CIRCULAR AND TRANSVERSE WAVE SOLUTIONS OF NON-LINEAR HYPERBOLIC SYSTEMS OF EQUATIONS

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1. Introduction

Circularly and transversely polarised (henceforth called circular and transverse) waves have been shown to occur as solutions of non-linear equations governing a wide range of physical phenomena, including finite elasticity (**1**), magnetohydrodynamics (**2**), and gyromagnetism (**3**), but only when the material properties of the medium are isotropic with respect to the direction of wave propagation. This paper is an attempt to unify and generalise these results.

In Section 2, we obtain the most general form of governing equation for an isotropic system. We show in Section 3 that circular discontinuities do not occur in such systems, and study a less general system in which they do occur. The analysis is repeated for simple waves in Section 4.

We obtain in Section 6 a sufficient condition for all non-circular discontinuities and simple waves in such a system to be transverse. As far as is known, this condition is satisfied in all isotropic systems which have so far been studied.

In Section 5, we consider an isotropic system whose governing equations are of a rather more special form. Circular waves of a correspondingly more general type can propagate in this system, and the relationship between the form of the governing equations, and the nature of the circular wave solutions, is clarified.

2. The governing equations for an isotropic system

We consider plane-wave propagation in a continuous homogeneous medium, and use a rectangular cartesian system of axes, with coordinates x, y, z. The direction of wave propagation is chosen to be parallel to the x-axis, so that quantities are functions of x and the time t alone. The dependent variables can be divided into two groups, (a) those which are invariant with respect to rotations about the axis of x, which we denote by u_i, $i = 1, \ldots, p$ (i.e. scalars, x-components of vectors, and xy- and xz-components of tensors), and (b) pairs representing (y, z)-components of vectors, and (yx, zx)-, (yy, zy)-, and (yz, zz)-components of tensors, which we denote by (w_{2i-1}, w_{2i}), $i = 1, \ldots, q$. The magnitude $(w_{2i-1}^2 + w_{2i}^2)^{\frac{1}{2}}$ of each of these pairs is unaltered by rotations about the x-axis.

We suppose that the governing equations can be put into the form of a system of n conservation laws (4)

$$V_{,t}+F_{,x} = 0, \quad F = F(V), \tag{1}$$

of a reducible type in which the dependence of F on the variables x, t is implicit through the dependence of V on x, t, where $n = p+2q$ and the column n-vectors V and F (written here for convenience as transposed row vectors) are given by

$$V = (u_1, \ldots, u_p, w_1, w_3, \ldots, w_{2q-1}, w_2, \ldots, w_{2q})^T$$

and

$$F(V) = (\phi_1, \ldots, \phi_p, f_1, f_3, \ldots, f_{2q-1}, f_2, \ldots, f_{2q})^T.$$

We wish to investigate systems of equations governing phenomena which are isotropic with respect to the direction of wave propagation, and this requires that the equations (1) be invariant with respect to rotations of coordinates through an arbitrary angle α about the x-axis. If u_i' and w_i' are the values of u_i and w_i in such a rotated coordinate system, it follows from the definitions above that

$$\left.\begin{aligned} u_i' &= u_i, \\ \begin{pmatrix} w_{2i-1}' \\ w_{2i}' \end{pmatrix} &= \begin{pmatrix} \cos\alpha, & \sin\alpha \\ -\sin\alpha, & \cos\alpha \end{pmatrix}\begin{pmatrix} w_{2i-1} \\ w_{2i} \end{pmatrix} \end{aligned}\right\} \tag{2}$$

Invariance of the system (1) under this transformation therefore requires that

$$\left.\begin{aligned} \phi_i(\underline{u}, \underline{w}') &= \phi_i(\underline{u}, \underline{w}), \\ \begin{pmatrix} f_{2i-1}(\underline{u}, \underline{w}') \\ f_{2i}(\underline{u}, \underline{w}') \end{pmatrix} &= \begin{pmatrix} \cos\alpha, & \sin\alpha \\ -\sin\alpha, & \cos\alpha \end{pmatrix}\begin{pmatrix} f_{2i-1}(\underline{u}, \underline{w}) \\ f_{2i}(\underline{u}, \underline{w}) \end{pmatrix}, \end{aligned}\right\} \tag{3}$$

where $\underline{u} = (u_1, \ldots, u_p)$, $\underline{w} = (w_1, \ldots, w_{2q})$ and similarly for $\underline{w}'$.

