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# LETTERS TO THE EDITOR

## A NOTE ON TWO MEASURES OF DEPENDENCE AND MIXING SEQUENCES

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**Abstract**

In this note we establish an inequality between the maximal coefficient of correlation and the $\varphi$-mixing coefficient which is symmetric in its arguments. Motivated by this inequality, we introduce a mixing coefficient which is the product of two $\varphi$-mixing coefficients.

We also study an invariance principle under conditions imposed on this new mixing coefficient. As a consequence of this result it follows that the invariance principle holds when either the direct-time process or its time-reversed process is $\varphi$-mixing; when both processes are $\varphi$-mixing the invariance principle holds for sequences of $L_2$-integrable random variables under a mixing rate weaker than that used by Ibragimov.

### MAXIMAL COEFFICIENT OF CORRELATION

Let $(\Omega, K, P)$ be a probability space and $K_1$ and $K_2$ two $\sigma$-algebras contained in the $\sigma$-algebra $K$. Define the measures of dependence between $K_1$ and $K_2$ as follows:

$$\varphi(K_1, K_2) = \sup_{\{A \in K_1, P(A) \neq 0, B \in K_2\}} |P(B \mid A) - P(B)|$$

and

$$\rho(K_1, K_2) = \sup_{\left\{\substack{X \in L_2(K_1), \\ Y \in L_2(K_2)}\right\}} \frac{|E(X-EX)(Y-EY)|}{E^{\frac{1}{2}}(X-EX)^2 E^{\frac{1}{2}}(Y-EY)^2}.$$

The following well-known inequality ([5], Theorem 17.2.3, p. 309) relates the two measures of dependence.

Suppose $X$ is a random variable $K_1$-measurable and $Y$ a random variable $K_2$-measurable and $E^{1/p}|X|^p < \infty$, $E^{1/q}|Y|^q < \infty$, where $1/p + 1/q = 1$. Then

$$|EXY - EX \cdot EY| \leqq 2(\varphi(K_1, K_2) E|X|^p)^{1/p} (E|Y|^q)^{1/q} \tag{1}$$

whence

$$\rho(K_1, K_2) \leqq 2\varphi^{\frac{1}{2}}(K_1, K_2). \tag{2}$$

We notice that in (2) $\varphi$ is not symmetric in its arguments whereas $\rho$ is. We shall

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establish the following symmetric inequality which improves (1):

$$|EXY - EX \cdot EY| \leqq 2(\varphi(K_1, K_2)E|X|^p)^{1/p}(\varphi(K_2, K_1)E|Y|^q)^{1/q}, \tag{3}$$

whence

$$\rho(K_1, K_2) \leqq 2\varphi^{\frac{1}{2}}(K_1, K_2)\varphi^{\frac{1}{2}}(K_2, K_1). \tag{4}$$

*Proof of* (3). The proof of (3) follows in the same way as the proof of (1). We approximate $X$ and $Y$ by $X = \sum_i a_i I(A_i)$, $Y = \sum_j b_j I(B_j)$, where $(A_i)_i$ and $(B_j)_j$ are respectively, finite decompositions of $\Omega$ into disjoint elements of $K_1$ and $K_2$ and $I(A)$ denotes the indicator function of $A$. Using Hölder's inequality we obtain

$$\begin{aligned}
|EXY - EX \cdot EY| &\leqq \Big(\sum_i |a_i|^p P(A_i)\Big)^{1/p} \\
&\quad \times \Big[\sum_i P(A_i)\Big(\sum_j |b_j|\,|P(B_j \mid A_i) - P(B_j)|\Big)^q\Big]^{1/q} \\
&\leqq (E|X|^p)^{1/p}\Big[\sum_i P(A_i) \times \Big(\sum_j |b_j|^q\,|P(B_j \mid A_i) - P(B_j)|\Big) \\
&\quad \times \Big(\sum_j |P(B_j \mid A_i) - P(B_j)|\Big)^{q/p}\Big]^{\frac{1}{2}} \leqq (E|X|^p)^{1/p}(E|Y|^q)^{1/q} \\
&\quad \times \max_i \Big(\sum_j |P(B_j \mid A_i) - P(B_j)|\Big)^{1/p} \max_j \Big(\sum_i |P(A_i \mid B_j) - P(A_i)|\Big)^{\frac{1}{2}}.
\end{aligned}$$

If $C_i^+$ (or $C_i^-$) is the union of those $B_j$ for which $P(B_j \mid A_i) - P(B_j)$ is positive, (or non-positive) then

$$\sum_j |P(B_j \mid A_i) - P(B_j)| \leqq |P(C_i^+ \mid A_i) - P(C_i^+)| + |P(C_i^- \mid A_i) - P(C_i^-)| \leqq 2\varphi(K_1, K_2).$$

Similarly

$$\sum_i |P(A_i \mid B_j) - P(A_i)| \leqq 2\varphi(K_2, K_1)$$

so (3) holds for simple random variables, and by passing to the limit the inequality remains valid for every $X \in L_p(K_1)$ and $Y \in L_q(K_2)$.

