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EQUATIONAL CLASSES OF DISTRIBUTIVE PSEUDO-COMPLEMENTED LATTICES

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1. Introduction. A pseudo-complemented lattice is a lattice L with zero such that for every $a \in L$ there exists $a^* \in L$ such that, for all $x \in L$, $a \wedge x = 0$ if and only if $x \leqq a^*$. a^* is called a pseudo-complement of a. It is clear that for each element a of a pseudo-complemented lattice L, a^* is uniquely determined by a. Thus * can be regarded as a unary operation on L. Moreover, each pseudo-complemented lattice contains the unit, namely 0^*. It follows that every pseudo-complemented lattice L can be regarded as an algebra $(L; (\vee, \wedge, {}^*, 0, 1))$ of type $(2, 2, 1, 0, 0)$. In this paper, we consider only distributive pseudo-complemented lattices. For simplicity, we call such a lattice a p-algebra. Thus, a p-algebra is an algebra $(A; (\vee, \wedge, {}^*, 0, 1))$ of type $(2, 2, 1, 0, 0)$ such that $(A; (\vee, \wedge))$ is a distributive lattice, 0 and 1 are the zero (smallest element) and the unit (largest element) of A, respectively, and * is the pseudo-complementation. A p-algebra satisfying the equation $x^* \vee x^{**} = 1$ is called Stone algebra (see [**6**]).

In this paper, we shall show that the class of all p-algebras is generated by its finite members and a complete description of the lattice of equational classes of p-algebras is given. We shall also show that each class of the lattice can be described by a single equation, in addition to the equations characterizing p-algebras, which generalizes the equation for Stone algebras. Furthermore, we shall characterize each class in the lattice in terms of prime filters and in terms of minimal prime ideals generalizing Nachbin's result for Boolean algebras [**8**] and the results on Stone algebras by Grätzer and Schmidt [**6**] and Varlet [**10**].

2. Characterizations of the classes $(E_n)^*$.

THEOREM 1. *An algebra* $(A; (\vee, \wedge, {}^*, 0, 1))$ *of type* $(2, 2, 1, 0, 0)$ *is a p-algebra if and only if* $(A; (\vee, \wedge, 0, 1))$ *is a distributive lattice with zero* 0 *and unit* 1 *and satisfying the following equations:*

(1) $a \wedge a^* = 0$,

(2) $a \vee a^{**} = a^{**}$,

(3) $(a \vee b)^* = a^* \wedge b^*$,

(4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,

(5) $0^* = 1$.

In particular, the class of all p-algebras is equational.

Proof. It is well known that p-algebras satisfy the conditions. Assume,

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conversely, that (1)–(5) are satisfied in a distributive lattice A with zero 0 and unit 1. We have to show that for all $a, x \in A$, $a \wedge x = 0$ if and only if $x \leqq a^*$. Clearly, by (1), $x \leqq a^*$ implies $a \wedge x = 0$. Assume now that $a \wedge x = 0$; then we have

$$\begin{aligned}
x &\leqq x^{**} && \text{(by (2))}\\
&= x^{**} \wedge 1 \\
&= x^{**} \wedge 0^* && \text{(by (5))}\\
&= x^{**} \wedge (a^* \wedge a^{**})^* && \text{(by (1))}\\
&= x^{**} \wedge (a \vee a^*)^{**} && \text{(by (3))}\\
&= (x \wedge (a \vee a^*))^{**} && \text{(by (4))}\\
&= ((x \wedge a) \vee (x \wedge a^*))^{**} && \text{(by distributivity)}\\
&= (x \wedge a^*)^{**} && \text{(since } x \wedge a = 0)\\
&= x^{**} \wedge a^{***} && \text{(by (4))}\\
&= x^{**} \wedge a^* && \text{(since } a^* = a^{***} \text{ by (2) and (3))}\\
&\leqq a^*.
\end{aligned}$$

Remark. Balbes and Horn [**1**] have shown that an algebra $(L; (\wedge, {}^*, 0))$ of type (2, 1, 0) is a pseudo-complemented semi-lattice if and only if it satisfies the following equations:
(1) $a \wedge b = b \wedge a$,
(2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
(3) $a \wedge a = a$,
(4) $0 \wedge a = 0$,
(5) $a \wedge (a \wedge b)^* = a \wedge b^*$,
(6) $a \wedge 0^* = a$,
(7) $0^{**} = 0$.
In particular, the class of all p-algebras is equational.