We define v_i and θ_i, $i = 1, \ldots, q$, by

$$\left.\begin{aligned} w_{2i-1} &= v_i \cos\theta_i, \\ w_{2i} &= v_i \sin\theta_i. \end{aligned}\right\} \tag{4}$$

and write $\underline{v} = (v_1, \ldots, v_q)$ and $\underline{\theta} = (\theta_1, \ldots, \theta_q)$. If we write (2) and (3) in terms of $\underline{u}$, $\underline{v}$, and $\underline{\theta}$, it becomes clear that apart from a possible dependence on expressions of the form $\theta_j-\theta_k$, $j \neq k$, the most general form of ϕ_i and f_i is given by

$$\left.\begin{aligned} \phi_i &= \phi_i(\underline{u}, \underline{v}), \\ f_{2i-1} &= \sum_{r=1}^{q} f_i^{(r)}(\underline{u}, \underline{v})w_{2r-1}, \\ f_{2i} &= \sum_{r=1}^{q} f_i^{(r)}(\underline{u}, \underline{v})w_{2r}. \end{aligned}\right\} \tag{5}$$

We define circular waves as waves across which $\underline{u}$ and the magnitude v_i of *one* of the pairs (w_{2i-1}, w_{2i}) remain unchanged. There is no loss of generality in choosing v_1 as the magnitude which remains constant. In the next two sections we consider in turn circular discontinuities and circular simple waves.

3. Circular discontinuities in an isotropic system

Lax (4) has developed the concept of a "weak solution" of a system of conservation laws. He shows that across a plane surface of discontinuity which is a weak solution of (1), travelling with velocity s and connecting uniform states on either side of the discontinuity, the generalised Rankine-Hugoniot relations

$$s[V] = [F] \tag{6}$$

must be satisfied, where $[X]$ denotes the jump in a quantity X across the discontinuity.

We wish to test whether relations (6) have a circular discontinuity as a possible solution, so we substitute

$$[u_1] = \dots = [u_p] = [v_1] = 0 \tag{7}$$

into (6). The first p equations give

$$[\phi_1] = \dots = [\phi_p] = 0 \tag{8}$$

and the remaining $2q$ equations

$$\left.\begin{aligned} s[w_{2i-1}] &= \sum_{r=1}^{q} [f_i^{(r)}(\underline{u}, \underline{v})w_{2r-1}], \\ s[w_{2i}] &= \sum_{r=1}^{q} [f_i^{(r)}(\underline{u}, \underline{v})w_{2r}]. \end{aligned}\right\} \tag{9}$$

It follows from (8) that in general

$$\phi_i = \phi_i(\underline{u}, \underline{v}_1). \tag{10}$$

Equations (9) then represent $2q$ equations in the $2q$ unknowns $[\theta_1]$, s, $[w_3]$, ..., $[w_{2q}]$, which are to be obtained in terms of the conditions ahead of the discontinuity. There exists in general, therefore, a finite number of discrete sets of solutions to (9) satisfying (7), These are of little interest, however, since the probability of conditions being exactly right for the occurrence of one of these sets of solutions in a physical situation is zero.

For a solution of (6) to be physically significant, we require it to exist for a continuous finite range (however small) of conditions behind the discontinuity; that is to say, we require a degree of freedom, which could perhaps be the jump in θ_1. If this is chosen arbitrarily, equations (9) will in general be incompatible. But we shall see that when each $f_i^{(r)}$ depends on $\underline{u}$ and v_1 only, solutions exist. We therefore restrict our attention to such systems, for which

$$f_i^{(r)} = f_i^{(r)}(\underline{u}, v_1). \tag{11}$$

The isotropic systems considered in (1), (2) and (3) are all of this form.

Using (7), equations (9) give

$$\left.\begin{aligned} s[w_{2i-1}] &= f_i^{(r)}(\underline{u}, v_1)[w_{2r-1}],\\ s[w_{2i}] &= f_i^{(r)}(\underline{u}, v_1)[w_{2r}], \end{aligned}\right\} \tag{12}$$

with summation over r.

