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GEOMETRIC STRUCTURE IN THE TEMPERED DUAL OF $\mathrm{SL}_\ell(F)$: TORAL CASE

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Abstract

We investigate the tempered representations derived from the principal series of $\mathrm{SL}_\ell(F)$ and their geometric structure. In particular, we give the parameterization for special representations and prove the tempered part of the Aubert–Baum–Plymen conjecture for the toral cases of $\mathrm{SL}_\ell(F)$.

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1. Introduction

In the representation theory of reductive p-adic groups, the issue of the reducibility of induced representations is quite intricate. For the special linear group $\mathrm{SL}_n(F)$ over a p-adic field, this issue has been studied intensively (see [7, 8, 10, 12]). In [1, 2], Aubert *et al.* propose a geometric conjecture related to reductive p-adic groups. They conjecture that there exists a continuous bijection between the (compact) extended quotients and the smooth (respectively tempered) dual of reductive p-adic groups. In other words, they use extended quotients as a model to describe the location of reducible points. We recall the definition of extended quotients here.

DEFINITION 1.1. Let X be a Hausdorff topological space. Let Γ be a finite group acting on X by a left continuous action. Let

$$\tilde{X} = \{(x, \gamma) \in X \times \Gamma \mid \gamma x = x\},$$

with group action on $\tilde{X}$ given by

$$g \cdot (x, \gamma) = (gx, g\gamma g^{-1})$$

for $g \in \Gamma$. Then the extended quotient is given by

$$X//\Gamma := \tilde{X}/\Gamma = \bigsqcup_{\gamma \in \Gamma} X^\gamma / Z(\Gamma), \tag{1.1}$$

where γ runs through the representatives of conjugacy classes of Γ.

Let G be a reductive p-adic group and let M be a Levi subgroup of G. Let $\mathfrak{s} = [M, \sigma]_G$ be the point in the Bernstein spectrum which contains the cuspidal pair (M, σ), where σ is a cuspidal representation of M. Let $W(M)$ denote the Weyl group of M and $W^{\mathfrak{s}}$ denote the subgroup $\{w \in W(M) \mid w \cdot \mathfrak{s} = \mathfrak{s}\}$ and call it the isotropy group attached to $\mathfrak{s}$. To conform to the notation in [2], we denote by $\Psi^{\mathrm{t}}(M)$ the set of unitary unramified characters of M. Indeed, $\Psi^{\mathrm{t}}(M)$ has the structure of a compact torus. Attached to $\mathfrak{s}$, we set

$$E^{\mathfrak{s}} := \{\psi \otimes \sigma \mid \psi \in \Psi^{\mathrm{t}}(M)\}.$$

By a cocharacter we mean a morphism $\mathbb{C}^\times \to T^\vee$ of algebraic groups, where $T^\vee$ is a maximal torus of the Langlands dual group $G^\vee$. The q-projection $\pi^{\mathfrak{s}}_{\sqrt{q}}$ is constructed from a finite set of cocharacters which are dependent on $\mathfrak{s}$ (see [3]). The space of tempered representations of G determined by $\mathfrak{s}$ will be denoted by $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$, and the infinitesimal character will be denoted inf.ch (see [5]). The following conjecture is the tempered part of the Aubert–Baum–Plymen (ABP) conjecture in [2].

Conjecture 1.2. *There exists a continuous bijection* $\mu^{\mathfrak{s}} \colon E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$ *with* $(\mathrm{inf.ch}) \circ \mu^{\mathfrak{s}} = \pi^{\mathfrak{s}}_{\sqrt{q}}$.

We will study the geometric structure of the tempered dual of $\mathrm{SL}_\ell(F)$ through the extended quotient. In particular, this structure can be studied in terms of symmetric groups and the special representations are totally determined by the conjugacy classes of the symmetric groups. Here is the main theorem.

Theorem 1.3. *The tempered part of the ABP conjecture is valid for the toral case of* SL_ℓ, *where* ℓ *is prime.*