For p-algebras we consider the following equations ($n \geqq 1$):

$$(\mathrm{E}_n) \qquad (x_1 \wedge \ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_i^* \wedge \ldots \wedge x_n)^* = 1.$$

It is clear that for $n = 1$, the equation (E_n) becomes

$$(\mathrm{E}_1) \qquad x^* \vee x^{**} = 1.$$

Thus p-algebras that satisfy the equations (E_n) are generalizations of Stone algebras.

We denote by $(\mathrm{E}_n)^*$ the class of all p-algebras which satisfy (E_n).

The problem of characterizing the class of p-algebras satisfying the equation (E_1) was first raised by M. H. Stone, and since then several solutions have been offered, the first being by Grätzer and Schmidt [**6**] who named this class of p-algebras Stone algebras. Later solutions were given by Varlet [**10**], Frink [**4**], Grätzer [**5**], and Bruns [**3**]. In the following theorems, several characterizations of $(\mathrm{E}_n)^*$ are given.

THEOREM 2. *For a p-algebra A, the following two conditions are equivalent* $(n \geqq 1)$:

(1) $A \in (\mathrm{E}_n)^*$,

(2) *for every prime filter P in A there exist at most n (distinct) maximal proper filters containing P.*

Proof. (1) $\Rightarrow$ (2). Assume that (2) is not true. Then there would exist a prime filter P and $n + 1$ distinct maximal (proper) filters $M_1, \ldots, M_{n+1}$ containing P. By distributivity and maximality, we have, for $i = 1, 2, \ldots, n + 1$, $\bigcap_{j \neq i} M_j \not\subseteq M_i$. Take

$$a_i \in \bigcap_{j \neq i} M_j - M_i \qquad (i = 1, 2, \ldots, n).$$

Then

$$a_i^* \in M_i \qquad (i = 1, 2, \ldots, n)$$

and

$$a_i \in M_j \qquad (i = 1, \ldots, n; j = 1, \ldots, n + 1; i \neq j).$$

It follows that

$$a_1 \wedge \ldots \wedge a_n \in M_{n+1}, \; a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n \in M_i \quad (i = 1, 2, \ldots, n);$$

therefore

$$(a_1 \wedge \ldots \wedge a_n)^* \notin P$$

and

$$(a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* \notin P \qquad (i = 1, 2, \ldots, n).$$

Since P is prime, it follows that

$$(a_1 \wedge \ldots \wedge a_n)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* \notin P.$$

But $1 \in P$, thus the equation (E_n) is not satisfied.

(2) $\Rightarrow$ (1). Assume that A does not satisfy the equation (E_n). Then there would exist $a_1, \ldots, a_n \in A$ such that

$$c = (a_1 \wedge \ldots \wedge a_n)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* < 1.$$

By Stone's lemma, there exists a prime filter P not containing c. Put

$$b_{n+1} = a_1 \wedge \ldots \wedge a_n$$

and

$$b_i = a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n \qquad (i = 1, 2, \ldots, n)$$

and define $F_i = P \vee [b_i, 1]$ $(i = 1, 2, \ldots, n + 1)$. We have $b_j \notin F_i$ for $i \neq j$, since $b_j \in F_i$ would imply $0 = b_i \wedge b_j \in F_i$ and there would exist $p \in P$

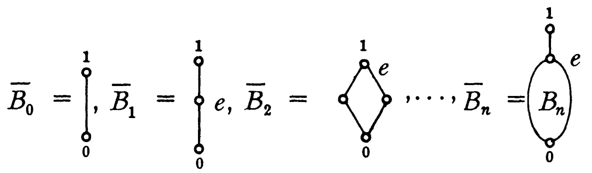
such that $p \wedge b_i = 0$, i.e. $p \leqq b_i^*$, contradicting the fact that $c \notin P$. This shows that each F_i is proper. Also $F_i \vee F_j = A$ $(i \neq j)$ by the definition of F_i. Let M_i be a maximal proper filter containing F_i $(i = 1, 2, \ldots, n+1)$. Then $M_1, \ldots, M_{n+1}$ are $n+1$ distinct maximal proper filters containing P, i.e. (2) is not fulfilled.

