PATTERNS OF APERIODIC PULSATION

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Abstract. Techniques for deriving amplitude equations for stellar pulsation are outlined. For the simplest such equations with multiple instabilities, the derivation of a map for the patterns of pulsation phases is described. This map gives the time between two successive maxima of pulsation in terms of the time between the previous pair, under suitable conditions. The phase differences can be regular, chaotic or hyperchaotic.

1. Introduction

My aim in this paper is to sketch how we can try to understand chaos in stellar pulsation through simple models. The results are still a long way from insight into the bursting and intermittency that we see in many cosmic bodies from the sun to the galaxies, but I hope it is a beginning. The story has evolved slowly, starting with a three paragraph abstract (Baker et al., 1966) published some time ago. I reproduce the first paragraph of that paper in the Appendix. Though different in approach, the present paper is offered as a substitute for a complete paper, which was not written. Nor was the equation on which the work was based published. Nevertheless, I did set out to respond to skeptics who were unwilling to believe that a simple model could produce such complexity or that this could represent real stellar behavior. This paper is a summary of that answer, achieved with the help of many. It describes how to go from the primitive pulsation equations to an algebraic description of chaotic phase variations of pulsation, essentially completely analytically.

In retrospect, it must be admitted that our original derivation of a third-order differential equation for pulsation theory was rough, for all our brash confidence in it. However, we now know how to derive such equations by more formally correct procedures, and I will review that aspect of the problem here before describing how to go on to the reductions that permit analysis of the behavior of their solutions. Such analysis is the part of the theory that has not been discussed in pulsation theory as yet, as for as I know. It shows how even a pulsation that looks periodic can contain chaotic phase variations, as was implied by numerical integrations by Baker and M. C. Depassier on the nonlinear one-zone model. Their theoretical O-C diagrams, reported in a lecture at Columbia in seventies, showed the kind of mild chaos that can now be understood by asymptotic methods. Such results are robust, as is shown in the work of several groups using simple equations to model stellar pulsation theory, and as you will see from other papers in this volume. In particular, the work summarized by Buchler makes possible

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detailed comparison between the numerical solutions of the full pulsation equations and of the simplified systems.

For those who are already willing to accept the relevance of simple equations, either as models for complicated behavior or as asymptotically correct limits of the full equations, there may be no need to review the derivations. Such readers can simply skip right to the analysis showing how chaos arises, starting with §5. There they will see how to use singular perturbation theory to obtain results that can complement the usual numerical exploration of chaotic solutions. Indeed, I do not describe numerical results at all. Instead I make my points through asymptotic methods that lead to simple algebraic recursion formulae for pulsational phase shifts.

2. Amplitude Equations

In this section I want to recall when and why a description of stellar pulsation in terms of simplified models may be possible. Let us think about the equations of stellar radial pulsation in general from, using $\mathbf{U}(\mathcal{M}_r, t)$ to designate the column vector whose components are the usual dependent physical variables of pulsation theory, $\rho, T, r, \dot{r}, \ldots$ and choosing $\mathcal{M}_r$, the mass within radius r, as the independent variable. We describe the form of the equations for radial pulsation as

$$\partial_t \mathbf{U} = \mathcal{F}(\mathbf{U}, \partial_{\mathcal{M}_r}). \tag{1}$$

A hydrostatic state of the system, $\mathbf{U}_0$, satisfies the condition

$$\mathcal{F}(\mathbf{U}_0, \partial_{\mathcal{M}_r}) = \mathbf{0}. \tag{2}$$

We suppose that solutions fulfilling this condition are distinct and we presume that we may ignore the slow changes caused by stellar evolution. These could be incorporated in the treatment at the cost of some complications, but I leave them out for simplicity and as an illustration of the main point of this section.

In considering the dynamics in the neighborhood of one of these static solutions, we introduce the disturbance vector, $\mathbf{u} = \mathbf{U} - \mathbf{U}_0$. We can then generally rearrange (1) into the form

$$\partial_t \mathbf{u} = \mathbf{L}\mathbf{u} + \mathbf{N}(\mathbf{u}), \tag{3}$$

where $\mathbf{L}$ and $\mathbf{N}$ are linear and nonlinear operators whose structure will generally depend on $\mathbf{U}_0$. This dependence may often be expressed in terms of control parameters that are functionals of the static mode, such as surface gravity and helium abundance.

We first look at the linear theory, which begins with the omission of the terms involved in $\mathbf{N}$. Then we may seek solutions of the form $\mathbf{u} =$

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$a(t)\Phi(\mathcal{M}_r)$. For many of the solutions of the linear theory, the time dependence is simply like $\exp(\lambda t)$ where λ is an allowed value of the characteristic value problem

$$\mathbf{L}\Phi = \lambda\Phi. \tag{4}$$

In general, λ is complex and we write $\lambda = \mu + i\omega$ where μ and ω are real. We restrict ourselves to the case of mild instability, in which no values of μ much greater than zero. The assumption limits the strict range of applicability of our results. Nevertheless, they do provide us with a locally accurate view of the possible behavior and permit its rapid qualitative exploration.

