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ON LEVINE'S DECOMPOSITION OF CONTINUITY

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ABSTRACT. A strong version of Levine's decomposition of continuity leads to the result that a closed graph weakly continuous function into a rim-compact space is continuous. This result implies a closed graph theorem: every almost continuous closed graph function into a strongly locally compact space is continuous. An open problem of Shwu-Yeng T. Lin and Y.-F. Lin asks if every almost continuous closed graph function from a Baire space to a second countable space is necessarily continuous. This question is answered in the negative by an example.

1. **Introduction.** In 1961 N. Levine [4] introduced weak continuity and weak* continuity for a function $f:X\to Y$ from a topological space X into a topological space Y and showed that f is continuous if and only if f is both weakly continuous and weak* continuous. The purpose of this note is to replace the weak* continuity by a weaker condition, local weak* continuity, and exercise the applicability of the resulting stronger decomposition theorem. As an application, a closed graph theorem of Paul E. Long and Earl E. McGehee Jr. [7] is slightly extended. An open problem of Shwu-Yeng T. Lin and Y.-F. Lin involving a similar closed graph theorem is answered in the negative by an example.

2. **Definitions and preliminary results.** Unless otherwise specified, no special properties will be assumed for the topological spaces X and Y, or for a function $f:X\to Y$ from X into Y. For a subset A of a topological space $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and interior of A respectively.

DEFINITION 1. [4]. A function $f:X\to Y$ is weakly continuous at the point x in X if and only if for each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U)\subseteq \mathrm{Cl}(V)$. A function $f:X\to Y$ is weakly continuous if and only if f is weakly continuous at each point x in X.

THEOREM 1. [4] *A function* $f:X\to Y$ *is weakly continuous if and only if* $f^{-1}(V)\subseteq \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ *for each open subset* V *of* Y.

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THEOREM 2. *A function $f:X\to Y$ is weakly continuous if and only if there is an open basis B for the topology on Y such that $f^{-1}(V)\subseteq \operatorname{int}(f^{-1}(\operatorname{Cl}(V)))$ for each V in B.*

THEOREM 3. [8] *Every weakly continuous function $f:X\to Y$ into a Hausdorff space Y has a closed graph, $G(f)$.*

In 1966 T. Husain [2] defined almost continuity and thus revived this condition which had been studied by H. Blumberg in 1922 [1] for real-valued functions on Euclidean space.

DEFINITION 2. [2] A function $f:X\to Y$ is almost continuous if and only if for each x in X and for each neighbourhood V of $f(x)$, $\operatorname{Cl}(f^{-1}(V))$ is a neighbourhood of x.

THEOREM 4. *A function $f:X\to Y$ is almost continuous if and only if $f^{-1}(V)\subseteq \operatorname{int}(\operatorname{Cl}(f^{-1}(V)))$ for each open subset V of Y.*

DEFINITION 3. [4] A function $f:X\to Y$ is weak* continuous if and only if $f^{-1}(\operatorname{Fr}(V))$ is closed in X for each open subset V of Y where $\operatorname{Fr}(V)=\operatorname{Cl}(V)-\operatorname{Int}(V)=\operatorname{Cl}(V)-V$ is the frontier (or boundary) of V.

Weak continuity and weak* continuity are independent conditions [4], weak continuity and almost continuity are independent conditions [10], and almost continuity and weak* continuity are independent conditions. Since every function into a discrete space is weak* continuous, Example 1 below shows that weak* continuity does not imply almost continuity. Example 3 of this paper demonstrates that almost continuity does not imply weak* continuity.

EXAMPLE 1. Let $X=R$ be the usual space of real numbers, $Y=R$ be the set of real numbers with the discrete topology, and let $f:X\to Y$ be the identity function.

3. **Levine's decomposition of continuity.** Levine's decomposition theorem [4] states that a function $f:X\to Y$ is continuous if and only if f is weakly continuous and weak* continuous. The applicability of this result will be enhanced by replacing weak* continuity by local weak* continuity as defined below.

DEFINITION 4. A function $f:X\to Y$ is locally weak* continuous if and only if there is an open basis B for the topology on Y such that $f^{-1}(\operatorname{Fr}(V))$ is closed for each V in B.