These may be written in the form

$$(b_{ij}-s\delta_{ij})r_{2j-1} = (b_{ij}-s\delta_{ij})r_{2j} = 0,$$

where
$$b_{ij} = f_i^{(j)}, \tag{13}$$
and
$$r_i = [w_i].$$

Thus solutions of (12) exist whenever the velocity s is an eigenvalue of the matrix $B = (b_{ij})_{q,q}$. We shall see that every eigenvalue of the matrix B is an eigenvalue of the matrix A, defined in (14). In hyperbolic systems the eigenvalues of A are real and distinct, so we have that *q distinct types of circular discontinuity can propagate.*

4. Circular simple waves in an isotropic system

The system of conservation laws (1) may be written in the form

$$\left.\begin{aligned} V_{,t}+A(V)V_{,x} &= 0,\\ \text{where} \qquad A = (a_{lm})_{n,n} &= \frac{\partial F}{\partial V}. \end{aligned}\right\} \tag{14}$$

From the definitions of F and V, and equations (10) and (11), we can show that A may be represented in the form of a partitioned matrix

$$A = \begin{bmatrix} A_1 & C_1\cos\theta_1 & C_1\sin\theta_1 \\ A_2 & B+C_2\cos\theta_1 & C_2\sin\theta_1 \\ A_3 & C_3\cos\theta_1 & B+C_3\sin\theta_1 \end{bmatrix}, \tag{15}$$

where B is defined by (13) and

$$\left.\begin{aligned} &A_1 = (\phi_i, u_j)_{p,p}, && A_2 = (f_{2i-1}, u_j)_{q,p}, && A_3 = (f_{2i}, u_j)_{q,p},\\ &C_1 = (\alpha_i\delta_{ij})_{p,q}, && C_2 = (\beta_i\delta_{ij})_{q,q}, && C_3 = (\gamma_i\delta_{ij})_{q,q},\\ &\alpha_i = \phi_{i,v_1}, && \beta_i = f_{i,v_1}^{(r)}w_{2r-1}, && \gamma_i = f_{i,v_1}^{(r)}w_{2r}. \end{aligned}\right\} \tag{16}$$

In this notation, $(g_{ij})_{r,s}$ represents the $r\times s$ matrix whose ij-th element is g_{ij}.

If the system (14) is hyperbolic, then it has exactly n simple wave solutions corresponding to the n eigenvalues of A. The theory of these simple waves is described in detail by Courant and Hilbert (5) and Jeffrey and Taniuti (6). We shall show that exactly q simple waves are circular.

If $\lambda^{(k)}$ is the k-th eigenvalue of B, and $v_j^{(k)}$ the corresponding right eigenvector, we have that

$$(b_{ij}-\lambda^{(k)}\delta_{ij})v_j^{(k)} = 0.$$

A straightforward calculation shows that

$$(a_{lm}-\lambda^{(k)}\delta_{lm})\mu_m^{(k)} = 0,$$

where

$$\mu_m^{(k)} = \begin{cases} 0, & m = 1, \dots, p, \\ \sin\theta_1 v_j^{(k)}, & m = p+j, j = 1, \dots, q, \\ -\cos\theta_1 v_j^{(k)}, & m = p+q+j, j = 1, \dots, q. \end{cases} \tag{17}$$

Thus, every eigenvalue of B is an eigenvalue of A, and so the velocity of propagation of the discontinuity is equal to an eigenvalue of A on each side of the discontinuity. Jeffrey and Taniuti (6) have called such discontinuities " *exceptional* ", in contrast to " intermediate discontinuities ", for which the velocity of propagation is equal to an eigenvalue on one side, and " genuine shocks ", for which the velocity of propagation is not equal to an eigenvalue on either side of the discontinuity.

Across each simple wave, $n-1$ functions of the dependent variables, called the generalised Riemann invariants, are constant. In the simple wave corresponding to the eigenvalue $\lambda^{(k)}$, these invariants are obtained by solving the system of differential equations

$$\frac{du_1}{\mu_1^{(k)}} = \dots = \frac{du_p}{\mu_p^{(k)}} = \frac{dw_1}{\mu_{p+1}^{(k)}} = \dots = \frac{dw_2}{\mu_{p+q+1}^{(k)}} = \dots = \frac{dw_{2q}}{\mu_n^{(k)}}.$$

Thus from (17) the first p invariants are

$$u_i = \text{constant}, \; i = 1, \dots, p,$$

while another invariant is obtained from

$$\frac{dw_1}{dw_2} = \frac{\mu_{p+1}^{(k)}}{\mu_{p+q+1}} = -\tan\theta_1 = -\frac{w_2}{w_1},$$

as v_1 = constant.