Near to the critical situation, in which the parameters of the problem are carefully chosen so that the allowed values of λ have either $\mu = 0$ or $\mu < 0$, we distinguish between slow modes ψ_n with $|\mu|$ small and fast modes ϕ_m with $\mu < 0$ and of order unity. We assume that there are a few (N) show modes and an infinite number of fast ones. Postponing the issue of completeness to the next section, we decompose the solution vector into normal modes as

$$\mathbf{u}(\mathcal{M}_r, t) = \sum_{n=1}^{N} a^n(t)\psi_n(\mathcal{M}_r) + \sum_{m=1}^{\infty} b^m(t)\phi_m(\mathcal{M}_r). \tag{5}$$

On introducing this expansion into (3) and projecting out the coefficients, we get equations of the general form

$$\frac{d}{dt}\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathcal{M} & \mathbf{0}\\ \mathbf{0} & \mathcal{K}\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix} + \begin{pmatrix}\mathbf{F}(\mathbf{a},\mathbf{b})\\ \mathbf{G}(\mathbf{a},\mathbf{b})\end{pmatrix}, \tag{6}$$

where the $N \times N$ submatrix $\mathcal{M}$ has as proper values all the λ with small $|\mu|$ and the infinite matrix $\mathcal{K}$ has the paper values corresponding to the rapidly decaying modes. We denote the amplitude vectors (a^n) and (b^m) as $\mathbf{a}$ and $\mathbf{b}$.

For a moment, think of a^n and b^m as the abundances of chemical or nuclear species in some well-mixed medium and (6) as the rate equations for the various reactions. The rapidly reacting species (b^m) go quickly to equilibrium, but any changes in the equilibrium state are slow since they are dictated by the instantaneous abundances (a^n) of the slow reactors. So $\mathbf{b}$ is a function of $\mathbf{a}$ and the evolution equation of the system becomes an equation in terms of $\mathbf{a}$ alone. This reasoning, familiar in chemical kinetics, applies to normal modes of the pulsation problem as well, and it is known as center manifold theory (Carr, 1981). It is the same idea as we used when we assumed at the outset that the star is in equilibrium, even though we know that it is slowly evolving.

Formally then, we have $\mathbf{b} = \mathcal{B}[\mathbf{a}]$, and, on introducing this into (6), we get the "amplitude equation"

$$\dot{\mathbf{a}} = \mathcal{M}\mathbf{a} + \mathbf{g}(\mathbf{a}), \tag{7}$$

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where $\mathbf{g}(\mathbf{a}) = \mathbf{F}(\mathbf{a}, \mathcal{B}[\mathbf{a}])$. Since the stable modes would simply die out if they were not driven by the nonlinear couplings to the slow modes, we understand that $\mathcal{B}$ and $\partial_a \mathcal{B}$ vanish for $\mathbf{a} = \mathbf{0}$.

The form of $\mathcal{M}$ comes from the linear theory and we take this up next. The specific details of g can be derived from the full problem so it is typically rather complicated to calculate it in numerical detail, but we can work out its general form, once $\mathcal{M}$ is known. In this respect, the situation is like fluid dynamics. We know the form of Navier-Stokes equations quite well, and we do not recalculate the viscosity every time we wish to use the equations for a different particular fluid. In this spirit, we can derive the derivation of the general equations that come up in all nonlinear instability problem, irrespective of details. I have summarized these matters before (Spiegel, 1985), so I shall recall only the main ideas in the next two sections.

3. The Linear Problem

To clarify the physics of ordinary vibrational instability, Baker (1966) isolated the key terms describing the process in his one-zone model. He started from the partial differential equations, or p.d.e.s. for radical motion. By finite differencing, in fact, he reduced these p.d.e.s to ordinary differential equations for the case with only one radial zone. We extended his derivation to the nonlinear case and I set out to write the present review by taking the published abstract of that extension (Baker et al., 1966) as the abstract for this paper. That did not quite work so, inspired by the fifteenth century rabbi who wrote "In may end is my beginning," I have put the first paragraph of the '66 abstract in the appendix.

In seeking to explain aperiodic pulsations, we set out to find a third order nonlinear equation like the one that had come up in overstable convection theory (Moore and Spiegel, 1966). I will not recall our physically motivated derivations here, but shall show how to proceed in the applied mathematical way to get similar results.

The radial oscillations of a spherical star obey an equation of the form

$$\partial_t^2 r = -\mathcal{G}_r + 4\pi r^2 \frac{\partial p}{\partial \mathcal{M}_r} \tag{8}$$

where $\mathcal{G}_r = G\mathcal{M}_r/r^2$, $d\mathcal{M}_r = 4\pi r^2 \rho dr$ and p and ρ are pressure and density. Through the pressure and the equation of state, this equation is coupled into the heat equation which contains the luminosity at radius r. Since the latter is a first-order equation, we arrive at a system of third order in time. In the one-zone approximation, one removes the radial dependence from the right hand sides by coarse finite differencing and obtains a third-order ordinary differential equation, or o.d.e. That kind of equation can be derived

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systematically from the full equations by introducing a suitable choice of critical conditions, as we shall indicate.

