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STRONGLY MONOTONE SOLUTIONS OF RETARDED DIFFERENTIAL EQUATIONS

BY
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1. **Introduction.** Let us consider the retarded differential equation

$$x^{(n)}(t)+(-1)^{n+1}p(t)\varphi(x[\sigma(t)])=0, \qquad t\geq t_0 \qquad (n\geq 1) \tag{1}$$

for which the following assumptions are made:

(i) $p:[t_0,\infty)\to[0,\infty)$ is continuous and not identically zero for all large t.

(ii) $\sigma:[t_0,\infty)\to\mathbb{R}$ is continuous, strictly increasing,

$$\sigma(t)\leq t \quad \text{for every} \quad t\geq t_0 \quad \text{and} \quad \lim_{t\to\infty}\sigma(t)=\infty.$$

(iii) $\varphi:\mathbb{R}\to\mathbb{R}$ is continuous, non-decreasing and

$$(\forall x\in\mathbb{R})\varphi(-x)=-\varphi(x).$$

A solution x of the differential equation (1) defined on some half-line $[t_x,\infty)$ is called *strongly monotone* if and only if

$$(\forall t\geq t_x)x(t)\neq 0 \quad \text{and} \quad x^{(i)}(t)x^{(i+1)}(t)\leq 0 \qquad (i=0,1,\ldots,n-1)$$

and

$$\lim_{t\to\infty}x^{(i)}(t)=0$$

Our purpose here is to give a necessary and sufficient condition in order that equation (1) have a (positive) strongly monotone solution x. As applications of this main result we obtain:

a) A sufficient condition in order that the first order linear retarded differential equation

$$x'(t)+p(t)x[\sigma(t)]=0 \tag{2}$$

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have a strongly monotone solution. By this result we extend a recent one due to Ladas ([4] Th. 2 and Corollary 1) which concerns the oscillatory behaviour of equation (2).

b) A sufficient condition in order that the linear retarded differential equation

$$x^{(n)}(t)+(-1)^{n+1}p(t)x[\sigma(t)]=0 \tag{3}$$

have a strongly monotone solution in the case where

$$\int^{\infty} t^{n-1}p(t)\,dt=\infty$$

We note that in this case, if the function $\sigma(t)$ is "small enough", then equation (3) does not have any strongly monotone solution while the associated to it ordinary differential equation may have such a solution (cf. [1]–[4], [7]–[11]).

c) A sufficient condition in order that equation (3) have a strongly monotone solution in the case where

$$\int^{\infty} t^{n-1}p(t)\,dt<\infty$$

We note that in this case the associated to (3) ordinary differential equation

$$x^{(n)}(t)+(-1)^{n+1}p(t)x=0$$

does not have any strongly monotone solution.

Finally we are referred to the problem of establishing conditions under which equation (1) or more general equations do not have any strongly monotone solutions.

Recently Lovelady [5] has given some results concerning the existence of strongly monotone solutions of equation (3) in the case where $\int^{\infty} t^{n-1}p(t)\,dt=\infty$. We remark that these results are of quite different nature from ours and are referred to the comparison of (3) and the linear ordinary differential equation

$$x^{(n)}(t)+(-1)^{n+1}p(t)x(t)=(-1)^{n}(t-\sigma(t))p(t)$$

as well as of equations of the form (3) with different functions σ.

2. **The main result.** In order to obtain our main result we make use of the following very simple fixed point theorem (cf. [6] p. 65).

FIXED POINT THEOREM. *Let $\leq$ be a partial ordering with field A, and suppose that every $B\subseteq A$ has a least upper bound. Suppose that F maps A into A in such a way that for all x, y in A $x\leq y$ implies that $Fx\leq Fy$. Then $Fx=x$ for some $x\in A$.*

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We note that the proof of our theorem can also be carried out by using the Schauder's fixed point theorem. In this case the assumption concerning the monotonicity of φ can be dropped. But, since our applications are referred to non-decreasing functions φ, we prefer to give a proof based on the simple fixed point theorem which is stated above.

THEOREM. *Equation* (1) *has a strongly monotone solution if and only if there exist positive and non-increasing solutions h and g of the inequalities*

$$(E_1)\qquad h(t) \geq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\varphi(h[\sigma(s)])\, ds$$

$$(E_2)\qquad g(t) \leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\varphi(g[\sigma(s)])\, ds$$

defined on some half-line $[T, \infty)$ *and such that* $\lim_{t\to\infty} h(t) = 0$ *and* $h(t) \geq g(t)$ *for every* $t \geq T$.

