \ldots, c_n$ $(n \in N)$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame sequence. The corresponding pre-frame operator U has an index, and $\dim(N_U) \leq \dim(N_T)$, $\operatorname{codim}(R_U) \leq \operatorname{codim}(R_T)$ and $\operatorname{ind}(U) = \operatorname{ind}(T)$.

The part of Theorem 3.5 concerning the relation between various dimensions is particulary interesting in the case where T is a *Fredholm operator*, meaning that both $\dim(N_T)$ and $\operatorname{codim}(R_T)$ are finite. In this case Theorem 3.5 says that a perturbation can increase the dimension of the spanned space, but the excess will decrease with the same amount. This general result can be illustrated by an easy example in R^3: Let $\{e_i\}_{i=1}^{3}$ be an orthonormal basis for R^3 and let

$$\{f_i\}_{i=1}^{3} = \{e_1, 0, 0\}, \quad \{g_i\}_{i=1}^{3} = \left\{e_1, \frac{1}{2}e_2, 0\right\}.$$

$\{f_i\}_{i=1}^{3}$ spans a one-dimensional subspace, and the excess is 2. $\{g_i\}_{i=1}^{3}$ is a perturbation of $\{f_i\}_{i=1}^{3}$ in the sense of Theorem 3.5, $\{g_i\}_{i=1}^{3}$ spans a 2-dimensional subspace, and the excess is 1.

For more results relating different perturbation conditions and excess, we refer to [CC2].

In concrete applications of the theorems discussed here it seems to be most convenient to find a value for μ such that $\|\sum_{i=1}^{n} c_i(f_i - g_i)\| \leq \mu(\sum_{i=1}^{n} |c_i|^2)^{1/2}$ for all sequences $\{c_i\}_{i=1}^{n}$. This is what we did in Proposition 3.3, and this is also the key principle in [FZ]. In terms of the operators T, U this corresponds to estimating $\|T - U\|$.

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