We recall that a Boolean algebra is an algebra $(B; (\vee, \wedge, ', 0, e))$ of type (2, 2, 1, 0, 0) such that $(B; (\vee, \wedge, 0, e))$ is a distributive lattice with zero element 0 and unit e, and $'$ is the complementation, i.e. for each $a \in B$, we have $a \wedge a' = 0$, $a \vee a' = e$. One can construct a p-algebra $(\bar{B}; (\vee, \wedge, *, 0, 1))$ from a given Boolean algebra $(B; (\vee, \wedge, ', 0, e))$ by adjoining a new unit 1 as follows: Put $\bar{B} = B \cup \{1\}$, where $x < 1$ for all $x \in B$ and define

$$x^* = \begin{cases} x', & \text{if } 0 \neq x \in B; \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x = 1. \end{cases}$$

It is clear that $(\bar{B}; (\vee, \wedge, *, 0, 1))$ is, in fact, a p-algebra. $(\bar{B}, (\vee, \wedge, *, 0, 1))$ (or simply $\bar{B}$) is called the p-algebra obtained from the Boolean algebra B by adjoining a new unit.

Let $\bar{B}_n$ $(n \geqq 0)$ be the p-algebras obtained from the 2^n-element Boolean algebras B_n by adjoining a new unit 1, e.g.



We have the following result.

LEMMA 1. *Let A be a p-algebra, P a prime filter in A, and let $M_1, \ldots, M_n$ $(n \geqq 0)$ be all distinct maximal (proper) filters properly containing P; let $a_1, \ldots, a_n$ be the atoms of the 2^n-element Boolean algebra B_n $(n \geqq 0)$. Define φ: $A \to \bar{B}_n$ by*

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in P; \\ \bigvee \{a_i | \, x \in M_i\}, & \text{if } x \notin P. \end{cases}$$

Then φ is a p-algebra homomorphism of A onto $\bar{B}_n$.

Proof. (1) $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$.
It is trivial if $\varphi(x \vee y) = 1$. If $\varphi(x \vee y) \leqq e$, then $x \vee y \notin P$ and

$$\begin{aligned} \varphi(x) \vee \varphi(y) &= (\bigvee\{a_i | \, x \in M_i\}) \vee (\bigvee\{a_j | \, y \in M_j\}) \\ &= \bigvee\{a_i | \, x \in M_i \text{ or } y \in M_i\} = \bigvee\{a_i | \, x \vee y \in M_i\} = \varphi(x \vee y). \end{aligned}$$

Similarly, we can show that

(2) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$,
(3) $\varphi(0) = 0$, $\varphi(1) = 1$ by definition of φ,
(4) $\varphi(x^*) = \varphi(x)^*$.

If $x \in P$, then clearly $\varphi(x)^* = 0 = \varphi(x^*)$. If $x \notin M_i$ $(i = 1, 2, \ldots, n)$, then $x^* \in P$, for otherwise we would have $p \nleqq x^*$ for all $p \in P$, i.e. $p \wedge x \neq 0$ for all $p \in P$, which would imply that the filter $P \vee [x, 1]$ is proper. But then every maximal (proper) filter $M \supseteq P \vee [x, 1]$ would be different from all M_i $(i = 1, 2, \ldots, n)$, a contradiction. Hence $\varphi(x)^* = 1 = \varphi(x^*)$. Assume that $x \in M_i - P$ for some i. Since the pseudo-complementation in $\bar{B}_n$ of an element y satisfying $0 < y \leqq e$ is the complement of y in the Boolean algebra $B_n = [0, e]$ and since this is the join of all atoms not contained in y, we have

$$\varphi(x)^* = (\bigvee\{a_i | \, x \in M_i\})^* = \bigvee\{a_i | \, x \notin M_i\} = \bigvee\{a_i | \, x^* \in M_i\} = \varphi(x^*).$$

(5) φ is onto.

We have to show that for every $a \in \bar{B}_n$, there exists $x \in A$ such that $\varphi(x) = a$. It is trivial for $a = 0$ or 1. If $n = 0$, then P is a maximal (proper) filter in A and φ is a p-algebra homomorphism of A onto $\bar{B}_0$. Now assume that $n > 0$. If $a = e$, then take $x_i \in M_i - P$ and put $x = \bigvee_{i=1}^n x_i$. Clearly, $x \in M_i$ $(i = 1, 2, \ldots, n)$ and $x \notin P$. It follows that $\varphi(x) = \bigvee\{a_i | \, x \in M_i\} = e$. Finally, if $0 < a < e$, then there exist a_i, a_j such that $a_j \leqq a$, $a_i \nleqq a$ $(i, j = 1, 2, \ldots, n; i \neq j)$ and $\bigcap\{M_j | \, a_j \leqq a\} \nsubseteq M_i$ for all i such that $a_i \nleqq a$. Pick $y_i \in \bigcap\{M_j | \, a_j \leqq a\} - M_i$ for all i with $a_i \nleqq a$, and put $x = \bigwedge\{y_i | \, a_i \nleqq a\}$. Then $\varphi(x) = \bigvee\{a_j | \, x \in M_j\} = \bigvee\{a_j | \, a_j \leqq a\} = a$.