Proof. It is easy to verify that if equation (1) has a positive strongly monotone solution x, then x is also a solution of both inequalities (E_1) and (E_2). Thus it remains to prove the sufficiency part of the theorem.

Let $\tau \geq T$ be such that $\sigma(t) \geq T$ for every $t \geq \tau$ and let X be the set of all non-increasing functions x defined on $[\tau, \infty)$ and such that

$$g(t) \leq x(t) \leq h(t) \quad \text{for every} \quad t \geq \tau.$$

The set X is considered endowed with the usual point-wise ordering $\leq$:

$$x_1 \leq x_2 \Leftrightarrow (\forall t \geq T) x_1(t) \leq x_2(t).$$

It is easy to verify that for every $A \subseteq X$, $\sup A$ belongs to X, i.e. that X is order complete.

Consider now the mapping $F: X \to X$ defined as follows:

$$(Fx)(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\varphi(\hat{x}[\sigma(s)])\, ds, \qquad t \geq \tau.$$

where $\hat{x}(t) = x(t)$ on (τ, ∞) and $\hat{x}(t) = h(t)$ on $[T, \tau]$.

By (iii), the definition of X and the conditions on g and h, for every $x \in X$ we have

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\varphi(g[\sigma(s)])\, ds \leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\varphi(\hat{x}[\sigma(s)])\, ds$$
$$\leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\varphi(h[\sigma(s)])\, ds$$

and consequently $Fx \in X$. Thus $FX \subseteq X$. In a similar way we see that for every x, y in X, $x(t) \leq y(t)$ on $[\tau, \infty)$ implies $F(x(t)) \leq F(y(t))$ on $[\tau, \infty)$, which means

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that the mapping F is non-decreasing with respect to the order of X. Thus, by the fixed point theorem, there exists an $x \in X$ such that $Fx = x$, i.e.

$$x(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s)\varphi(\hat{x}[\sigma(s)])\, ds \quad \text{for every} \quad t \geq \tau.$$

Since x is non-increasing it is obvious that the function

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s)\varphi(\hat{x}[\sigma(s)])\, ds, \quad t \geq \tau$$

is continuous and consequently x itself is continuous. It is obvious now that x is a strongly monotone solution of (1) on some half-line $[t_1, \infty)$ where $t_1 \geq \tau$ is such that $\sigma(t) \geq \tau$ for every $t \geq t_1$.

3. **Applications.** In what follows by the term "solution" for an inequality of the form (E_1) or (E_2) we shall always mean a positive, non-decreasing solution of the inequality, defined for all large t and tending to zero as $t \to \infty$.

PROPOSITION 1. *If*

(C_1) *for some* $T \geq t_0$

$$0 < \inf_{t \geq \sigma^{-1}(T)} \int_{\sigma(t)}^t p(s)\, ds \equiv m_T \leq M_T \equiv \sup_{t \geq \sigma^{-1}(T)} \int_{\sigma(t)}^t p(s)\, ds \leq 1/e$$

then equation (2) *has a strongly monotone solution.*

NOTE. Condition (C_1) is obviously satisfied if

$$0 < \liminf_{t \to \infty} \int_{\sigma(t)}^t p(s)\, ds \leq \limsup_{t \to \infty} \int_{\sigma(t)}^t p(s)\, ds < 1/e$$

Proof. Consider the inequalities:

$$h(t) \geq \int_t^\infty p(s)h[\sigma(s)]\, ds \tag{4}$$

and

$$g(t) \leq \int_t^\infty p(s)g[\sigma(s)]\, ds \tag{5}$$

By requiring (4) to have the solution $h(t) = \exp(-\lambda \int_T^{\sigma^{-1}(t)} p(s)\, ds)$, $t \geq T$, $\lambda > 0$, we get

$$\int_t^\infty p(s)\exp\left(-\lambda \int_T^s p(u)\, du\right) ds = \frac{1}{\lambda} \exp\left(-\lambda \int_T^t p(s)\, ds\right) \leq \exp\left(-\lambda \int_T^{\sigma^{-1}(t)} p(s)\, ds\right)$$

or

$$\frac{\exp\left(\lambda \int_t^{\sigma^{-1}(t)} p(s)\, ds\right)}{\lambda} \leq 1, \qquad t \geq T.$$