Thus the simple wave corresponding to each eigenvalue $\lambda^{(k)}$, $k = 1, \dots, q$, is circular. Since the system (14) is hyperbolic, the eigenvalues of A, and hence of B, are real and distinct. It is easily verified that no other simple waves possess these invariants, so it follows that *q distinct types of circular simple wave can propagate*.

5. A further result on circular waves

By using techniques identical in principle to the above, we may prove the following result.

We suppose that wave propagation in a certain medium is governed by the system of $n = p+2mq$ hyperbolic conservation laws

$$V_{,t}+F_{,x} = 0, \quad F = F(V),$$

in which V and F are given by

$$V = \begin{bmatrix} u_1 \\ \vdots \\ u_p \\ w_1 \\ \vdots \\ w_{2mq} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \\ f_1 \\ \vdots \\ f_{2mq} \end{bmatrix},$$

and the functions ϕ_i and f_i are of the form

$$\phi_i = \phi_i(\underline{u}, v_1, \ldots, v_m),\ i = 1, \ldots, p,$$

$$\left.\begin{aligned} f_{2m(i-1)+1} &= f_i^{(r)}(\underline{u}, v_1, \ldots, v_m) w_{2m(r-1)+1}, \\ &\vdots \\ f_{2mi} &= f_i^{(r)}(\underline{u}, v_1, \ldots, v_m) w_{2mr}, \end{aligned}\right\} i = 1, \ldots, q,$$

where

$$v_i = (w_{2i-1}^2 + w_{2i}^2)^{\frac{1}{2}}, \quad i = 1, \ldots, m.$$

Then exactly q distinct circularly polarised discontinuities and simple waves can propagate, across each of which the quantities $u_1, \ldots, u_p, v_1, \ldots, v_m$ are unaltered. The velocities of these waves are given as before, by the eigenvalues of the matrix

$$B = (f_i^{(j)})_{q,q}.$$

6. Transverse waves

In the isotropic systems which have so far been studied in detail, all discontinuities and simple waves which are not circular are transverse, in the sense that the direction of the pair (w_1, w_2) remains unaltered across them. In this section we show that this result is not true in general, but obtain a sufficient condition for it to hold for a given system.

We consider the transformation

where
$$\left.\begin{aligned} &u_1, \ldots, u_p, v_1 \to \psi_1, \ldots, \psi_p, v_1, \\ &\psi_i = \phi_i(\underline{u}, v_1) - \lambda u_i, \quad i = 1, \ldots, p, \end{aligned}\right\} \tag{18}$$

and λ is an arbitrary parameter which is not an eigenvalue of A_1 or B. Since the jacobian $J = |A_1 - \lambda I|$ of the transformation is non-zero, the transformation is non-singular.

For any function G of $\underline{u}$, v_1, and λ, we define

$$\hat{G}(\underline{\psi}, v_1, \lambda) = G(\underline{u}(\underline{\psi}, v_1, \lambda), v_1, \lambda). \tag{19}$$

We define the $q \times q$ matrix $N^{(\lambda)} = (n_{ij}^{(\lambda)})_{q,q}$ as

$$N^{(\lambda)} = (B - \lambda I)^{-1}. \tag{20}$$

In this section we prove that for both discontinuities and simple waves, all non-circular waves are transverse if

$$\frac{\partial}{\partial v_1}\left(\frac{\hat{n}_{ij}^{(\lambda)}}{\hat{n}_{11}^{(\lambda)}}\right) = 0, \quad j = 2, \ldots, q, \tag{21}$$

that is, if the ratios $n_{ij}^{(\lambda)}/n_{11}^{(\lambda)}$ may be written as functions of $\underline{\phi} - \lambda\underline{u}$ and λ alone, with no explicit dependence on v_1.