Baker's linear third-order equation for the radial displacement, r', has three parameters as coefficients and, if we look for temporal behavior of the perturbations of the form $r' \propto \exp(\lambda t)$, these three coefficients appear in the equation of λ:

$$\lambda^3 = \alpha\lambda^2 + \beta\lambda + \gamma. \tag{9}$$

In the same way, there will be parameters in the complete models that can be tuned to bring on various instabilities. As I have said, we will work near to the critical situation of marginal stability since that makes things tractable analytically. There is no need to do that in numerical studies, once we have seen the kind of behavior we want to look for.

The simplest situation is the one in which one mode passed through marginality as we tune a parameter. That is, for real $\lambda = \mu, \mu$ passes through zero as the control parameter passed through a critical value. In (9), this critical situation occurs for $\gamma = 0$ with appropriate values of α and β, but we could just as well consider μ itself as the control parameter; this may be harder to do in practice, but it makes the principles easier to think about. The nonlinear outcome is an approach to a new steady state, as in the familiar Landau equation.

We could also have a simple passage to instability when, for a complex pair of proper values $\lambda = \mu \pm i\omega, \mu$ passes through zero. For ω bounded away from zero, we again pass through marginality by tuning a single parameter, expressed most simply as μ itself, and we find overstability. I will not dwell on this case either, for it is familiar and gives rise to limit cycles in the nonlinear regime.

To achieve a richer behavior, we can tune two parameters. For example, we may arrange to have two values of μ, for distinct conjugate pairs of oscillatory modes, pass through criticality together. Or we can tune both the μ and the ω for a single conjugate pair. In either case, there is a double zero for λ at criticality. When we move slightly off criticality, we need to control two parameters to lift this degeneracy since the two values of λ need two parameters to be fully characterized. The simplest description can be extracted from (9) when β and γ tend to zero for $\alpha < 0$. To study that case, we divide (9) by $\lambda - \alpha$ and, for small λ/α, find to leading order that $\lambda^2 + p\lambda + q = 0$ gives the proper values of the slow modes, where $p = (\gamma + \alpha\beta)/\alpha^2$ and $q = \gamma/\alpha$. The nonlinear amplitude equation associated with this case must be a second-order differential equation.

The point of this example is to suggest how an equation like (9) can in turn be similarly extracted from the linear theory when there are three show modes. Suppose that in the full problem there exists some equation $\mathcal{T}(\lambda) = 0$ for all the proper values of λ, even if it is known only numerically.

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In the neighborhood of a critical point for N modes, I can always do the same trick on $\mathcal{T}$ to get an N^{th}-order polynomial whose roots give me the λs for the slow modes near to marginality (Coullet and Spiegel, 1983). The calculation of the coefficients in this polynomial may need to be numerically assisted in real situations, but, near to criticality, such an equation is valid and does not rely on rough arguments for its justification.

In the tricritical case, (9) we need the freedom to control three parameters in order to make three modes become slow at once. I will not enumerate the diverse possibilities of three mode instabilities (see Tresser, 1984) but will concentrate on the situation where has a triple zero at marginality.

As we see from the previous section, our linear problem has the form

$$\partial_t \mathbf{u} = \mathbf{L}\mathbf{u}. \tag{10}$$

At marginality, the normal modes are null vectors such that $\mathbf{L}\mathbf{\Phi} = \mathbf{0}$. However, since $\mathbf{L}$ is not self-adjoint, it will sometimes not have as many null vectors as there are zeros of λ. In this case, the conventional normal modes do not constitute a complete set and we need to add some more basis modes.

To see this, we consider three matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11a,b,c}$$

Matrix (a) has three characteristic vectors associated with the proper value zero, (b) has only two and (c) has only one. The critical conditions they describe each have triple zeros for λ, but they correspond to different dynamical situations: (a) arises in the Lotka-Volterra equations (or predator-prey equations) with three species, (b) figures in the Lorenz equations, and (c) is the one we chose, both in our convective and pulsational examples, so I will stay with that one here, even though it is the rarest of the three cases.

To get a complete set in case (c), we need to have two additional slowly growing solutions of the linear problem. That is provided by the general slow solution of (10), $\varphi_1(\mathcal{M}_r) + t\varphi_2(\mathcal{M}_r) + \frac{1}{2}t^2\varphi_3(\mathcal{M}_r)$ where $\mathbf{L}\varphi_1 = \varphi_2$, $\mathbf{L}\varphi_2 = \varphi_3$ and $\mathbf{L}\varphi_3 = 0$. Hence the transpose of matrix (c) is a matrix representation of $\mathbf{L}$ at criticality. (I put the transpose in (11) since that is what operates on the vector a^n in the amplitude equation.)

Slightly off criticality we want the proper values of λ to be the roots of (9). Hence, we need to modify matrix (c) so that it includes the three parameters α, β, γ of that equation when they are nonzero. There are many ways that are satisfactory of doing this and one standard choice, or normal form, is the following.

In the neighborhood of criticality we write the slowly evolving part of the solution as $\mathbf{u} = \sum_{n=1}^{3} a^n(t)\psi_n(\mathcal{M}_r)$. In order to fulfill (10) we then demand that

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$$\dot{\mathbf{a}} = \mathcal{M}\mathbf{a} \tag{12}$$

where $\mathcal{M}$ is an appropriate three by three matrix. That is, we need to choose a matrix whose proper values are the ones we have found to represent our situation, either numerically or otherwise. In particular, for our present example, we may use the Jordan-Arnold form (Gilmore, 1981)

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & \beta & \alpha \end{pmatrix}. \tag{13}$$

This leads to the cubic (9) and it reduces to the matrix (11c) at criticality.