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The above inequality is equivalent to

$$\frac{\exp\left(\lambda \int_{\sigma(t)}^{t} p(s)\, ds\right)}{\lambda} \le 1, \qquad t \ge \sigma^{-1}(T)$$

and obviously it holds if

$$\frac{e^{\lambda M_T}}{\lambda} \le 1$$

For $\lambda = 1/M_T$ the expression $e^{\lambda M_T}/\lambda$ takes its minimum $M_T e$. Thus, by (C_1), (4) has the solution

$$h(t) = \exp\left(-\frac{1}{M_T} \int_{T}^{\sigma^{-1}(t)} p(s)\, ds\right), \qquad t \ge T.$$

Working similarly we find that (5) has a solution of the form

(6) $$g(t) = \exp\left(-\lambda \int_{T}^{\sigma^{-1}(t)} p(s)\, ds\right), \qquad t \ge T$$

if

$$\frac{e^{\lambda m_T}}{\lambda} \ge 1.$$

Since $\lim_{\lambda \to \infty} e^{\lambda m_T}/\lambda = \infty$ we can choose λ in (6) so that $\lambda \ge 1/M_T$.

It is obvious now that $h(t) \ge g(t)$ for every $t \ge T$ and consequently (2) has a strongly monotone solution x on $[T, \infty)$ such that

$$g(t) \le x(t) \le h(t) \quad \text{for every} \quad t \ge T.$$

REMARK 1. Recently Ladas ([4] Th. 1) has proved that in the particular case $\sigma(t) = t - \tau$, $\tau > 0$ a constant, the condition

(7) $$\liminf_{t \to \infty} \int_{t-\tau}^{t} p(s)\, ds > \frac{1}{e}$$

is sufficient in order that all solutions of (2) to be oscillatory. Since under the condition $\int^{\infty} p(t)\, dt = \infty$ every non-oscillatory solution of (2), is strongly monotone, we can say equivalently that under condition (7) equation (2) has no strongly monotone solutions. Also in the same paper it is proved that in the particular case $p > 0$, a constant, condition (7) is also necessary in order that all solutions of (2) to be oscillatory. Using our previous results it is obvious now that we can state the following:

PROPOSITION 2. *Condition* (7) *is sufficient, while the negation of condition* (C_1) *is necessary in order that all solutions of the equation*

$$x'(t) + p(t)x(t - \tau) = 0$$

to be oscillatory.

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PROPOSITION 3. *Suppose that* $\int^{\infty} p(t)\,dt=\infty$. *If*

$$(C_2)\ \text{for some } T\geq t_0 \qquad \frac{n!}{r_T^{n-1}}\leq M_T\leq \frac{n}{e}(q_T^{n-1})^{1/n}$$

where

$$r_T=\inf_{t\geq\sigma^{-1}(T)}[t-\sigma(t)],\qquad q_T=\inf_{t\geq\sigma^{-1}(T)}p(t),\qquad M_T=\sup_{t\geq\sigma^{-1}(T)}\int_{\sigma(t)}^{t}p(s)\,ds$$

then equation (3) *has a strongly monotone solution.*

NOTE. Condition (C_2) is obviously satisfied if $\liminf_{t\to\infty}[t-\sigma(t)]=r>0$, $\liminf_{t\to\infty}p(t)=q>0$ and

$$\frac{n!}{r^{n-1}}<\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\,ds<\frac{n}{e}(q^{n-1})^{1/n}.$$

Proof. Consider the two inequalities

$$h(t)\geq\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}p(s)h[\sigma(s)]\,ds \tag{8}$$

$$g(t)\leq\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}p(s)g[\sigma(s)]\,ds \tag{9}$$

It is enough to prove that (8) and (9) have solutions h and g defined on some half-line $[t_1,\infty)$ and such that

$$h(t)\geq g(t)\quad \text{for every}\quad t\geq t_1$$

Since $\sigma^{-1}(t)\geq t$ for every $t\geq t_0$, if the inequality

$$g(t)\leq\frac{[\sigma^{-1}(t)-t]^{n-1}}{(n-1)!}\int_{\sigma^{-1}(t)}^{\infty}p(s)g[\sigma(s)]\,ds,\qquad t\geq T \tag{10}$$

has a solution g, then the same function is also a solution of (9). Thus it is easy to verify that for

$$\lambda_0=r_T^{n-1}/(n-1)!$$

(10), and consequently (9), has the solution

$$g(t)=\exp\left(-\lambda_0\int_T^{\sigma^{-1}(t)}p(s)\,ds\right),\qquad t\geq T \tag{11}$$