6.1. We now prove this result for discontinuities. We first use the definition (20) to obtain certain identical relations involving some of the elements of $N^{(s)}$, where s is an arbitrary parameter which is not an eigenvalue of A_1 or B. It follows from (20) that

$$n_{ij}^{(s)}(f_j^{(r)} - s\delta_{jr}) = \delta_{ir}, \quad i, r = 1, \ldots, q.$$

If we divide by $n_{11}^{(s)}$, assuming this to be non-zero, put $i = 1$, take the jump of each side of this equation across any discontinuity, and multiply by $n_{11}^{(s)}$, we obtain

$$n_{ij}^{(s)+}[f_j^{(r)}] = F_r^{(s)} + g^{(s)}\delta_{ir}, \tag{22}$$

where

$$F_r^{(s)} = -n_{11}^{(s)+}(f_j^{(r)-} - s\delta_{jr})[n_{ij}^{(s)}/n_{11}^{(s)}], \tag{23}$$

$$g^{(s)} = -n_{11}^{(s)+}[1/n_{11}^{(s)}],$$

and superscripts + and − denote quantities evaluated behind and ahead of the discontinuity.

We now return to the original system (6), and obtain conditions under which all non-circular discontinuities are transverse. We therefore require s to be the velocity of a non-circular discontinuity. The jump relations take the form

$$\left.\begin{aligned} s[u_i] &= [\phi_i], \quad i = 1, \ldots, p, \\ s[w_i] &= [f_i], \quad i = 1, \ldots, 2q. \end{aligned}\right\} \tag{24}$$

The second set may be written

$$\left.\begin{aligned} s[w_{2i-1}] &= f_i^{(r)+}[w_{2r-1}] + [f_i^{(r)}]w_{2r-1}^-, \\ s[w_{2i}] &= f_i^{(r)+}[w_{2r}] + [f_i^{(r)}]w_{2r}^-. \end{aligned}\right\} \; i = 1, \ldots, q.$$

After some rearrangement, we can use (20) to show that

$$[w_1] = -n_{ij}^{(s)+}[f_j^{(r)}]w_{2r-1}^-,$$

$$[w_2] = -n_{ij}^{(s)+}[f_j^{(r)}]w_{2r}^-.$$

We now use the identity (22) to obtain

$$\frac{[w_1]}{[w_2]} = \frac{(F_r^{(s)} + g^{(s)}\delta_{ir})w_{2r-1}^-}{(F_t^{(s)} + g^{(s)}\delta_{it})w_{2t}^-}. \tag{25}$$

The condition for a discontinuity to be transverse is that $[w_1]/[w_2] = w_1^-/w_2^-$. From (25), this is so if

$$F_r^{(s)} = 0, \quad r = 2, \ldots, q.$$

Since $[n_{ij}^{(s)}/n_{11}^{(s)}] = 0$ when $j = 1$, we have from (23) that this is so if and only if

$$[n_{ij}^{(s)}/n_{11}^{(s)}] = 0, \quad j = 2, \ldots, q. \tag{26}$$

If we write these in terms of ψ and v_1, and note that from the first of (24), $[\psi] = 0$, we see that (26) can hold if and only if v_1 does not appear explicitly in $n_{ij}^{(s)}/n_{11}^{(s)}$. Equation (26) is therefore equivalent to (21) with λ replaced by s.

6.2. We now prove the result (21) for simple waves. From the definition (20), we have the identity

$$n_{ij}^{(\lambda)}(f_j^{(r)} - \lambda\delta_{jr}) = \delta_{ir}, \quad i, r = 1, \ldots, q.$$

If we divide this by $n_{11}^{(\lambda)}$, assuming this to be non-zero, put $i = 1$, apply the transformation (18), differentiate with respect to v_1, and multiply by $n_{11}^{(\lambda)}$, we obtain an identity analogous to (22), valid for all λ except those which are eigenvalues of B and A_1, this being

$$\hat{n}_{ij}^{(\lambda)}\hat{f}_{j,v_1}^{(r)} = F_r^{(\lambda)} + g^{(\lambda)}\delta_{ir}, \tag{27}$$

where

$$F_r^{(\lambda)} = -\hat{n}_{11}^{(\lambda)}(\hat{f}_j^{(r)} - \lambda\delta_{jr})\hat{h}_{j,v_1}^{(\lambda)}, \tag{28}$$

$$g^{(\lambda)} = \hat{n}_{11,v_1}^{(\lambda)}/\hat{n}_{11}^{(\lambda)},$$

and

$$\hat{h}_j^{(\lambda)} = \hat{n}_{ij}^{(\lambda)}/\hat{n}_{11}^{(\lambda)}. \tag{29}$$