Near to the tricritical point for three slow modes, we have reduced the linear theory for the amplitudes of the triplet of slow modes to (12)–(13). This illustrative case is the one whose structure coincides with the one-zone model but the procedure is general. Now nonlinear terms must be brought into the theory to represent the coupling to stable modes, which inhibits the continued growth of the unstable modes. The situation has an analogy to particle physics. The damped modes are like virtual particles. When the slow modes begin to develop, they excite stable, or "virtual," modes that otherwise would die off quickly. Typically, the effect of these excited modes is to modify, or renormalize, the effective growth rates of the unstable modes through nonlinear terms that keep the amplitudes of the unstable modes from growing indefinitely. That is the content of the nonlinear generalization of (12).

4. The Pulsational Amplitude Equation

As we discussed in §2, the nearly marginal modes satisfy a nonlinear version of (12), which we write as

$$\dot{\mathbf{a}} = \mathcal{M}\mathbf{a} + \mathbf{g}(\mathbf{a}), \tag{14}$$

where $\mathbf{g}$ is strictly nonlinear. The question now is, what is the form of $\mathbf{g}$? Remarkably, this is fixed by our choice of $\mathcal{M}$, which is why we gave it so much attention in the previous section.

It is simplest to obtain $\mathbf{g}$ at marginality and that provides an adequate approximation for many purposes. The center manifold theorem applies at marginality and it says that

$$\mathbf{u}(\mathcal{M}_r, t) = \mathbf{v}(\mathcal{M}_r, \mathbf{a}(t)). \tag{15}$$

This is the usual starting point of the Bogoliubov method of asymptotic theory and it offers a revealing way to look at and develop nonlinear stability theory (Coullet and Spiegel, 1981, 1983). The time dependence of the solutions is carried by the internal variable $\mathbf{a}$ which gyrates in its own space according to (14). So we can turn (3) into

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$$\mathcal{L}\mathbf{v} = \mathbf{N}(\mathbf{v}) - \mathbf{g} \cdot \partial_{\mathbf{a}}\mathbf{v} \tag{16}$$

where

$$\mathcal{L} = \mathcal{M}\mathbf{a} \cdot \partial_{\mathbf{a}} - \mathbf{L}. \tag{17}$$

For convenience in writing powers of the amplitudes, we introduce the notation $a^1 = A$, $a^2 = B$, $a^3 = C$. For example, at criticality, the first term in (17), a scalar operator, is

$$\Sigma := \mathcal{M}\mathbf{a} \cdot \partial_{\mathbf{a}} = B\partial_A + C\partial_B. \tag{18}$$

This is a representation of the angular momentum operator $\mathbf{J}_+$ of quantum mechanics for the case of particles of spin one. Moreover, the second term in $\mathcal{L}$ has $\mathcal{M}$ in its representation, and that is the matrix representation of $\mathbf{J}_+$. So both parts of $\mathcal{L}$ have similar structures, one operating on functions of position ($\mathcal{M}_r$) and the other operating on functions of the internal variable ($\mathbf{a}$). Hence $\mathcal{L}$ itself is a representation of $\mathbf{J}_+$ (Spiegel, 1985), so it is an annihilation operator.

We are thinking about a star in conditions that are close to those described by $\alpha = \beta = \gamma = 0$, when the instability is weak. So we expect that the components of $\mathbf{a}$ are not too large and that Taylor series in those components may be used. The procedure then is to make Taylor expansions of $\mathbf{v}$ and $\mathbf{g}$ in terms of monomials $A^k B^l C^m$. The linear solution satisfies $\mathcal{L}\mathbf{v}_1 = \mathbf{0}$, hence we readily find $\mathbf{v}_1 = \mathbf{a} \cdot \psi = A(t)\varphi_1 + B(t)\varphi_2 + C(t)\varphi_3$. For each higher order, we have to solve an equation of the form

$$\mathcal{L}\mathbf{v}_S = \mathbf{I}_S - \mathbf{g}_S \cdot \psi, \tag{19}$$

where $S = k+l+m$ and $\mathbf{I}_S$ is a function of $\mathbf{v}_{S-1}, \mathbf{v}_{S-2}, \ldots$ and $\mathbf{g}_{S-1}, \mathbf{g}_{S-2}, \ldots$.

In solving this sequence of inhomogeneous equations order by order, we need to group the possible terms in the "wave functions" $\mathbf{v}_S$ into multiplets of terms that are not mixed by the action of $\mathcal{L}$. The terms within each multiplet are transformed into one another by the action of $\mathcal{L}$, as we just saw, for the effect of $\mathbf{L}$ on the φ_n. However, the last member of the multiplet is annihilated and the first one is not regenerated. For any multiplet, if there is a term in $\mathbf{I}_S$ that corresponding to this first member, it cannot be generated by operation with $\mathcal{L}$. Hence, a solution of the perturbation theory can be produced only if that term can be eliminated by a suitable term in $\mathbf{g}_S$. This requirement dictates the terms that must occur in $\mathbf{g}_S$ and that is how $\mathbf{g}$ is determined, up to some arbitrary gauge choices. The coefficients of the terms in $\mathbf{g}_S$ are reminiscent of Clebsch-Gordan coefficients.