Now, we require (8) to have a solution h of the same form i.e.

$$h(t)=\exp\left(-\lambda\int_T^{\sigma^{-1}(t)}p(s)\,ds\right),\qquad t\geq T$$

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Successive integrations by parts give

$$\begin{aligned}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\,p(s)h[\sigma(s)]\,ds &= \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\,p(s)\exp\left(-\lambda\int_T^s p(u)\,du\right)ds\\ &=\frac{1}{\lambda}\int_t^\infty\frac{(s-t)^{n-2}}{(n-2)!}\exp\left(-\lambda\int_T^s p(u)\,du\right)ds\\ &\le \frac{1}{q_T\lambda}\int_t^\infty\frac{(s-t)^{n-2}}{(n-2)!}\,p(s)\exp\left(-\lambda\int_T^s p(u)\,du\right)ds\\ &\le\cdots\le\frac{1}{q_T^{n-1}\lambda^n}\exp\left(-\lambda\int_T^t p(s)\,ds\right).\end{aligned}$$

(If $n=1$ then we have to make a simple integration.)

Thus it is enough to have

$$\frac{1}{q_T^{n-1}\lambda^n}\exp\left(\lambda\int_t^{\sigma^{-1}(t)} p(s)\,ds\right)\le 1 \quad \text{for every} \quad t\ge T$$

or

$$\frac{1}{q_T^{n-1}\lambda^n}\,e^{\lambda M_T}\le 1.$$

Moreover for $\lambda=\lambda_1=n/M_T$ the expression $e^{\lambda M_T}/q_T^{n-1}\lambda^n$ takes its minimum $1/n^n q_T^{n-1}\,e^n M_T^n$ which by (C_2) is less than or equal to 1. Hence inequality (8) has the solution

$$h(t)=\exp\left(-\lambda_1\int_T^{\sigma^{-1}(t)} p(s)\,ds\right),\qquad t\ge T \tag{12}$$

Finally, since by (C_2) $\lambda_0\ge\lambda_1$, by (11) and (12) we have that

$$h(t)\ge g(t) \quad \text{for every} \quad t\ge T.$$

It is easy to verify that under the condition

$$\int^\infty t^{n-1}p(t)\,dt<\infty \tag{C_3}$$

the linear ordinary differential equation

$$x^{(n)}+(-1)^{n+1}p(t)x=0 \tag{13}$$

does not have any strongly monotone solution. Indeed, if g were a solution of the associated to (13) inequality of the form (E_2), then we should have

$$g(t)\le\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}\,p(s)g(s)\,ds\le g(t)\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}\,p(s)\,ds$$

or

$$\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}\,p(s)\,ds\ge 1 \quad \text{for all large } t.$$

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But this is impossible since by (C_3)

$$\lim_{t\to\infty}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\,p(s)\,ds=0.$$

However if the delay $t-\sigma(t)$ is "large enough", then the associated to (13) retarded differential equation (3) may have strongly monotone solutions, as the following proposition shows.

PROPOSITION 4. *Consider the retarded differential equation* (3) *subject to the condition* (C_3). *Suppose moreover that* $p(t)>0$ *for every* $t\ge t_0$ *and*

(C_4) *for some* $\lambda>0$

(i) *the function* $e^{-\lambda t}/p(t)$ *is non-increasing for all large* t.
(ii) $\liminf_{t\to\infty} p(t)\exp[\lambda(t-\sigma(t))]>\lambda^n$
(iii) $\liminf_{t\to\infty} p(t)e^{\lambda t}/t\equiv K>0$
(iv) $\limsup_{t\to\infty}\dfrac{t}{\sigma(t)}\displaystyle\int_{\sigma(t)}^\infty \frac{s^{n-1}p(s)}{(n-1)!}\,ds<1.$

Then equation (3) *has a strongly monotone solution.*

NOTE. It is obvious that if $\limsup_{t\to\infty} t/\sigma(t)<\infty$ then (iv) is a consequence of (C_3). (For example: $\sigma(t)=t/\nu$, $\nu>1$.)