We now return to the original system (14) and define λ to be an eigenvalue of A which is not an eigenvalue of B. We note that if we assume that not all α_i, $i = 1, \ldots. p$, are zero, then λ is also not an eigenvalue of A_1. If $\underline{r}$ is the corresponding right eigenvector, we have that

$$(A - \lambda I)\underline{r} = 0.$$

From (15), this may be split up into three vectorial equations

$$\left.\begin{aligned} (\phi_{i,u_j} - \lambda\delta_{ij})r_j + k_1\alpha_i = 0, \quad i = 1, \ldots, p,\\ f_{i,u_j}^{(r)}w_{2r-1}r_j + (f_i^{(j)} - \lambda\delta_{ij})r_{p+j} + k_1\beta_i = 0, \quad i = 1, \ldots, q,\\ f_{i,u_j}^{(r)}w_{2r}r_j + (f_i^{(j)} - \lambda\delta_{ij})r_{p+q+j} + k_1\gamma_i = p, \quad i = 1, \ldots, q, \end{aligned}\right\} \tag{30}$$

where

$$k_1 = r_{p+1}\cos\theta_1 + r_{p+q+1}\sin\theta_1.$$

From $(30)_1$, we have

$$r_i = -k_1 m_{ij}^{(\lambda)}\alpha_j, \quad i = 1, \ldots, p, \tag{31}$$

where we define $M^{(\lambda)} = (m_{ij}^{(\lambda)})_{q,q}$ by

$$M^{(\lambda)} = (A_1 - \lambda I)^{-1}. \tag{32}$$

If we substitute into $(30)_{2,3}$, we can use (20) to show that

$$\left.\begin{aligned} \frac{1}{k_1} r_{p+1} &= n_{ij}^{(\lambda)}(f_{j,u_k}^{(r)} m_{kl}^{(\lambda)} \alpha_l - f_{j,v_1}^{(r)}) w_{2r-1}, \\ \frac{1}{k_1} r_{p+q+1} &= n_{ij}^{(\lambda)}(f_{j,u_k}^{(r)} m_{kl}^{(\lambda)} \alpha_l - f_{j,v_1}^{(r)}) w_{2r}. \end{aligned}\right\} \tag{33}$$

But from (18), for any function G of $\underline{u}$ and v_1, we have

$$\left(\frac{\partial G}{\partial \underline{u}^T}, \frac{\partial G}{\partial v_1}\right) = \left(\frac{\partial \hat{G}}{\partial \underline{\psi}^T}, \frac{\partial \hat{G}}{\partial v_1}\right)\begin{pmatrix} A_1 - \lambda I, & \underline{\alpha} \\ 0 & , 1 \end{pmatrix},$$

where the superscript T denotes transposition, and $\underline{\alpha}$ is given by (16).

Hence we obtain

$$\left(\frac{\partial \hat{G}}{\partial \underline{\psi}^T}, \frac{\partial \hat{G}}{\partial v_1}\right) = \left(\frac{\partial G}{\partial \underline{u}^T}, \frac{\partial G}{\partial v_1}\right)\begin{pmatrix} M^{(\lambda)}, & -M^{(\lambda)}\underline{\alpha} \\ 0 \;, & 1 \end{pmatrix},$$

where $M^{(\lambda)}$ is given by (32).

Thus, for $G = f_j^{(r)}$, we see that

$$\hat{f}_{j,v_1}^{(r)} = f_{j,v_1}^{(r)} - f_{j,u_k}^{(r)} m_{kl}^{(\lambda)} \alpha_l. \tag{34}$$

Using (34) and (27), we find from (33) that

$$\frac{r_{p+1}}{r_{p+q+1}} = \frac{(F_r^{(\lambda)} + g^{(\lambda)}\delta_{ir}) w_{2r-1}}{(F_s^{(\lambda)} + g^{(\lambda)}\delta_{is}) w_{2s}}. \tag{35}$$

The condition for a simple wave to be transverse is that one of the generalised Riemann invariants be θ_1. This requires that $r_{p+1}/r_{p+q+1} = w_1/w_2$, and from (35) this is so if

$$F_r^{(\lambda)} = 0, \quad r = 2, \ldots, q.$$

From (28), since $\hat{h}_{1,v_1}^{(\lambda)} = 0$, this is so if and only if

$$\hat{h}_{j,v_1}^{(\lambda)} = 0, \quad j = 2, \ldots, q,$$

which, in view of (29). is identical to (21).

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