These asymptotic methods thus permit a systematic derivation of the amplitude equations (Arneodo et al., 1982, 1985ab), however there are many equivalent ways to write them. We can arrange the results so that $\mathbf{g}$ has only a third component. Of these, four are quadratic terms, six are cubic terms, and so forth. One way to express the outcome is

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$$\dddot{A} - f(A, \dot{A}, \ddot{A})\ddot{A} - g(A, \dot{A})\dot{A} - h(A)A = 0. \tag{20}$$

where

$$f(A, \dot{A}, \ddot{A}) = \alpha + \alpha_1 A + \alpha_2 A\ddot{A} + \alpha_3 A^2 + O(A^3), \tag{21}$$

$$g(A, \dot{A}) = \beta + \beta_1 A + \beta_2 \dot{A} + \beta_3 A^2 + \beta_4 A\dot{A} + \beta_5 \dot{A}^2 + O(A^3), \tag{22}$$

$$h(A) = \gamma + \gamma_1 A + \gamma_2 A^2 + O(A^3). \tag{23}$$

In this equation, we see that the nonlinear terms act to saturate or renormalize the growth rates of the linear theory when the amplitudes become large enough. There are a few too many of these nonlinear terms to keep track of and, in most studies, special choices of the coefficients in them have been made to limit the complications. In any case, when we are near enough to tricriticality, the dominant nonlinear term is either A^2 or A^3, depending on symmetries (Arneodo, et al., 1985b).

In the study of overstable convection that I referred to Moore and Spiegel (1966), the instability caused by the α term was turned off by the choice $\alpha < 0$. So the nonlinear terms that protect against that instability did not play a qualitatively significant role and were not needed. That was why we got by with $f = \alpha$. As to the choices of g and h, we kept them arbitrary for a time, but specialized to a simple choice for specific calculations. Some writers have assumed that this choice also represented the equation referred to in the 1966 abstract about stellar pulsation theory (Baker et al., 1966), perhaps on account of our loose description of it.

The equation referred to in the Appendix is a nonlinear one zone model, and though it is a special case of (20), it is not included in the possibilities offered by (21)–(23). To bring (21)–(23) into that form, we need one more step, Padé resummation. That is, if we rewrite g and h as rational functions, we get an equation like that derived for the one zone case by qualitative arguments. I will not write that out here. But I would like to add that the earlier study of overstable convection was also motivated by an interest in solar oscillations. We suspected that sound waves were subject to the same kind of convective overstability as rotating and magnetic convection (Chandrasekhar, 1961), hence sought a generic model of convective overstability.

5. Pattern Maps

I have indicated how, for a star with N slowly evolving modes, we may derive an Nth order o.d.e. by asymptotic methods. In general, these give an amplitude equation of the form

$$\dot{\mathbf{a}} = \mathcal{M}\mathbf{a} + \mathbf{g}(\mathbf{a}). \tag{24}$$

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The simplest case for stellar pulsation arises with a complex conjugate pair of oscillatory modes going unstable. In that case, (24) corresponds to two equations, but they are complex conjugates of each other. So it really describes a first-order equation for a complex amplitude, hence a two dimensional phase space is involved. With more slow modes, (24) becomes of higher order. A common situation has two oscillatory modes and, in the richest example that I will refer to, we have a four dimensional phase space. For illustration, I will sometimes refer to a case of intermediate complexity, an asymptotic version of the third order system already discussed at length. However, the results I describe in this section apply also to the more familiar case of pulsation theory with two pairs of oscillatory modes.

Rather than speak of the general third-order system, I will refer to the asymptotic limit that it takes as we approach tricriticality. That limit can be found by careful application of the Poincaré-Linsteadt method for $N = 3$. Whether we do this for the general pulsation equations or for the amplitude equation (14), we get the same answer (Arneodo et al., 1985b) and this can be expressed either by specifying

$$\mathbf{a} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ A^2 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & \beta & \alpha \end{pmatrix}. \tag{25}$$

or simply writing it out as

$$\dddot{A} - \alpha\ddot{A} - \beta\dot{A} - \gamma A = A^2. \tag{26}$$

In either form, I will use this special case for illustration.

For this particular equation, we may refer to a theorem of Shil'nikow to conclude that, for suitable choices of the parameters, (26) will have an infinite number of unstable periodic solutions. This is, in effect, the statement that there will be chaotic behavior as the system wanders from one to another periodic solution (Baker, Moore and Spiegel, 1971). To see this explicitly, the arguments of Shil'nikov may be used to extract a Poincaré map for the problem to good accuracy in terms of elementary functions (Arneodo et al., 1985). From that we see that there are sound asymptotic arguments that show that the solutions of the stellar oscillation equations may be chaotic if there exist static tricritical states. The same kind of discussion can be made for any number of other polycritical situations.

Another approach to studying the behavior of the solutions of the amplitude equations is suggested by the observation of Coullet and Elphick (1987) that the effective particle method of nonlinear field theory may be used on (26) to obtain results very like those following from the Shil'nikov arguments. An advantage of this alternative approach is that the results come out in terms of measurable quantities. In view of this, it has seemed worthwhile to

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systemize this procedure for general use (Elphick et al., 1990a), though it has been so far used mainly for studying p.d.e.s.

