BULL. AUSTRAL. MATH. SOC. 35B65, 35J65
VOL. 54 (1996) [515–520]

ON THE BEHAVIOUR OF THE SUP- AND INF-CONVOLUTIONS OF A FUNCTION NEAR THE BOUNDARY

TIMOTHY R. CRANNY

The study of nonclassical solutions for elliptic and parabolic PDE's often involves the use of regularisation processes such as the sup- and inf-convolutions. In this note we study the behaviour of these regularised functions near the boundary of the domain, and derive constraints on the appropriate second-order sub- and superdifferentials on and near the boundary. Potential applications to regularity results are also noted.

In the study of viscosity solutions for elliptic PDE problems on a domain $\Omega \subset \mathbb{R}^n$, one common way to circumvent the lack of smoothness of a nonclassical subsolution u is to consider instead the more regular sup-convolution u_ε^+. Such regularisations were first given by Lasry and Lions [5], and for a fixed $\varepsilon > 0$ are defined by

$$u_\varepsilon^+(x) = \sup_{y\in\overline{\Omega}}\Big\{u(y) - \frac{|x-y|^2}{2\varepsilon}\Big\}, \tag{1}$$

with the natural equivalent for viscosity supersolutions and the inf-convolution u_ε^-.

As shown in [5], u_ε^+ and u_ε^- are twice differentiable almost everywhere in $\overline{\Omega}$, with

$$\left|Du_\varepsilon^+\right|, \left|Du_\varepsilon^-\right| \leqslant C\varepsilon^{-1/2} \quad -\varepsilon^{-1}I \leqslant D^2u_\varepsilon^+, \ \ \varepsilon^{-1}I \geqslant D^2u_\varepsilon^-, \tag{2}$$

holding in the sense of distributions for some constant $C < \infty$.

The importance of the convolutions in viscosity theory comes from the observation in [4] that if a smooth function ϕ touches u_ε^+ from above at a point x_0, and y_0 is a point in $\overline{\Omega}$ at which the supremum in the definition of $u_\varepsilon^+(x_0)$ is taken, then the function $\phi(x_0 - y_0 + y)$ touches u from above at y_0. The behaviour of such convolutions near the boundary of the domain plays an important role in the formulation of viscosity boundary conditions, but the effect of the boundary is to introduce complications which are not found in the interior case. We describe here some effects which are a direct consequence of the involvement of the boundary.

If $\partial\Omega$ is differentiable at $y \in \partial\Omega$ we shall let $T_{\partial\Omega}(y)$ denote the tangent hyperplane to $\partial\Omega$ at y. For convenience we shall say a vector $\mathbf{v} \in T_{\partial\Omega}(y)$ if the inclusion holds when $\mathbf{v}$ is taken to be based at y.

Received 21st August, 1995.

Theorem. *Let Ω be a domain in $\mathbb{R}^n$ with differentiable boundary $\partial\Omega$, and consider $u \in USC(\overline{\Omega})$. If $x_0 \in \Omega$ is a point where u_ε^+ is twice differentiable, and the supremum used to define $u_\varepsilon^+(x_0)$ corresponds to a point $y_0 \in \partial\Omega$, then:*

1. *Every eigenvector of $D^2 u_\varepsilon^+(x_0)$ which is not in $T_{\partial\Omega}(y_0)$ corresponds to an eigenvalue of $-\varepsilon^{-1}$, the minimum allowed by semiconvexity.*
2. *Every such eigenvector is also an eigenvector of $D^2\psi(y_0)$ where ψ touches u from above at y_0 (relative to $\overline{\Omega}$), and $\psi(\cdot)$ may be chosen to make the corresponding eigenvalue arbitrarily large and negative.*

The equivalent remarks hold for the inf-convolution u_ε^- of $u \in LSC(\overline{\Omega})$.

Proof: The second half of the result is a consequence of the first half and Lemma 2.14 of Jensen [**2**].