THEOREM 3. *Let A be a p-algebra. Then the following two conditions are equivalent* $(n \geqq 1)$:

(1) $A \in (\mathrm{E}_n)^*$.

(2) *A is isomorphic with a subdirect product of copies of $\bar{B}_0, \bar{B}_1, \ldots, \bar{B}_n$.*

Proof. (1) $\Rightarrow$ (2). Take $a, b \in A$ with $a \neq b$. We have to show that there exists a p-algebra homomorphism φ of A onto $\bar{B}_k$ $(0 \leqq k \leqq n)$ such that $\varphi(a) \neq \varphi(b)$. We can assume, without loss of generality, that $a \nleqq b$. By Stone's lemma, there exists a prime filter P such that $a \in P$ and $b \notin P$. By Theorem 2, there exist at most n distinct maximal (proper) filters containing P, say $M_1, \ldots, M_k$ $(0 \leqq k \leqq n)$. By Lemma 1, there exists a p-algebra homomorphism φ of A onto $\bar{B}_k$ $(0 \leqq k \leqq n)$ such that $\varphi(a) \neq \varphi(b)$.

(2) $\Rightarrow$ (1). It is trivial that $\bar{B}_0$ satisfies (E_n) $(n \geqq 1)$. Since each $\bar{B}_n$ $(n \geqq 1)$ has exactly n maximal filters, namely the principal filters generated by atoms of $\bar{B}_n$, it follows from Theorem 2 that $\bar{B}_1, \ldots, \bar{B}_n$ all satisfy (E_n). Consequently, $A \in (\mathrm{E}_n)^*$.

Notation. $\mathscr{B}_{-1}$ = the class of all one-element p-algebras;

$\mathscr{B}_n = \mathrm{HSP}(\bar{B}_n)$ = the equational class of p-algebras generated by $\bar{B}_n$ $(n \geqq 0)$;

$\mathscr{B}_F$ = the class of all finite p-algebras;

$\mathscr{B}_\infty$ = the class of all p-algebras;

$\mathscr{P}_n = \{A | \, A \in \mathscr{B}_\infty$ and every prime filter in A is contained in at most n distinct maximal proper filters$\}$ $(n \geqq 1)$;

$\mathscr{I}_n = \{A \mid A \in \mathscr{B}_\infty$ and every proper prime ideal in A contains at most n distinct mimimal prime ideals$\}$ $(n \geqq 1)$.

COROLLARY. *For* $n \geqq 1$, *we have*

$$(\mathrm{E}_n)^* = \mathscr{B}_n = \mathscr{P}_n.$$

Proof. $(\mathrm{E}_n)^* = \mathscr{P}_n$ follows directly from Theorem 2. Since $\bar{B}_0, \ldots, \bar{B}_n$ are subalgebras of $\bar{B}_n$, it follows from Theorem 3 that $(\mathrm{E}_n)^* \subseteq \mathscr{B}_n$. But $\mathscr{B}_n \subseteq (\mathrm{E}_n)^*$ by virtue of the fact that $\bar{B}_n \in (\mathrm{E}_n)^*$. Thus we have $(\mathrm{E}_n)^* = \mathscr{B}_n$.

The following theorem gives another characterization of $(\mathrm{E}_n)^*$.

THEOREM 4. *Let L be a distributive lattice with* 0 and 1. *Then the following three conditions are equivalent:*

(1) *every prime filter in L is contained in at most n distinct maximal* (*proper*) *filters;*

(2) *every* (*proper*) *prime ideal in L contains at most n distinct minimal prime ideals;*

(3) *The lattice-theoretical join of any* $n + 1$ *distinct minimal prime ideals in L is L.*

Proof. (1) $\Leftrightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). If not, then there would exist $n + 1$ distinct minimal prime ideals $P_1, \ldots, P_{n+1}$ such that $\bigvee_{i=1}^{n+1} P_i < L$. It follows from Stone's lemma that there exists a prime filter F disjoint from $\bigvee_{i=1}^{n+1} P_i$. Clearly, $L - F$ is a prime ideal containing P_i $(i = 1, \ldots, n + 1)$, a contradiction.