In studying the behavior of the amplitude equations, it is useful to isolate the fixed points and the homoclinic orbits of the system. The former are solutions with $\dot{\mathbf{a}} = \mathbf{0}$ and, in particular, we focus on $\mathbf{a} = \mathbf{0}$. A homoclinic orbit is one that is biasymptotic to a fixed point, that is, on this orbit, the system approaches the fixed point as $t \to \pm\infty$. A picture of $A(t)$ for a homoclinic orbit of (10) is shown in the figure. Both the fixed point and the homoclinic orbit are solutions with infinite period.

For the example in the figure, we have selected the parameters in the system so that the linear theory in the neighborhood of the origin has one monotonically growing solution, $\lambda = \mu_1 > 0$, and a damped oscillation, $\lambda = \mu_2 \pm i\omega$, $\mu_2 < 0$. Associated with each μ_i, there is a proper vector. Starting from initial conditions near the origin of phase space, the solution takes off along the vector associated with μ_1 with A growing like $\exp(\mu_1 t)$. After a time (during which it may take a turn or two around the fixed point with $A = -\gamma$) it spirals back in toward the origin coming into the plane formed by the two proper vectors associated with μ_2. In the figure, we see the exponential rise and the oscillatory decay suggested by our choice of proper values of λ.

I shall call the solution shown in the figure a principal homoclinic orbit since it corresponds to one turn around the other fixed point of (26). Secondary homoclinic solutions can also be found in which the solutions make several loops around the other fixed point before returning to the origin. Those solutions look somewhat like the figure, except that the peak has fine structure. The secondary homoclinic orbits may be considered as closely spaced groupings of the primary homoclinic orbit and need not be treated separately for our purposes. When solutions of (26) are obtained numerically, they typically show series of pulses, which may be regularly or irregularly spaced according to the parameter choices. The pulses have a characteristic shape that is approximated by the principal homoclinic orbit of the figure.

The general case, (24), is much the same. If there are only a couple of real modes in the problem, both the rise and the fall of $A(t)$ must be monotonic. However, for some examples with coupled oscillatory modes, we can expect oscillations in both the rise and decay. All these details can make a difference to the appearance of the signal (Elphick et al., 1990b). When there are even more modes in the problem, there can be several fundamental homoclinic orbits and the plot thickens. Not much calculation has been done in those cases, and I will say no more of them.

Suppose that we are in a low-dimensional situation with a simple homoclinic solution $A = H(t)$. The phase of this pulse is arbitrary. That is, if $H(t)$ is a solution of (26) or (24), so is $H(t-\tau)$, where τ is a constant. In fact, τ

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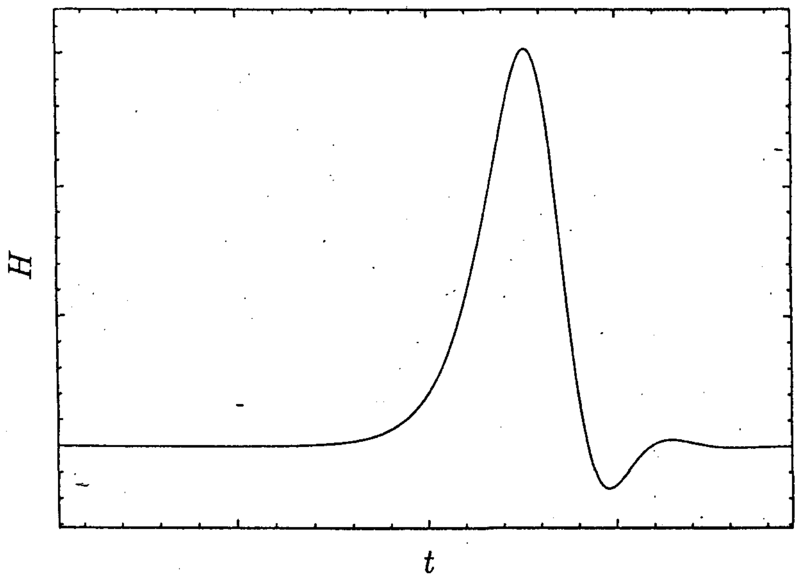

Fig. 1. A(t) for a homoclinic orbit of (26), adapted from Elphick et al., (1991).

is the parameter of the group of transformations that shift the origin of the time and the presence of this group means that the solution is indifferent to the value of τ. Let us choose the phase so that $H(t-\tau)$ has its maximum at $t=\tau$. We are going to look for a more general solution with a sequence of pulses at times τ_n for $n = 1, 2, \ldots$. But before writing this down, we need to note a few things.

First, we have to decide at which level we want to work. We can study the behavior of just $A(t)$ or go to the full amplitude vector **a**. Indeed there are times when we might wish to look for waves in the original p.d.e.s. of stellar pulsation theory and work with them to describe the spatio-temporal nature of the full solutions. For the present illustration, I shall stick with $A(t)$, as it is the simplest thing to do.