2. On the R-group

Let F be a local nonarchimedean field of characteristic 0. Let ℓ be prime and put $G = \mathrm{SL}_\ell(F)$. We will focus on the toral case of G. Hence, we fix the standard Levi subgroup M to be a maximal torus T. In this paper, we will use the framework in [10]. Let $\tilde{M}$ denote the corresponding Levi subgroup of $\tilde{G} = \mathrm{GL}_\ell(F)$, so that $M = \tilde{M} \cap \mathrm{SL}_\ell(F)$. Let $\sigma \in E_2(M)$ and $\pi_\sigma \in E_2(\tilde{M})$ with $\pi_\sigma \supset \sigma$, where $E_2(M)$ means the collection of equivalence classes of irreducible discrete series representations of M. Recall that $W(M)$ is the Weyl group of M. Since M is maximal torus T here, the group $W(M)$ is just W, the Weyl group of G with respect to T. Let

$$\begin{aligned}
\bar{L}(\pi_\sigma) &:= \{\eta \in \widehat{F^\times} \mid \eta \cdot \pi_\sigma \simeq^{w} \pi_\sigma \text{ for some } w \in W\},\\
X(\pi_\sigma) &:= \{\eta \in \widehat{F^\times} \mid \eta \cdot \pi_\sigma \simeq \pi_\sigma\}.
\end{aligned}$$

For the Bernstein variety $\mathfrak{s} = [M, \sigma]_G$, we let $\pi_\sigma|_M \supset \sigma$. Then we write π_σ in the form $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell$. In fact, the form of the representation π_σ is the same as the form of σ. This means

$$\sigma = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell.$$

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Hence, we can rewrite this representation in the form

$$\sigma = \pi_1^{\otimes n_1} \otimes \pi_2^{\otimes n_2} \otimes \cdots \otimes \pi_k^{\otimes n_k}, \tag{2.1}$$

where $\sum n_i = \ell$ and π_i is not the twist of π_j by an unramified character for all $i \neq j$.

Now we focus on the representations induced by (2.1) from the Levi subgroup M to G, namely $\mathrm{Ind}_M^G(\sigma)$, and classify them. First, we assume that π_σ is not of the form $\pi \otimes \eta\pi \otimes \cdots \otimes \eta^{\ell-1}\pi$, where η is a ramified character of order ℓ, and not of the form $\pi^{\otimes \ell}$ (ℓ-tuple). We will return to these two cases later. Recall that the R-group $R(\sigma)$ is isomorphic to the quotient $\bar{L}(\pi_\sigma)/X(\pi_\sigma)$ [10, Theorem 2.4]. For details of the R-group of SL_n, refer to [7, 8, 10].

Lemma 2.1. *The group $R(\sigma)$ is trivial when π_σ satisfies the above assumption.*

Proof. By (2.1),

$$\sigma = \pi_1^{\otimes n_1} \otimes \pi_2^{\otimes n_2} \otimes \cdots \otimes \pi_k^{\otimes n_k}.$$

Since, by assumption, we have $n_i < \ell$ and ℓ is prime, we know that n_i is coprime to ℓ for $i \in 1, 2, \ldots, k$, that is, n_i cannot divide ℓ. Let $\eta \in \bar{L}(\pi_\sigma)$. This implies that there exists $w \in W$ such that $\eta \cdot \pi_\sigma \cong^w \pi_\sigma$. If w is trivial, then $\bar{L}(\pi_\sigma) = X(\pi_\sigma)$. Hence, $R(\sigma) = 1$. From now on, we assume that w is not trivial. Then there must exist π_i and π_j such that $\eta\pi_i \cong \pi_j$ for $i \neq j$. Thus, $n_i = n_j$. In other word, this means that there are several cycles in π_σ after twisting by a character η. The number of cycles is equal to $o(\eta)$, the order of the character η. This implies that $o(\eta)$ divides ℓ. Since ℓ is prime, we get $o(\eta) = 1$ or $o(\eta) = \ell$. If $o(\eta) = 1$, we have $R(\sigma) = 1$ straightforwardly. Otherwise, when $o(\eta) = \ell$, the representation of π_σ is of the form $\pi \otimes \eta\pi \otimes \eta^2\pi \otimes \cdots \otimes \eta^{\ell-1}\pi$. This contradicts our assumption. □

Corollary 2.2. *The isotropy group $W^{\mathfrak{s}}$ attached to σ is given by*

$$W^{\mathfrak{s}} = W(\sigma) = \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}.$$

Now we go back to the other two cases. It is not hard to prove the following two lemmas.

Lemma 2.3. *The R-group $R(\sigma)$ attached to $\pi_\sigma = \pi \otimes \eta\pi \otimes \cdots \otimes \eta^{\ell-1}\pi$ is given by $\mathbb{Z}/\ell\mathbb{Z}$ and the isotropy subgroup $W^{\mathfrak{s}}$ is $\mathbb{Z}/\ell\mathbb{Z}$.*

Lemma 2.4. *The R-group $R(\sigma)$ attached to $\pi_\sigma = \pi^{\otimes \ell}$ is given by $\mathfrak{S}_\ell$ and $W^{\mathfrak{s}} = W = \mathfrak{S}_\ell$.*

Then we have Table 1.