(3) $\Rightarrow$ (2). If not, there would exist a prime (proper) ideal P containing $n + 1$ distinct minimal prime ideals $Q_1, \ldots, Q_{n+1}$. Hence $\bigvee_{i=1}^{n+1} Q_i \subseteq P < L$, contradicting (3).

Combining the results of Theorems 2, 3, 4 and the corollary of Theorem 3, we have the following.

THEOREM 5. *Let A be a p-algebra. Then the following conditions are equivalent* $(n \geqq 1)$:

(1) $A \in (\mathrm{E}_n)^*$;

(2) $A \in \mathscr{B}_n$;

(3) $A \in \mathscr{P}_n$;

(4) $A \in \mathscr{I}_n$;

(5) *the lattice-theoretical join of any* $n + 1$ *distinct minimal prime ideals in* A *is* A.

3. The lattice. In this section we shall show that the lattice of equational classes of p-algebras is a chain of type $\omega + 1$. We shall also prove that the class of all p-algebras is generated by its finite members. As a first result in this respect, we have the following.

THEOREM 6. *$\mathscr{B}_{-1} \subset \mathscr{B}_0 \subset \mathscr{B}_1 \subset \ldots \subset \mathscr{B}_n \subset \ldots \subset \mathscr{B}_\infty$ ("$\subset$" meaning proper inclusion).*

Proof. It is clear that $\mathscr{B}_{-1} \subset \mathscr{B}_0 \subset \mathscr{B}_1$. For $n \geqq 1$, we have clearly $\mathscr{B}_n \subseteq \mathscr{B}_{n+1}$. It remains to show that $\mathscr{B}_n \neq \mathscr{B}_{n+1}$. But this follows immediately from the fact that $\bar{B}_{n+1} \in \mathscr{B}_{n+1}$ contains a prime filter $\{1\}$ which is contained in exactly $n + 1$ distinct maximal filters.

LEMMA 2. *Let A be a p-algebra and $e_1, \ldots, e_m$ $(m \geqq 1)$ elements of A satisfying*

$$e_i \wedge e_j = 0 \qquad (i \neq j;\, i, j = 1, 2, \ldots, m),$$

$$e_i^{**} = e_i \qquad (i = 1, 2, \ldots, m).$$

Put

$$S = \{0, 1\} \cup \left\{ \bigwedge_{\alpha \in J} e_\alpha^* \,\middle|\, \phi \neq J \subseteq \{1, \ldots, m\} \right\} \cup \left\{ \left(\bigwedge_{\alpha \in J} e_\alpha^* \right)^* \,\middle|\, \phi \neq J \subseteq \{1, \ldots, m\} \right\}$$

and $T = \{x_1 \vee \ldots \vee x_n |\, x_i \in S, n \geqq 1\}$. Then T is the subalgebra of A generated by $\{e_1, \ldots, e_m\}$. In particular, T is finite.

Proof. It is obvious that S is closed under *.

We claim that S is closed under $\wedge$. To do this, it suffices to show that

$$\left(\bigwedge_{\alpha \in J_1} e_\alpha^* \right) \wedge \left(\bigwedge_{\beta \in J_2} e_\beta^* \right)^* \in S \quad \text{and} \quad \left(\bigwedge_{\alpha \in J_1} e_\alpha^* \right)^* \wedge \left(\bigwedge_{\beta \in J_2} e_\beta^* \right)^* \in S,$$

where J_1 and J_2 are non-empty subsets of $\{1, \ldots, m\}$. Indeed, we have

$$\begin{aligned}
\left(\bigwedge_{\alpha \in J_1} e_\alpha^* \right) \wedge \left(\bigwedge_{\beta \in J_2} e_\beta^* \right)^* &= \left(\bigvee_{\alpha \in J_1} e_\alpha \right)^* \wedge \left(\bigvee_{\beta \in J_2} e_\beta \right)^{**} \\
&= \left(\bigvee_{\alpha \in J_1} e_\alpha \right)^{***} \wedge \left(\bigvee_{\beta \in J_2} e_\beta \right)^{**} \\
&= \left(\left(\bigvee_{\alpha \in J_1} e_\alpha \right)^* \wedge \left(\bigvee_{\beta \in J_2} e_\beta \right) \right)^{**} \\
&= \left(\left(\bigwedge_{\alpha \in J_1} e_\alpha^* \right) \wedge \left(\bigvee_{\beta \in J_2} e_\beta \right) \right)^{**} \\
&= \left(\bigvee_{\beta \in J_2} \left(e_\beta \wedge \bigwedge_{\alpha \in J_1} e_\alpha^* \right) \right)^{**} \\
&= \begin{cases} 0, & \text{if } J_2 \subseteq J_1; \\ \left(\bigwedge_{\beta \in J} e_\beta^* \right)^*, & \text{if } J_2 \nsubseteq J_1, \quad \text{where } J = J_2 - J_1. \end{cases}
\end{aligned}$$