Proof. We consider the inequalities (8) and (9) as in the proof of Proposition 3. If $T_1\ge t_0$ is such that

$$p(t)\exp[\lambda(t-\sigma(t))]\ge\lambda^n \quad \text{for every} \quad t\ge T_1$$

then, by substitution in (9), we can easily derive that this inequality has the solution

$$g(t)=\frac{K}{2}\,e^{-\lambda\sigma^{-1}(t)}/p[\sigma^{-1}(t)],\quad t\ge T_1.$$

(The fact that $\lim_{t\to\infty} g(t)=0$ is an easy consequence of (C_4), (iii).)

Also, if $T_2>\max\{t_0,0\}$ is such that

$$\frac{t}{\sigma(t)}\int_{\sigma(t)}^\infty s^{n-1}p(s)\,ds<1 \quad \text{for every} \quad t\ge T_2$$

then we also have

$$\frac{\sigma^{-1}(t)}{t}\int_t^\infty \frac{s^{n-1}}{(n-1)!}\,p(s)\,ds<1 \quad \text{for every} \quad t\ge T_2$$

and consequently

$$\sigma^{-1}(t)\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\frac{p(s)}{s}\,ds<1 \quad \text{for every} \quad t\ge T_2 \tag{14}$$

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By (14) it is easy to see that (8) has the solution

$$h(t)=\frac{1}{\sigma^{-1}(t)}, \qquad t\geq T_2.$$

Now, if $T\geq \max\{T_1, T_2\}$ is such that

$$\frac{p(t)}{t}e^{\lambda t}\geq \frac{K}{2} \quad \text{for every} \quad t\geq T$$

then we have

$$\frac{1}{\sigma^{-1}(t)}\geq \frac{K}{2}\frac{e^{-\lambda\sigma^{-1}(t)}}{p[\sigma^{-1}(t)]} \quad \text{for every} \quad t\geq T.$$

The proof of our proposition is now obvious.

EXAMPLE. Consider the equation

$$x'(t)+\frac{1}{t^{3/2}}x(t-\log t)=0, \qquad t\geq 1 \tag{15}$$

We have $\lim_{t\to\infty} t/t-\log t=1$ and consequently the inequality

$$h(t)\geq \int_t^\infty s^{-3/2}h[s-\log s]\,ds, \qquad t\geq 1$$

has the solution

$$h(t)=\frac{1}{\sigma^{-1}(t)}, \qquad t\geq 1$$

where σ^{-1} denotes here the inverse function of $\sigma(t)=t-\log t$, $t\geq 1$. On the other hand, it is easy to check that for $\lambda>\frac{3}{2}$ all conditions (C_4), [(i)–(iii)] are also satisfied. Hence (15) has a strongly monotone solution.

REMARK 2. In the case where (C_3) is satisfied it is known (cf. (1)) that equation (1) has a (positive) solution x defined on some half-line $[t_x, \infty)$ and such that

$$\begin{aligned} &(\forall t\geq t_x)x^{(i)}(t)x^{(i+1)}(t)\leq 0 && (i=0,1,\ldots,n-1)\\ &\lim_{t\to\infty} x(t)>0 \quad \text{and} \quad \lim_{t\to\infty} x^{(i)}(t)=0 && (i=1,2,\ldots,n-1)\end{aligned} \tag{16}$$

Thus, under the conditions of proposition 4, we conclude that *equation* (3) *has a strongly monotone solution and a solution x which satisfies* (16).

4. **Discussion.** The problem of establishing conditions under which (1) or more general equations do not have any strongly monotone solution was the subject of some interesting recent papers ((1)–(4), (7)–(11)). We note that in some of these results (7), (10), (11) it is proved that, under certain conditions,

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inequality (E_1) does not have any (positive, non-increasing and tending to zero) solution. In this paper we also pointed out a result in this direction which is obtained by proving that, under condition (C_3), inequality (E_2) does not have any solution. It seems to us that our Theorem can be used in many directions for obtaining results concerning the existence or non-existence of strongly monotone solutions. It is also obvious that this theorem can easily be stated so that to include more general cases of (non-linear) differential equations with deviating arguments since some rather restrictive assumptions are not used in the proof of this theorem (for example the conditions which are referred to the function $\sigma(t)$ except that σ is continuous and $\lim_{t\to\infty} \sigma(t)=\infty$).

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