Then we need to decide something about the parameter range. The relative magnitudes of the rise and decay times of the pulses, as measured by μ_1 and μ_2 are important. Let $\mu_1 = \mu$ and $\mu_2 = -\mu\xi$, where μ and ξ are positive. In this discussion, I shall assume that ξ is of order unity; if it were not, the results would be different. Thus there are many other possible behaviors, which complicated (or enriches, if you prefer) these problems.

Finally, suppose that the mean spacing between pulse times is T. Then we can define a key parameter of the theory:

$$\varepsilon = e^{-T\mu}. \tag{27}$$

Throughout this discussion, I assume that $\varepsilon \ll 1$, that is, that the pulses do not overlap significantly. This is a situation often perceived in numerical

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studies; other situations may arise, but they are not so easily characterized.

When the pulses in the solutions are distinct, they generally look like $H(t)$, which we may use a building block for a series of pulses. That is, we can look for a solution of the form

$$A(t) = \sum_m H(t - \tau_m) + \varepsilon\mathcal{R}(t, \varepsilon), \tag{28}$$

where the total number of pulses may be infinite, and $\varepsilon\mathcal{R}$ is the error made in trying to use a linear superposition of solutions in a nonlinear problem. The form of this error term expresses the belief that, as the spacing of the pulses increases, the error diminishes.

The procedure now is to base a perturbation expansion on (28) for small ε. We develop $\mathcal{R} = \sum_k \varepsilon^k \mathcal{R}_k$ and get a series of linear equations for the $\mathcal{R}_k$. We want to find finite solutions for the $\mathcal{R}_k$, and especially for $\mathcal{R}_0$, so that the error term, $\varepsilon\mathcal{R}$, should tend to zero as $\varepsilon \to 0$. However, if we proceed naively, this hope is not realized. That is, if we simply choose a set $\{\tau_m\}$ we find that $\mathcal{R}_0$ diverges — the perturbation theory is singular. Even though the phase of the simple homoclinic orbit is arbitrary, the phases of individual pulses cannot be chosen independently of each other. Up to a constant phase shift for the whole solution, the choice of the $\{\tau_m\}$ is not free and must be determined from the equations.

As is usual in singular perturbation problems, we broaden our outlook and let τ_k be a function of εt. Then, when we look at the problem of calculating $\mathcal{R}_0$, we find that the condition that this be possible in finite terms is a condition on $\dot{\tau}(\varepsilon t)$, where we use the dot to imply derivative with respect to argument. This solvability condition for $\mathcal{R}_0$ may be obtained in the normal way on multiplying by a null vector of the adjoint linear operator as spelled out in the references I have cited. The matter can be resolved in simple terms when certain simplifying approximations are included. We focus on the neighborhood of the m^{th} pulse and make the approximation that only the interactions with its nearest neighbors, the $(m \pm 1)^{th}$ pulses, matter. Then, we get coupled equations of motion for all the τ_m.

We consider (24) in two, three or four dimensions, where there are at most two conjugate pairs of proper values of $\mathcal{M}$, say $\mu_1 \pm i\omega_1$ and $\mu_2 \pm i\omega_2$. As before, $\mu_1 = \mu$ is positive and $\mu_2 = -\mu\xi$, with $\xi \sim 1$. For a three-dimensional equation like (26), we can have only one of the two frequencies nonzero but, more generally, both the rising and the decaying edges of the pulse may be oscillatory if there are enough slow modes in the original problem. For pulse shapes with either oscillatory or monotonic rise and decay, we find (Elphick et al., 1990)

$$\dot{\tau}_m = \zeta_1 e^{-\mu\Delta_{m+1}} \cos(\omega_1\Delta_{m+1} + \theta_1) + \zeta_2 e^{-\mu\xi\Delta_m} \cos(\omega_2\Delta_m + \theta_2), \tag{29}$$

where

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$$\Delta_m = \tau_m - \tau_{m-1}, \tag{30}$$

and the ξ_i and θ_i ($i = 1, 2$) are constants that depend on the pulse shape.

This is the same kind of result that you would get for solitary waves described by p.d.e.s with translational invariance. That leads to the interpretation of (29) as an equation of motion for the m^{th} pulse under the influence of its nearest neighbors. In this dynamics, εt is the time and τ is like position. The pulses have no inertia in this example, so the velocity is given directly by the forces. Since the problem is one-dimensional, the force does not show an algebraic dependence on the separation but it does have an exponential cutoff, or range. There is also the interesting effect of the ripples in the force law coming from the oscillatory character of the modes. This permits bound states of pulse pairs to form at differing phase separations.

According to this system of equations of motion, the pulses tend to move into a locked pattern with constant velocity, $\dot{\tau}_m = V$, so that (29) turns into a *pattern map* that gives a deterministic relation between the phase difference of a pair of successive pulses and that of the previous pair. Once the system is relaxed into this uniformly progressing state, the possible phase patterns can be diverse (Elphick et al., 1990b). If $\omega_1 = \omega_2 = 0$, the pulses will be uniformly spaced, as in the case of a second-order system. When both ω_1 and ω_2 are nonzero, which is possible when we have a fourth order o.d.e. as a model, we can get hyperchaotic patterns (Glendinning and Tresser, 1985). Of course, when we deal with the kind of third-order systems suggested by (26), we can have oscillations on only one side of the pulse. For example, when the oscillatory tail is the leading one, with $\omega_1 = 0$ and $\omega_2 = \omega$, we get

$$Z_{m+1} = C - K Z_m^{\xi} \cos(\omega \ln Z_m - \Theta), \tag{31}$$

where

$$Z_m = e^{-\mu \Delta_m} \tag{32}$$

and C, K and Θ are constants involving the pulse shape and V. This is a map that gives the possible patterns of phases. Its fixed points correspond to periodic solutions with constant phased and the number of these tends to infinity as $C \to 0$.