Suppose now $(X_n, n = 0, \pm 1, \pm 2, \cdots)$ is a stationary sequence of random variables and denote by $F_n^m = \sigma(X_k, n \leqq k < m)$. For each $n \in N$ define

$$\varphi(n) = \varphi(F_{-\infty}^0, F_n^\infty)$$
$$\rho(n) = \rho(F_{-\infty}^0, F_n^\infty).$$

The sequence $(X_n)_{n \in Z}$ is said to be $\varphi$-mixing, or $\rho$-mixing, respectively, as $\varphi(n) \to 0$ or $\rho(n) \to 0$. It is known that there are sequences of random variables that are not $\varphi$-mixing, while their reverses are, (see [6], p. 414). For instance let $(X_n, n = 0, \pm 1, \pm 2, \cdots)$ be a stationary Markov chain with transition matrix $A_{1,j} = 2^{-j}$ and $A_{i,i-1} = 1$ for $j, i \geqq 1$. This sequence is not $\varphi$-mixing, but its reversed-time sequence, with transition matrix $B_{i,1} = B_{i,i+1} = \frac{1}{2}$ for all $i$, is $\varphi$-mixing. Therefore it seems natural to ask if the properties valid for $\varphi$-mixing sequences are valid for sequences of random variables with the time-reversed sequence $\varphi$-mixing, and the fact that both the direct and the reversed sequence are $\varphi$-mixing can improve on the $\varphi$-mixing rate in certain limit theorems.

The new relation between $\rho$ and $\varphi$ suggests that instead of the mixing coefficient $\varphi(n)$ we can consider another one, namely the product

$$\varphi(n)\varphi^r(n) = \varphi(F_{-\infty}^0, F_n^\infty)\varphi(F_n^\infty, F_{-\infty}^0).$$

The following theorem gives an invariance principle for stationary sequences of $L_2$-integrable random variables under conditions imposed on this new mixing coefficient. From this result we deduce that the invariance principle obtained by Ibragimov [4], Theorem (3.2), also holds for stationary sequences of $L_2$-integrable random variables whose time-reversed sequences satisfy a $\varphi$-mixing condition. When both the direct-time sequence and its reverse are $\varphi$-mixing the $\varphi$-mixing rate used in [4], Theorem (3.2), is improved (for instance for reversible $\varphi$-mixing sequences). This theorem also yields a functional form for Corollary 5.3. (i) of [3], which is a central limit theorem for sequences of random variables whose reversed-time sequences are $\varphi$-mixing. At the same time the mixing rate used there (polynomial) is improved (logarithmic).

Let $S_n = \sum_{i=1}^{n} X_i$, and let $[t]$ denote the greatest integer $\leqq t$.

*Theorem. Let* $(X_n, n = 0, \pm 1, \pm 2, \cdots)$ *be a stationary sequence of centered random variables which have* $L_2$*-moments and* $ES_n^2 \to \infty$. *Suppose also that*

$$\sum_i [\varphi(2^i)\varphi^r(2^i)]^{\frac{1}{2}} < \infty. \tag{5}$$

*Then there exists* $\sigma^2, 0 < \sigma^2 < \infty$ *such that* $\lim_n ES_n^2/n = \sigma^2$, *and the normalised sample paths* $W_n(t) = S_{[nt]}/n^{\frac{1}{2}}\sigma$, $(0 \leqq t \leqq L)$ *converge in distribution to the standard Brownian motion process* $W(t)$, $(0 \leqq t \leqq 1)$.