To prove the first claim, let us fix $\varepsilon > 0$ and consider a point x_0 where u_ε^+ is twice differentiable and $y_0 \in \partial\Omega$ the (unique) point used to define $u_\varepsilon^+(x_0)$. It follows from Jensen [**2**] that $Du_\varepsilon^+(x_0) = (y_0 - x_0)/\varepsilon$. Since u_ε^+ is twice differentiable at x_0, there exists a smooth function $\phi(\cdot)$ such that

$$-\circ\left(|x-x_0|^2\right) \leqslant (u_\varepsilon^+ - \phi)(x) \leqslant \circ\left(|x-x_0|^2\right), \tag{3}$$

so for any fixed $\delta \in (0,1)$, the function $\phi_\delta(x) \stackrel{\text{def}}{=} \phi(x) + \delta\,|x-x_0|^2$ satisfies

$$-\delta\,|x-x_0|^2 - \circ\left(|x-x_0|^2\right) \leqslant (u_\varepsilon^+ - \phi_\delta)(x) \leqslant -\delta\,|x-x_0|^2 + \circ\left(|x-x_0|^2\right). \tag{4}$$

Note that unlike $u_\varepsilon^+ - \phi$, the function $u_\varepsilon^+ - \phi_\delta$ attains a strict local maximum at x_0 while remaining semiconvex. We may now follow the standard arguments to 'tilt' this maximum to nearby points by looking for the maximum of $(u_\varepsilon^+ - \phi_\delta)(x) + \langle p, x - x_0\rangle$ for small $p \in \mathbb{R}^n$ of our choosing.

At this point the choice of ϕ as the smooth approximation of u_ε^+ at x_0 gives some valuable extra information. Notice that if $u_\varepsilon^+ - \phi_\delta$ were *precisely* $-\delta\,|x-x_0|^2$, then the maximum of $(u_\varepsilon^+ - \phi_\delta)(x) + \langle p, x-x_0\rangle$ would be taken at x where $2\delta(x-x_0) = p$, and we would have complete freedom in determining x by our choice of p. Straightforward calculations show that (4) is sufficient to give us $2\delta(x-x_0) = p + \circ(p)$, or equivalently, $p = 2\delta(x-x_0) + \circ(|x-x_0|)$. (To see this, it is simplest to show that for fixed x^* sufficiently close to x_0, the function $(u_\varepsilon^+ - \phi_\delta)(x) + 2\delta\langle x^* - x_0, x - x_0\rangle$ has a local maximum at some x satisfying $|x - x^*| = \circ(|x^* - x_0|)$.)

This means that in a slightly more restrictive sense we may choose the direction in which we move the supremum, since for any unit vector $\vec{v} \in \mathbb{R}^n$ we may choose p so as to make $\vec{v} - (\overline{x} - x_0)/\,|\overline{x} - x_0|$ arbitrarily small, while simultaneously ensuring that $|p|$ is sufficiently small and u_ε^+ is twice differentiable at $x \in \Omega$.

https://doi.org/10.1017/S0004972700017287 Published online by Cambridge University Press

If we denote by y_x the unique point in $\overline{\Omega}$ used to define $u_\varepsilon^+(x)$, it follows that $Du_\varepsilon^+(x) = (y_x - x)/\varepsilon$. By the definition of x, we have

$$
\begin{aligned}
Du_\varepsilon^+(x) &= D\phi_\delta(x) - p \\
&= D\phi(x) + 2\delta(x - x_0) - p \\
&= D\phi(x) + 2\delta(x - x_0) - 2\delta(x - x_0) - o(|x - x_0|) \\
&= D\phi(x) - o(|x - x_0|).
\end{aligned} \tag{5}
$$

Since $Du_\varepsilon^+(x_0) = D\phi(x_0) = (y_0 - x_0)/\varepsilon$, we therefore have

$$Du_\varepsilon^+(x) - Du_\varepsilon^+(x_0) = D\phi(x) - D\phi(x_0) - o(|x - x_0|), \tag{6}$$

that is $$\varepsilon^{-1}(y_x - x - y_0 + x_0) = D\phi(x) - D\phi(x_0) - o(|x - x_0|),$$

so

$$y_x = y_0 + (x - x_0) + \varepsilon(D\phi(x) - D\phi(x_0)) - o(|x - x_0|). \tag{7}$$