$$\left(\bigwedge_{\alpha\in J_1} e_\alpha^*\right)^* \wedge \left(\bigwedge_{\beta\in J_2} e_\beta^*\right)^* = \left(\bigvee_{\alpha\in J_1} e_\alpha\right)^{**} \wedge \left(\bigvee_{\beta\in J_2} e_\beta\right)^{**}$$
$$= \left(\left(\bigvee_{\alpha\in J_1} e_\alpha\right) \wedge \left(\bigvee_{\beta\in J_2} e_\beta\right)\right)^{**}$$
$$= (\bigvee \{e_\alpha \wedge e_\beta | \alpha \in J_1, \beta \in J_2\})^{**}$$
$$= (\bigvee \{e_\alpha | \alpha \in J_1 \cap J_2\})^{**}$$
$$= \left(\bigwedge_{\alpha\in J_1\cap J_2} e_\alpha^*\right)^*.$$

Evidently $0, 1 \in T$, $\{e_1, \ldots, e_m\} \subseteq S \subseteq T$. Also T is closed under $\vee$ by definition, and T is closed under $\wedge$ by distributivity. Furthermore, T is closed under *, since $(x_1 \vee \ldots \vee x_n)^* = x_1^* \wedge \ldots \wedge x_n^* \in S \subseteq T$. Thus we see that T is a subalgebra of A containing $\{e_1, \ldots, e_m\}$ and T is evidently the subalgebra generated by $\{e_1, \ldots, e_m\}$.

LEMMA 3. *Let $\mathscr{K}$ be an equational class of p-algebras and $\mathscr{K} \not\subseteq \mathscr{B}_n$ $(n \geqq -1)$. Then $\mathscr{B}_{n+1} \subseteq \mathscr{K}$.*

Proof. If $n = -1$, then $\mathscr{K}$ contains a non-trivial algebra A since $\mathscr{K} \not\subseteq \mathscr{B}_{-1}$. But then A contains $\bar{B}_0$ as a subalgebra and hence $\mathscr{B}_0 \subseteq \mathscr{K}$. If $\mathscr{K} \not\subseteq \mathscr{B}_0$, then there exists an algebra $A \in \mathscr{K}$ which is not Boolean. Hence there exists an element $a \in A$ such that $a \vee a^* < 1$ and it follows that $\{0, a \vee a^*, 1\}$ is a subalgebra of A isomorphic with $\bar{B}_1$, and this implies that $\mathscr{B}_1 \subseteq \mathscr{K}$. Now assume that $n \geqq 1$ and take $A \in \mathscr{K} - \mathscr{B}_n$. By the corollary of Theorem 3, there exist $a_1, \ldots, a_n \in A$ such that

$$(a_1 \wedge \ldots \wedge a_n)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* < 1.$$

Put

$$e_i = (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^{**}$$
$$= a_1^{**} \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n^{**} \qquad (i = 1, 2, \ldots, n),$$
$$e_{n+1} = \left(\bigwedge_{i=1}^{n} a_i\right)^{**} = \bigwedge_{i=1}^{n} a_i^{**}.$$

Clearly, $e_i \wedge e_j = 0$ $(i, j = 1, \ldots, n+1; i \neq j)$, $e_i^{**} = e_i$ $(i = 1, \ldots, n+1)$. By Lemma 2, the subalgebra B generated by $\{e_1, \ldots, e_{n+1}\}$ is finite. Now, for $1 \leqq i \leqq n$, we have $e_i^* = (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* \geqq a_i^{**}$ so that

$$(e_1^* \wedge \ldots \wedge e_n^*)^* \leqq (a_1^{**} \wedge \ldots \wedge a_n^{**})^* = \left(\bigwedge_{i=1}^{n} a_i\right)^{***} = e_{n+1}^*.$$

Moreover, we have $(e_1^* \wedge \ldots \wedge e_i \wedge \ldots \wedge e_n^*)^* = e_i^*$. We assert that $B \notin (\mathrm{E}_n)^*$. Indeed, if we put $x_i = e_i^*$ $(i = 1, 2, \ldots, n)$, we have