THEOREM 5. *A function $f:X\to Y$ is continuous if and only if f is weakly continuous and locally weak* continuous.*

Proof. If f is weakly continuous and locally weak* continuous and if B is an open basis for the topology on Y such that $f^{-1}(\operatorname{Fr}(V))=f^{-1}(\operatorname{Cl}(V))-f^{-1}(V)$ is closed for each V in B, then

$$f^{-1}(V)=(X-f^{-1}(\mathrm{Fr}(V)))\cap \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$$

is open for each V in B and f is continuous. The converse is clear.

The following example shows that locally weak* continuous functions exist which are not weak* continuous.

EXAMPLE 2. Let $X=Y=R$ be the space of real numbers with the usual topology. Let Q be the subset of rational numbers and let Z be the subset of integers. Let $g:Q\to Z$ be a set equivalence (or bijection). Define $f:X\to Y$ by $f(x)=g(x)$ if x is in Q and $f(x)=x$ if x is not in Q. Using the usual basis of open intervals f is seen to be locally weak* continuous. But f is not weak* continuous since $V=\bigcup\{(2n, 2n+1):n\in Z\}$ is open and $f^{-1}(\mathrm{Fr}(V))=f^{-1}(Z)=Q$ is not closed.

4. **Applications.** A topological space Y is rim-compact if and only if there is an open basis B for the topology on Y such that $\mathrm{Fr}(V)$ is compact for each V in B. Paul E. Long and Larry L. Herrington [6] proved that an almost continuous (S and S) closed graph function into a rim-compact space is necessarily continuous. This follows as a corollary to the following theorem since almost continuity (S and S) implies weak continuity [11].

THEOREM 6. *Let $f:X\to Y$ be a weakly continuous function with a closed graph $G(f)$. If Y is rim-compact then f is continuous.*

Proof. Let B be an open basis for the topology on Y such that $\mathrm{Fr}(V)$ is compact for each V in B. Since $G(f)$ is closed, $f^{-1}(\mathrm{Fr}(V))$ is closed for each V in B by Problem 6A of [3]. Thus f is locally weak* continuous and hence continuous by Theorem 5.

By Theorem 3 and Theorem 6, every weakly continuous function into a rim-compact Hausdorff space is continuous. But this is already known since weak continuity implies continuity when the range space is regular [4].

A topological space Y is said to be strongly locally compact if each point of Y has a closed compact neighbourhood. Every locally compact regular space is strongly locally compact and every strongly locally compact space is rim-compact. Since non-regular strongly locally compact spaces exist (Example 73 of [12]), a closed graph theorem of Paul E. Long and Earl E. McGehee Jr. [7] is slightly extended as follows.

THEOREM 7. *If $f:X\to Y$ is an almost continuous function into a strongly locally compact space Y and if f has a closed graph, then f is continuous.*

Proof. Let B be an open basis for the topology on Y such that $\mathrm{Cl}(V)$ is compact for each V in B. Since the graph of f is closed, $f^{-1}(\mathrm{Cl}(V))$ is closed for

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each V in B. Thus $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) \subseteq \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ for each V in B and by Theorem 4 and Theorem 2, f is weakly continuous. Since Y is rim-compact f is continuous by Theorem 6.

COROLLARY TO THEOREM 7. *If $f: X \rightarrow R$ is an almost continuous closed graph real-valued function then f is continuous.*

5. **Counterexample for an open problem.** Shwu-Yeng T. Lin and Y.-F. Lin [5] recently posed the following open problem. If $f: X \rightarrow Y$ is an almost continuous closed graph function from a Baire space X into a second countable space Y, then is f necessarily continuous? This question is answered in the negative by the following example.

EXAMPLE 3. Let $X = R$ be the usual space of real numbers and let Q be the subset of rational numbers. Let $Y = R$ be the space of real numbers topologized with the smallest extension of the usual topology for which Q is open in Y. Then X is a Baire space and if $\{V_n\}$ is a countable open basis for the usual topology on X then $\{V_n\} \cup \{V_n \cap Q\}$ is a countable open basis for the extension topology on Y showing that Y is a second countable space. Clearly Y is a Hausdorff (and Urysohn) space. Let $f: X \rightarrow Y$ be the identity function. By Theorem 2, f *is weakly* continuous since $f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ for each $V = (a, b)$ and for each $V = (a, b) \cap Q$ where $a < b$. Further, f is clearly open but f is discontinuous since $f^{-1}(Q) = Q$ is not open in X. It is noted in [10] that every open weakly continuous function is almost continuous. Thus f is almost continuous and by Theorem 3 the graph of f is closed. Further, by Theorem 5 f is not locally weak* continuous.

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