6. Discussion

We have seen what can be done in situations near to polycriticality, where several modes can be simultaneously marginal. Such behavior, which I like to call competing instabilities (Spiegel, 1972), leads to amplitude equations of an order dictated by the number of significant parameters afforded by the available slow modes. For illustration, we have used a case of tricriticality where, according to how we unfold the singularity, we can get three

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monotonic modes or one monotonic and two oscillatory modes. Near such polycritical conditions, we can legitimately reduce the full pulsation problem described by p.d.e.s to one of o.d.e.s. The dimension of the phase space in which we must work has thus been reduced from infinity to a few.

Having found the amplitude equation, we need to analyze the diversity of behavior it may describe. We saw in the previous section that, by suitable choice of the parameters of the stability theory, we can describe a solution as a series of pulses in amplitude with a set of equations for the phases of the pulses. At first glance, this does not seem to be progress for cases where we have to analyze a long series of pulses. But in fact, since the phases do lock in rather quickly (see Elphick et al., 1990a), we may often go right to the limit where they have established their relative phases and we are led to an algebraic equation for the successive phase differences in the form of a map of the real line onto itself. In other words, we have gone from the full p.d.e.s of pulsation theory to an algebraic formula without needing any detailed calculations unless we want to evaluate the coefficients in the formula.

Of course, for this to be quantitatively good, we need to adopt specially chosen parameter ranges. Still, it does bring out asymptotically correct features of the content of the pulsation theory for those ranges. As those a little familiar with such maps will readily see from the sample result, (31), we can conclude that the pulsation equations give chaotic behavior arbitrarily closely to the onset of a triple instability. Such results are by now familiar in nonlinear stability theory and they naturally carry over to stellar pulsation.

However special they may be, these results do represent real behavior of the basic equations and may even show us the simplest form of the complex patterns that stellar pulsation may exhibit. They tell us that, with a few slow modes in the problem, otherwise regular pulsations can be expected to exhibit erratic phase variations. Moreover, in the simplest situations we have considered, these fluctuations are deterministic. In the example of (31), we can even see that there is a quantized aspect to the successive phase shifts as a result of the oscillatory character of some of the modes. The way that happens is that the pulses are locked in preferentially to alternate minima of the forces implied by (29). The existence of such minima in the forces is a consequence of the oscillatory tails of the pulses.

The implication of all this is that pulsational phase diagrams can be expected to exhibit the variety of behavior contained in a map like (31). The pulses can cluster into pairs, then into pairs of pairs, and so on through a full hierarchy of clusters of pulses, as in the usual period-doubling cascade. There can also be completely chaotic and hyperchaotic progressions in the phases. The pattern map (31) even describes intermittent, or bursts, behavior, but that is not so robust.

The question of the stability of phase patterns is a difficult one that is

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governed by the differential equations (29) and not by the map. This is too complicated to include in this sketch. Instead, I reluctantly leave the matter here, recalling an incident from the life of J. S. Bach. It is said that he had already retired for the night, though one of his children was still practicing on the harpsichord. When the child stopped abruptly, leaving a chord unresolved, Bach got of bed, went downstairs in his nightcap, resolved the dissonance on the harpsichord, and went back up to sleep soundly. You will understand how this tale has inspired the work described here. And just as Bach was up early next morning composing, we must now turn seriously to the question of how we may reach the next level of complexity in the patterns that the bursters or solar cycle lead us to expect from the theory.

One of the simplest things that we can do to increase the complexity allowed by the pattern map (31) is to relax the assumption that the field of pulses is so widely spaced. We then need to include the interaction with the next nearest (in time) pulses. Then (31) becomes a two-term recursion formula, implying a two-dimensional map. This makes for more complicated patterns that resemble a noisy version of (31). But these are still a far cry from the more erratic patterns of variation that we see, for example, in the X-ray bursters. Models for those, I believe, will have higher dimension than those I have discussed here. I hope that a sequel to this meeting will provide a forum to discuss them.

I am very grateful to the Air Force Office of Scientific Research whose support has permitted the continuation of this work for application to the solar cycle. And I am happy to thank Neil Balmforth for his careful reading of the manuscript.

Appendix

Appendix. The Original Abstract

Recently Baker (1966) devised a model for pulsational instability based on the dynamics of the layer in the star where the instability originates. The advantage of this model is that it leads to an ordinary third-order differential equation whose linearized form can be discussed quite readily. In the present work the model is used to study finite-amplitude oscillations. The equation governing the one-zone model is rather like the simpler third-order equation studied by Moore and Spiegel (1966) in connection with nonlinear overstability, and it exhibits the same kinds of phenomena. In particular it gives rise in different cases to aperiodic oscillations, relaxation oscillations and "stillstands" in the displacement curves, in addition to well-behaved periodic solutions.

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