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$$\left(\bigwedge_{i=1}^{n} x_i\right)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_i^* \wedge \ldots \wedge x_n)^*$$
$$= \left(\bigwedge_{i=1}^{n} e_i^*\right)^* \vee \bigvee_{i=1}^{n} (e_1^* \wedge \ldots \wedge e_i \wedge \ldots \wedge e_n^*)^* \leqq e_{n+1}^* \vee \bigvee_{i=1}^{n} e_i^*$$
$$= \left(\bigwedge_{i=1}^{n} a_i\right)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* < 1.$$

By Theorem 2, there exist a natural number $k \geqq n + 1$ and a prime filter P in B which is contained in exactly k distinct maximal filters. Hence $\bar{B}_k \in \mathscr{K}$ by Lemma 1, and this implies that $\mathscr{B}_{n+1} \subseteq \mathscr{B}_k \subseteq \mathscr{K}$.

The next theorem shows that the class of all p-algebras is generated by its finite members.

THEOREM 7. $\mathscr{B}_\infty = \mathrm{HSP}(\mathscr{B}_F)$.

Proof. We have to show that every equation which does not hold in some p-algebra A does not hold in some finite p-algebra B. Let $\mathscr{F}_\tau(V)$ be an algebra of the type τ of p-algebras, absolutely freely generated by some countable set V. Let $p, q \in \mathscr{F}_\tau(V)$ and assume that the equation (p, q) does not hold in some p-algebra A, i.e. there is a homomorphism $\varphi\colon \mathscr{F}_\tau(V) \to A$ such that $\varphi(p) \neq \varphi(q)$. There exists a finite sequence of finite sets $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$ such that $F_0 \subseteq V$, $p, q \in F_n$ and for every $i = 1, 2, \ldots, n$ and every $a \in F_i$, one of the following holds:

(a) there exist $b, c \in F_{i-1}$ such that $a = b \vee c$ or $a = b \wedge c$;

(b) there exists $b \in F_{i-1}$ such that $a = b^*$;

(c) $a = 0$ or $a = 1$.

Define $M = \varphi(F_n) \cup \{\varphi(a^*) | a \in F_n\} \cup \{0_A, 1_A\}$. Let B be the sublattice (not sub-p-algebra) of A generated by M. Then B as a finite distributive lattice is pseudo-complemented, and hence can be regarded as a p-algebra. Furthermore, since the pseudo-complement of every element $x \in \varphi(F_n)$ belongs to M, the pseudo-complement of every element $x \in \varphi(F_n)$ is the same in both A and B. Now let $\psi\colon \mathscr{F}_\tau(V) \to B$ be a homomorphism extending $\varphi|F_0$. We shall show by induction on i that $\psi|F_i = \varphi|F_i$ for all $i = 0, 1, \ldots, n$. It is trivial for $i = 0$. Assume that it is true for $i - 1$ $(i \geqq 1)$. Take $a \in F_i$. We have to show that $\psi(a) = \varphi(a)$. It is trivial for $a = 0$ or $a = 1$. If $a = b \vee c$, where $b, c \in F_{i-1}$, then $\varphi(a) = \varphi(b \vee c) = \varphi(b) \vee_A \varphi(c) = \psi(b) \vee_A \psi(c) = \psi(b) \vee_B \psi(c) = \psi(b \vee c) = \psi(a)$. Similarly, $\varphi(a) = \psi(a)$ for $a = b \wedge c$, where $b, c \in F_{i-1}$. Finally, if $a = b^*$ for some $b \in F_{i-1}$, then $\varphi(a) = \varphi(b^*) = \varphi(b)^* = \psi(b)^* = \psi(b^*) = \psi(a)$. It follows, in particular, that $\psi(p) = \varphi(p) \neq \varphi(q) = \psi(q)$. This shows that the equation (p, q) does not hold in B.