Since $\phi \in C^2(\overline{\Omega})$, we have

$$
\begin{aligned}
D\phi(x) - D\phi(x_0) &= D^2\phi(x_0) \cdot (x - x_0) + o(|x - x_0|) \\
&= D^2u_\varepsilon^+(x_0) \cdot (x - x_0) + o(|x - x_0|),
\end{aligned} \tag{8}
$$

so (7) becomes

$$y_x = y_0 + \mathcal{A}(x_0) \cdot (x - x_0) + o(|x - x_0|), \tag{9}$$

where $\mathcal{A}(x_0) \stackrel{\text{def}}{=} I + \varepsilon D^2 u_\varepsilon^+(x_0)$. It follows directly from (2) that $\mathcal{A}$ is positive semi-definite, and again it should be emphasised that we have almost complete freedom in the choice of the direction of $x - x_0$.

Let us choose the coordinate system so as to diagonalise $D^2u_\varepsilon^+(x_0)$, thereby diagonalising $\mathcal{A}(x_0)$. Let e_1 be an (inward-pointing) eigenvector of $D^2u_\varepsilon^+(x_0)$ which is not in $T_{\partial\Omega}(y_0)$ (noting that there must be at least one).

If the eigenvalue λ_1 corresponding to e_1 satisfies $\lambda_1 > -\varepsilon^{-1}$, then e_1 is an eigenvector of $\mathcal{A}(x_0)$ with eigenvalue $1 + \varepsilon\lambda_1 > 0$. By choosing p appropriately, we may make $x - x_0$ sufficiently close to $-e_1$ in direction (with sufficiency determined by the spread of eigenvalues of $\mathcal{A}(x_0)$). Equation (9) then says that as we tilt from x_0 to x, the point at which the supremum in $u_\varepsilon^+(\cdot)$ is achieved moves from $y_0 \in \partial\Omega$ *out of* $\overline{\Omega}$. This is not possible, contradicting the possibility that $\lambda_1 \neq -\varepsilon^{-1}$. ∎

REMARKS. **1.** One can obviously replace the differentiability everywhere of $\partial\Omega$ with a weaker condition such as a (nonuniform) exterior sphere condition.
2. The above result implies that if u is semiconvex on $\partial\Omega$ then all but one of the eigenvectors of $D^2\psi(y_0)$ are in $T_{\partial\Omega}(y_0)$ and the remaining eigenvector is normal to $\partial\Omega$ at y_0. In any case, it is guaranteed that the 'normal' vector is an eigenvector of $D^2u_\varepsilon^+(x_0)$ with eigenvalue $-\varepsilon^{-1}$.

REGULARITY RESULTS. Equation (9) also provides some further interest in that Lemma 2.14 of Jensen [2] shows that the twice superdifferentiability of u_ε^+ at x implies the twice superdifferentiability of u at y_x (with all superdifferentiability taken with respect to $\overline{\Omega}$). Since u_ε^+ is twice differentiable almost everywhere, it is reasonable to hope that (9) might be used to obtain results implying the twice differentiability almost everywhere of viscosity solutions.

The best known results in this direction are those of Trudinger [6] where it is shown that a viscosity subsolution (respectively supersolution) u of $Fu = 0$ in Ω is twice superdifferentiable (respectively, subdifferentiable) almost everywhere in Ω if F satisfies (for some positive constant μ_0) either:

1. F is strictly elliptic and satisfies $|F(x,z,p,r)| \leqslant \mu_0(1+|p|+\|r\|)$, or
2. F is uniformly elliptic and satisfies $|F(x,z,p,r)| \leqslant \mu_0\left(1+|p|^2+\|r\|\right)$.

Results of this type are not yet available by our methods, but there is evidence to suggest that such an approach is feasible. For example, let us consider a given function $u \in USC\left(\overline{\Omega}\right)$, and define, for each $\varepsilon > 0$, $\mathcal{D}_\varepsilon$ to be the set (of full measure) in Ω upon which u_ε^+ is twice superdifferentiable. We hope to use (9) to give some indication about the set $q_\varepsilon(\mathcal{D}_\varepsilon)$ upon which we know that u is twice superdifferentiable. We have added the parameter ε everywhere necessary since the ultimate goal of this approach is to show that

$$(10) \qquad \bigcup_{\varepsilon>0} q_\varepsilon(\mathcal{D}_\varepsilon) \quad \text{has full measure in } \Omega.$$

It is obviously necessary for u to satisfy further constraints before (9) can be made useful. To do so we must use the differential operator F, and so must restrict the domain of q_ε to $\widetilde{\mathcal{D}}_\varepsilon \overset{\text{def}}{=} \mathcal{D}_\varepsilon \cap q_\varepsilon^{-1}(\Omega)$. Note that $\left\{x \in \mathcal{D}_\varepsilon \mid dist(x,\partial\Omega) > \varepsilon^{1/2}\right\} \subset \widetilde{\mathcal{D}}_\varepsilon$.