Here, we turn to the relation between partitions of n and the special representations where $n \in \mathbb{Z}$ and $n \geq 2$.

Lemma 2.5. *Suppose $\pi_\sigma = \pi \otimes \pi \otimes \cdots \otimes \pi$. Each special representation in the set $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$ is parameterized by a partition λ of n.*

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TABLE 1. R-groups in the toral case of SL_ℓ.

Case	π_σ	$R(\sigma)$	$W(\sigma)$	Condition
1	$\pi\otimes\pi\otimes\pi\otimes\cdots\otimes\pi$	1	$\mathfrak{S}_\ell$	
2	$\pi_1^{\otimes n_1}\otimes\pi_2^{\otimes n_2}\otimes\cdots\otimes\pi_k^{\otimes n_k}$	1	$\mathfrak{S}_{n_1}\times\mathfrak{S}_{n_2}\times\cdots\times\mathfrak{S}_{n_k}$	
3	$\pi_\sigma=\pi\otimes\eta\pi\otimes\cdots\otimes\eta^{\ell-1}\pi$	$\mathbb{Z}/\ell\mathbb{Z}$	$\mathbb{Z}/\ell\mathbb{Z}$	$\eta^\ell=1$

PROOF. First of all, we define the partitions of n. The symbol λ is called a partition of n if

$$\lambda=[\lambda_1,\lambda_2,\ldots,\lambda_r],\quad \sum\lambda_i=n \text{ and } \lambda_i\geq 1\in\mathbb{Z}.$$

The partition λ is *dominant* if $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r$. We denote by Λ the set of the partitions of n and by Λ^+ the set of dominant partitions. In fact, this makes sense because, in this case, the isotropy group is given by $\mathfrak{S}_n$, which acts on the Levi as permutation groups. Thus, we can rearrange a nondominant partition to a dominant one by suitable group actions.

EXAMPLE. $n=14$, [8, 1, 2, 3] is a partition and [8, 3, 2, 1] is a dominant partition.

For every dominant partition $\lambda=[\lambda_1,\lambda_2,\ldots,\lambda_k,\ldots,\lambda_r]\in\Lambda^+$, where $\lambda_k=1$ and $\lambda_{k-1}>1$, we can construct a corresponding special representation canonically. In other words, for such a partition, the representation is of the form

$$\mathrm{St}_{\lambda_1}(\pi)\otimes\mathrm{St}_{\lambda_2}(\pi)\otimes\cdots\otimes\mathrm{St}_{\lambda_{k-1}}(\pi)\otimes\underbrace{\pi\otimes\cdots\otimes\pi}_{n-\sum_{i=1}^{k-1}\lambda_i},$$

where $\mathrm{St}_{\lambda_i}(\pi)$ is the generalized Steinberg representation of GL_{λ_i} attached to π.

Then we can twist an unramified unitary character of

$$\mathrm{GL}_{\lambda_1}\times\mathrm{GL}_{\lambda_2}\times\cdots\times\mathrm{GL}_{\lambda_{k-1}}\times F^\times\times\cdots\times F^\times$$

to this representation by partition.

Then we have

$$\mathrm{St}_{\lambda_1}(z_1\pi)\otimes\mathrm{St}_{\lambda_2}(z_2\pi)\otimes\cdots\otimes\mathrm{St}_{\lambda_{k-1}}(z_{k-1}\pi)\otimes\underbrace{z_k\pi\otimes\cdots\otimes z_r\pi}_{n-\sum_{i=1}^{k-1}\lambda_i}.$$

It is clear that these representation are parameterized by the r-torus $\mathbb{T}^r$. Now we induce these representations to GL_n and then restrict the obtained representations to SL_n. Hence, the parameter space is given by the quotient $\mathbb{T}^r/\mathbb{T}$ because the elements of SL_n have determinant 1. □

Here, we mention that the cocharacters for the special representation can be taken to be

$$t\longmapsto(t^{a_1},t^{a_1-1},\ldots,t^{1-a_1},t^{-a_1},\ldots,t^{a_{k-1}},t^{a_{k-1}-1},\ldots,t^{1-a_{k-1}},t^{-a_{k-1}},1,\ldots,1)$$

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where a_i denotes the greatest integer a such that

$$a \leq (\lambda_i + 1)/2.$$

In fact, the structure of such cocharacters is well adapted with the extended quotient for this case. As we know, the extended quotient can be decomposed into the disjoint union of some ordinary quotients (we call such parts 'components') via the conjugacy classes of the action group. We know that the action on the quotient is given by a finite group, which is an isotropy subgroup of the Weyl group. In this case, such an isotropy subgroup is given by the symmetric group $\mathfrak{S}_n$. It is well known that the conjugacy classes of $\mathfrak{S}_n$ depend on the length of cycles in $\mathfrak{S}_n$.