THEOREM 8. *The chain of Theorem* 6 *is the whole lattice of equational classes of p-algebras.*

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Proof. We first show that $\mathscr{B}_\infty = \bigvee\{\mathscr{B}_n | n = 0, 1, \ldots\}$ ($\bigvee$ is the join in the lattice of equational classes of p-algebras). Let A be a finite p-algebra. Then A has finitely many maximal (proper) filters, and hence $A \in (\mathrm{E}_n)^* = \mathscr{B}_n$ for some n by Theorem 2. It follows that every finite p-algebra is contained in $\bigvee\{\mathscr{B}_n | n = 0, 1, \ldots\}$ which in turn means that, by Theorem 7,

$$\mathscr{B}_\infty = \bigvee\{\mathscr{B}_n | n = 0, 1, \ldots\}.$$

Finally, let $\mathscr{K}$ be an arbitrary equational class of p-algebras. If $\mathscr{K} \supseteq \mathscr{B}_n$ for all $n = 0, 1, \ldots,$ then, by what we have just proved, $\mathscr{K} = \mathscr{B}_\infty$. Otherwise there exists a largest natural number n such that $\mathscr{B}_n \subseteq \mathscr{K}$. But then we have $\mathscr{B}_n = \mathscr{K}$, for otherwise we would have $\mathscr{K} \nsubseteq \mathscr{B}_n$ which would imply that, by Lemma 3, $\mathscr{B}_{n+1} \subseteq \mathscr{K}$. This contradicts the choice of n.

COROLLARY. *The algebras $\bar{B}_n$ $(n \geqq 0)$ are exactly the finite subdirectly irreducible p-algebras.*

Proof. We first show that every p-algebra $\bar{B}$ obtained from an arbitrary Boolean algebra B by adjoining a new unit 1 is subdirectly irreducible. In fact, the binary relation $\theta_0 = \Delta \cup \{(1, e), (e, 1)\}$, where $\Delta = \{(x, x) | x \in \bar{B}\}$ and e is the unit of B, on $\bar{B}$, is clearly a p-algebra congruence relation. If θ is a p-algebra congruence on $\bar{B}$ such that $\theta > \Delta$, then $x \,\theta\, y$ for some $x, y \in \bar{B}$ with $x \neq y$. If either x or y is 1, then we have $e \,\theta\, 1$ and hence $\theta_0 \subseteq \theta$. If neither x nor y is 1, then $x, y \in B$ and thus we have $x \vee y^* \,\theta\, e$ and $x^* \vee y \,\theta\, e$. We claim that either $x \vee y^*$ or $x^* \vee y$ is not e, for otherwise we would have $x = x \wedge e = x \wedge (x^* \vee y) = x \wedge y$ and $y = y \wedge e = y \wedge (x \vee y^*) = y \wedge x$, i.e. $x = y$, a contradiction. This shows that $a \,\theta\, e$ for some $e \neq a \in B$. It follows that $a = a^{**} \,\theta\, e^{**} = 1$, hence $\theta_0 \subseteq \theta$. This shows that $\bar{B}$ is subdirectly irreducible. In particular, $\bar{B}_n$ $(n \geqq 0)$ are finite subdirectly irreducible p-algebras.

Next, let L be a finite subdirectly irreducible p-algebra. We claim that $L \cong \bar{B}_n$ for some n. Indeed, there is a natural number n such that

$$|\bar{B}_n| \leqq |L| < |\bar{B}_{n+1}|$$

($|A|$ is the cardinality of A). Put $\mathscr{L} = \mathrm{HSP}(L)$. If $\mathscr{B}_n \subset \mathscr{B}_n \vee \mathscr{L}$, then $\mathscr{L} \nsubseteq \mathscr{B}_n$ and hence $\mathscr{B}_{n+1} \subseteq \mathscr{L}$ by Lemma 3. In particular, $\bar{B}_{n+1} \in \mathrm{HSP}(L)$. By Corollary 3.4, $\bar{B}_{n+1} \in \mathrm{HS}(L)$ [**7**]. This is impossible. Thus we have $\mathscr{B}_n = \mathscr{B}_n \vee \mathscr{L}$, i.e. $\mathscr{L} \subseteq \mathscr{B}_n$. But then $L \in \mathrm{HS}(\bar{B}_n)$ and hence $L \cong \bar{B}_n$.

4. Concluding remark. A lattice L is said to be a relative Stone lattice if every closed interval of L is a Stone algebra [**6**]. By applying the methods of Grätzer and Schmidt [**6**] and Varlet [**10**], we obtain the following theorem generalizing the results of Grätzer and Schmidt [**6**] and Varlet [**10**].

THEOREM 9. *Let L be a distributive lattice in which every closed interval (as a sublattice) is pseudo-complemented. Then the following three conditions are equivalent $(n \geqq 1)$:*

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(1) *every closed interval* $[a, b]$ *in* L *satisfies the equation* (E_n);

(2) L *is the lattice-theoretical-join of any* $n + 1$ *pairwise incomparable prime ideals;*

(3) $\bar{B}_{n+1}$ *is not a lattice-homomorphic image of* L.

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