One significant result which is readily available is the following:

LEMMA. *Let u be a viscosity subsolution of $Fu = 0$ in Ω, where F is strictly elliptic and satisfies the constraint $|F(x,z,p,r)| \leqslant \mu_0\left(1+|p|^2+\|r\|\right)$ for some $\mu_0 > 0$. Then*

1. *For each $\varepsilon > 0$ the mapping $q_\varepsilon : \widetilde{\mathcal{D}}_\varepsilon \to \Omega$ is one-to-one.*

2. *The matrix $Dq_\varepsilon(x) = \mathcal{A}_\varepsilon(x)$ defined with respect to $\mathcal{D}_\varepsilon$ has minimal eigenvalue $\lambda_\varepsilon(x)$ satisfying $\lambda_\varepsilon(x) \geqslant \overline{\lambda} > 0$ for some $\overline{\lambda}$ independent of $\varepsilon > 0$.*

OUTLINE OF PROOF: The first result follows trivially from the strict ellipticity of F, since q_ε not one-to-one implies the existence of $(p, X), (q, Y) \in J^{2,+}_{\overline{\Omega}} u(y)$ with $p \neq q$. This easily leads to a contradiction of $Fu \geqslant 0$ in Ω.

The second part of the Lemma uses Lemma 2.14 of [**2**] to note that small eigenvalues of $\mathcal{A}$ imply large negative eigenvalues of $X \in \mathcal{S}^n$ for some $(p, X) \in J^{2,+}_{\overline{\Omega}} u(y)$. That same result also implies an upper bound on the positive eigenvalues of X, so the structure conditions on F and the fact that u is a viscosity subsolution rule out the possibility that the eigenvalues of $\mathcal{A}$ are too small. ∎

While it does not follow from the above alone that (10) holds, the uniform non-degeneracy of $\mathcal{A}$ suggests such a result is attainable. We conjecture that the above structure conditions are sufficient to ensure the twice superdifferentiability almost everywhere of viscosity subsolutions (and hence the twice differentiability almost everywhere of viscosity solutions), thereby extending the results in [**6**]. The conjecture is nontrivial in that q_ε cannot be extended to Ω in a continuous manner, necessitating the taking of limits or unions as $\varepsilon \to 0$.

NOTE ADDED IN PROOF:. After completion of this note it was pointed out by Dr. M. Kocan that similar issues are important in the study of viscosity solutions for PDE's with measurable coefficients. The papers [**3, 1**] rely in part upon results similar to those above, and provide examples of the usefulness of such results.

REFERENCES

[1] M.G. Crandall, M. Kocan, P. Soravia and A. Święch, *On the equivalence of various weak notions of solutions of elliptic pde's with measurable ingredients*, Proceedings of Conference on Elliptic and Parabolic PDE, Capri, (1994), (to appear).

[2] R. Jensen, 'Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations', *Indiana Univ. Math. J.* **38** (1989), 629–667.

[3] R. Jensen,, 'Uniformly elliptic pdes with bounded, measurable coefficients', *J. Fourier Anal. Appl.* (to appear).

[4] R. Jensen, P.-L. Lions and P.E. Souganidis, 'A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations', *Proc. Amer. Math. Soc.* **102** (1988), 975–978.

[5] J.M. Lasry and P.-L. Lions, A remark on regularisation in Hilbert spaces, *Israel J. Math.* **55** (1986), 257–266.

[6] N.S. Trudinger, 'On the twice differentiability of viscosity solutions of nonlinear elliptic equations', *Bull. Austral. Math. Soc.* **39** (1989), 443–447.

Centre for Mathematics and Its Applications
Australian National University
Canberra ACT 0200
Australia