*Proof.* By (4) and (5) we have $\sum_i \rho(2^i) < \infty$, and, using Theorem 1 in [2], or Theorem (4.1) in [7], we obtain that $ES_n^2/n$ converges to a positive constant $\sigma^2 > 0$. The theorem follows by applying Theorem 19.2 of [1]. First $W_n(t)$ has asymptotically independent increments (see the proof of Theorem 20.1 of [1]). Then, by Lemma (3.5) of [7] it follows that $(S_n^2/n, n \geqq 1)$ is uniformly integrable, so $W_n^2(t)$ is uniformly integrable for each $t$ and obviously $EW_n(t) = 0$ and $EW_n^2(t) \xrightarrow[n\to\infty]{} t$. It remains only to verify the tightness condition, namely that for each $\varepsilon > 0$, there exists $\lambda > 1$ and an integer $n_0$ such that $n \geqq n_0$ implies $P(\max_{1 \leqq i \leqq n} |S_i| > \lambda\sigma n^{\frac{1}{2}}) \leqq \varepsilon/\lambda^2$. Without loss of generality we assume $\sigma^2 = 1$. If $\varphi_n \to 0$ this condition was verified in [1], pp. 175–176. If $\varphi_n^r \to 0$, the proof follows the same lines with the difference that we now denote

$$E_i^n = \left\{ \max_{0 \leqq j < i} |S_n - S_j| < 3\lambda n^{\frac{1}{2}} \leqq |S_n - S_i| \right\} \in F_i^n.$$

So, we have successively:

$$\begin{aligned}
P\left(\max_{i \leqq n} |S_i| > 4\lambda n^{\frac{1}{2}}\right) &\leqq P(|S_n| > \lambda n^{\frac{1}{2}}) + P\left(\max_{i \leqq n-1} |S_n - S_i| > 3\lambda n^{\frac{1}{2}}\right) \\
&\leqq 2P(|S_n| > \lambda n^{\frac{1}{2}}) + \sum_{i=1}^{n-1} P(E_i^n \cap \{|S_i| > 2\lambda n^{\frac{1}{2}}\}) \leqq 2P(|S_n| > \lambda n^{\frac{1}{2}}) \\
&\quad + \sum_{i=1}^{p} P(|S_i| > 2\lambda n^{\frac{1}{2}}) + \sum_{i=p+1}^{n-1} P(|S_i - S_{i-p}| > \lambda n^{\frac{1}{2}}) + \sum_{i=p+1}^{n-1} P(E_i^n \cap \{|S_{i-p}| > \lambda n^{\frac{1}{2}}\}) \\
&\leqq 2P(|S_n| > \lambda n^{\frac{1}{2}}) + nP(S_p^* > \lambda n^{\frac{1}{2}}) + \sum_{i=p+1}^{n-1} P(E_i^n)(P(|S_{i-p}| > \lambda n^{\frac{1}{2}}) + \varphi^r(p))
\end{aligned}$$

where $p$ and $S_p^*$ were defined in [1], p. 175. This gives the desired result.

With a similar proof it is easy to see the following.

*Remark.* This theorem can be obtained for some non-stationary sequences of random variables $(X_n, n \geqq 1)$, namely, we can assume instead of stationarity that $(X_n^2, n \geqq 1)$ is uniformly integrable and $E\left(\sum_{i=kn}^{(k+1)n} X_i\right)^2 \Big/ ES_n^2 \to 1$ as $n \to \infty$ uniformly in $k$, the mixing coefficients $\varphi(n)$ and $\varphi^r(n)$ being defined by

$$\varphi(n) = \sup_m \varphi(F_0^m, F_{m+n}^\infty) \quad \text{and} \quad \varphi^r(n) = \sup_m \varphi(F_{m+n}^\infty, F_0^m).$$

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## References

[1] Billingsley, P. (1968) *Convergence of Probability Measures.* Wiley, New York.

[2] Bradley, R. C. (1981) A sufficient condition for linear growth of variances in a stationary random sequence. *Proc. Amer. Math. Soc.* **83,** 583–589.

[3] Hall, P. and Heyde, C. C. (1980) *Martingale Limit Theory and its Application.* Academic Press, New York.

[4] Ibragimov I. A. (1975) A note on the central limit theorem for dependent random variables. *Theory Prob. Appl.* **20,** 135–140.

[5] Ibragimov, I. A. and Linnik, Yu. V. (1971) *Independent and Stationary Sequences of Random Variables*, Walters-Noordhoff, Groningen, The Netherlands.

[6] Kesten, H. and O'Brien, G. L. (1976) Examples of mixing sequences. *Duke Math. J.* **43,** 405–415.

[7] Peligrad, M. (1982) Invariance principles for mixing sequences of random variables. *Ann. Prob.* **10,** 968–981.