3. Geometric structure

In this section, we discuss the extended quotient $E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ with respect to each Bernstein variety $\mathfrak{s} = [M, \sigma]_G$ and show the geometric conjecture for the toral case of $\mathrm{SL}_\ell(F)$. Hence, we will construct the explicit bijection between $E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ and $\mathrm{Irr}^{\mathrm{t}}(G)^{\mathfrak{s}}$. From now on, for convenience, we denote $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r}) \cap \mathrm{SL}_n$ by $n_1 + n_2 + \cdots + n_r$, where $\sum n_i = n$. For example, $1 + 1 + 1 + 1$ means

$$(\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1) \cap \mathrm{SL}_4.$$

3.1. Case 1: $\boldsymbol{\pi \otimes \pi \otimes \pi \otimes \cdots \otimes \pi}$. In the previous section, for the case $\pi_\sigma \cong \pi^{\otimes \ell}$, we have shown that $W^{\mathfrak{s}} = \mathfrak{S}_\ell$. The group $W^{\mathfrak{s}}$ is given by the symmetric group $\mathfrak{S}_\ell$. The number of conjugacy classes equals the number of cycle types in $\mathfrak{S}_\ell$.

We denote by B_ℓ the number of conjugacy classes of $\mathfrak{S}_\ell$. We consider the extended quotient $E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$. Recall the property of the extended quotient,

$$E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} = \bigsqcup_{\gamma} E^{\gamma}/Z(\gamma),$$

where γ runs through all the representatives $\{e = \gamma_0, \gamma_1, \ldots, \gamma_{B_\ell - 1}\}$ of conjugacy classes. We choose $\gamma_{B_\ell - 1}$ to be the length ℓ cycle. In other words, the cycle is $(123 \cdots \ell)$. Hence, we can decompose into B_ℓ components as follows:

$$E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} = E^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup E^{\gamma_1}/Z(\gamma_1) \sqcup E^{\gamma_2}/Z(\gamma_2) \sqcup \cdots \sqcup E^{\gamma_{B_\ell - 1}}/Z(\gamma_{B_\ell - 1}).$$

Then we investigate each component $E^{\gamma_k}/Z(\gamma_k)$ for $k = 1, 2, \ldots, B_\ell - 2$ in $E^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$. In fact, each γ_k represents a kind of partition of ℓ. We denote such partition by $[\lambda_1, \lambda_2, \ldots, \lambda_j]$, where $\sum \lambda_i = \ell$ and $\lambda_i > 0$ for any i. Since we can rearrange the order of such partitions, without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$. From above, we know that E^{γ_k} is the projective variety

$$E^{\gamma_k} = \Big\{ \Big[\Big(\underbrace{z_1, z_1, \ldots, z_1}_{\lambda_1}, \underbrace{z_2, \ldots, z_2}_{\lambda_2}, \ldots, \underbrace{z_j \ldots z_j}_{\lambda_j} \Big) \Big] \;\Big|\; z_1, z_2, \ldots, z_j \in \mathbb{T} \Big\}$$

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and

$$Z(\gamma_k) = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_j}.$$

In fact, the partition above is compatible with the structure of dominant partitions in Lemma 2.5. Hence, now we apply Lemma 2.5 and each component corresponds to a generalized Steinberg representation canonically. At the same time, γ_k runs through all the conjugacy class except γ_0 and $\gamma_{B_\ell - 1}$.

REMARK 3.1. The component $E^{\gamma_0}/Z(\gamma_0)$ is the ordinary quotient and the component $E^{\gamma_{B_\ell - 1}}/Z(\gamma_{B_\ell - 1})$ is isomorphic to ℓ points, namely $pt_1, pt_2, pt_3, \ldots, pt_\ell$.

$$E^{\gamma_{B_\ell - 1}}/Z(\gamma_{B_\ell - 1}) = pt_1 \sqcup pt_2 \sqcup \cdots \sqcup pt_\ell.$$

The paragraph above is all related to special representations. In what follows, we consider the unramified principal series. Let $t = (z_1, z_2, \ldots, z_\ell)$ be a point in $E^{\mathfrak{s}}/W^{\mathfrak{s}}$, except $(1, \omega, \ldots, \omega^{\ell-1})$, where $\omega^\ell = 1$, and let χ_t be the corresponding unramified unitary character.

CLAIM. *The induced representation* $\mathrm{Ind}_T^G(\chi_t \otimes \sigma)$ *is irreducible for such a point* t.

PROOF. Assume $\eta \in \bar{L}(\chi_t \otimes \pi_\sigma)$. This implies

$$\eta \cdot \chi_t \otimes \pi_\sigma \cong^w \chi_t \otimes \pi_\sigma \tag{3.1}$$

for some $w \in W$. The element w permutes the representations in π_σ. Hence, there exist several cycles $q_1, q_2, \ldots, q_m$. (In fact, it is not necessary to compute m.) According to (3.1), the length of these cycles must divide ℓ. Since ℓ is prime, there are two possibilities. The length of a cycle is 1 or ℓ. If the cycle has length 1, this implies $\eta = 1$. Otherwise, cycles are given by cyclic permutations. This contradicts our assumption. Hence, $\bar{L}(\chi_t \otimes \pi_\sigma)$ is trivial. This implies that $R(\chi_t \otimes \sigma)$ is trivial. In other word, $\mathrm{Ind}_M^G(\chi_t \otimes \pi_\sigma)$ is irreducible. □

Let $t = (1, \omega, \omega^2, \ldots, \omega^{\ell-1}) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$, where $\omega^\ell = 1$, and let χ_t be the corresponding unramified unitary character. Thus, $\bar{L}(\chi_t \otimes \pi_\sigma) = 1, \omega, \omega^2, \ldots, \omega^{\ell-1}$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, $R(\chi \otimes \sigma) = \mathbb{Z}/\ell\mathbb{Z}$. This implies that the induced representation $\mathrm{Ind}_M^G(\chi_t \otimes \pi_\sigma)$ is reducible and there are ℓ irreducible constituents. We match these representations by $(1, \omega, \omega^2, \ldots, \omega^{\ell-1}) \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$ and $pt_1, pt_2, \ldots, pt_{\ell-1}$. Furthermore, we map $\mathrm{St}_\ell(\pi)$ to pt_ℓ.

The cocharacters can be taken to be

$$h_c = 1 \quad \text{if } c = pt_i \text{ with } 1 \le i \le \ell - 1$$

and

$$h_c(t) = (t^a, t^{a-1}, \ldots, t^{1-a}, t^{-a}) \quad \text{if } c = pt_\ell,$$

where a denotes the greatest integer a such that $a \le (\ell + 1)/2$.

From the above, we conclude that part (3) of the conjecture is true for this case.

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3.2. Case 2: $\pi_\sigma = \pi_1^{\otimes n_1} \otimes \pi_2^{\otimes n_2} \otimes \cdots \otimes \pi_k^{\otimes n_k}$. Recall that the group $W^{\mathfrak{s}}$ is $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}$. The proof is analogous to that for case 1. We have a similar conclusion and so we omit the proof. Note that if there are elements η in $\bar{L}(\chi_t \otimes \pi_\sigma)$, the order $o(\eta)$ of character η must divide the number ℓ. Thus, $o(\eta)$ is 1 or ℓ. Since $k \geq 2$ (this means we have at least two different representations in π_σ), η cannot be of order ℓ. Thus, η is a character of order one.

In this case, all $\mathrm{Ind}_M^G(\chi_t \otimes \pi_\sigma)$ are irreducible. We mention that $E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} - E^{\mathfrak{s}}/W^{\mathfrak{s}}$ is mapped to special representations and $E^{\mathfrak{s}}/W^{\mathfrak{s}}$ is mapped to unramified principal series.

Remark 3.2. The necessary condition for reducibility is that n_i is a divisor of ℓ, except 1. Furthermore, if all $n_i = 1$, the representation π_σ will be in the form $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell$. The isotropy group $W^{\mathfrak{s}}$ is trivial. In this case, $\mathrm{Ind}_M^G(\chi_t \otimes \pi_\sigma)$ is irreducible for all $t \in E^{\mathfrak{s}}/W^{\mathfrak{s}}$. A special representation does not exist since the necessary condition is that there are at least two equal terms in $\chi_t \otimes \pi_\sigma$.

3.3. Case 3: $\pi_\sigma = \pi \otimes \eta\pi \otimes \cdots \otimes \eta^{\ell-1}\pi$. In this section, we will discuss the case when $\pi_\sigma = \pi \otimes \eta\pi \otimes \cdots \otimes \eta^{\ell-1}\pi$. From Table 1 we know that the R-group $R(\sigma)$ with respect to this is the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$. Jawdat and Plymen have proved that there exists a bijection between the extended quotient and the tempered dual with respect to this case (see [11, Theorem 5.3]).

From the discussion of the above three cases, we conclude that the Aubert–Baum–Plymen conjecture for the tempered dual of $\mathrm{SL}_\ell(F)$ is true in the toral case.

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