ON THE SPECTRUM OF ALMOST PERIODIC SOLUTIONS OF AN ABSTRACT DIFFERENTIAL EQUATION

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Abstract

If a linear operator A in a Banach space satisfies certain conditions, then the spectrum of any almost periodic solution of the differential equation $u' = Au + f$ is shown to be identical with the spectrum of f, where f is a Stepanov almost periodic function.

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Suppose X is a Banach space and J is the interval $-\infty < t < \infty$. A continuous function $f: J \to X$ is said to be (Bochner or strongly) almost periodic if, given $\varepsilon > 0$, there is a positive real number $\ell = \ell(\varepsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$\sup_{t \in J} \|f(t+\tau) - f(t)\| \leqq \varepsilon. \tag{1.1}$$

For $1 \leqq p < \infty$, a function $f \in L^p_{\text{loc}}(J; X)$ is said to be Stepanov almost periodic or S^p-almost periodic if, given $\varepsilon > 0$, there exists a positive real number $\ell = \ell(\varepsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$\sup_{t \in J} \left[\int_t^{t+1} \|f(s+\tau) - f(s)\|^p ds \right]^{1/p} \leqq \varepsilon. \tag{1.2}$$

It is known that, for an S^p-almost periodic X-valued function $f(t)$ and a real number λ, the mean value

$$m(e^{-i\lambda t} f(t)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-i\lambda t} f(t) dt \tag{1.3}$$

exists in X and is different from the null element θ of X for at most a countable set $\{\lambda_n\} n \geqq 1$, called the spectrum of $f(t)$ (see Theorem 9, page 79, Amerio-Prouse [1]). We denote by $\sigma(f(t))$ the spectrum of $f(t)$.

Our main result is as follows.

THEOREM 1. *Suppose A is a closed linear operator with domain $D(A)$ in a Banach space X, $(i\lambda - A)^{-1}$ exists for all real λ, and $f: J \to X$ is an S^p-almost periodic continuous function with $1 \leqq p < \infty$. If a continuously differentiable function u: $J \to D(A)$ is an almost periodic solution of the differential equation*

$$u'(t) = Au(t) + f(t) \quad \text{on } J, \tag{1.4}$$

then $\sigma(u(t)) = \sigma(f(t))$.

2. Proof of Theorem 1

We have

$$\begin{aligned} \frac{1}{T}\int_0^T e^{-i\lambda t}u'(t)dt &= \frac{1}{T}\Big[e^{-i\lambda t}u(t)\Big]_0^T + \frac{i\lambda}{T}\int_0^T e^{-i\lambda t}u(t)dt \\ &\to i\lambda m(e^{-i\lambda t}u(t)) \text{ as } T \to \infty. \end{aligned} \tag{2.1}$$

By (1.4) and (2.1), since A is a closed linear operator,

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda t}Au(t)dt &= \lim_{T\to\infty} A\left(\frac{1}{T}\int_0^T e^{-i\lambda t}u(t)dt\right) \\ &= i\lambda m(e^{-i\lambda t}u(t)) - m(e^{-i\lambda t}f(t)). \end{aligned} \tag{2.2}$$

So, again by the closedness of the operator A, since

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda t}u(t)dt = m(e^{-i\lambda t}u(t))$$

exists in X, we have

$$\begin{cases} m(e^{-i\lambda t}u(t)) \in D(A) \text{ and} \\ Am(e^{-i\lambda t}u(t)) = i\lambda m(e^{-i\lambda t}u(t)) - m(e^{-i\lambda t}f(t)). \end{cases} \tag{2.3}$$

Hence

$$(i\lambda - A)m(e^{-i\lambda t}u(t)) = m(e^{-i\lambda t}f(t)).$$

By our hypothesis, the operator $(i\lambda - A)$ is $1-1$ for all real λ. So it follows that

$$m(e^{-i\lambda t}u(t)) = \theta \text{ if and only if } m(e^{-i\lambda t}f(t)) = \theta, \tag{2.4}$$

which completes the proof of the theorem.

3. Now we establish the following result

THEOREM 2. *Let B be a bounded linear operator in a Banach space X such that*

(3.1) $$\|e^{tB}\| \leqq e^{at} \text{ for some } a < 0 \text{ and all } t \geqq 0.$$

Further, let $g: J \to X$ *be an* S^p*-almost periodic continuous function with* $1 \leqq p < \infty$. *Then the differential equation*

(3.2) $$v'(t) = Bv(t) + g(t) \text{ on } J$$

has a unique almost periodic solution $v(t)$. *Moreover, we have* $\sigma(v(t)) = \sigma(g(t))$.

PROOF. Since the S^p-almost periodicity of g implies the S^1-almost periodicity of g, it is sufficient to consider the case $p = 1$. The proof of the uniqueness and existence of an almost periodic solution $v(t)$ of the differential equation (3.2) given by Zaidman [3] for $p = 2$ in a Hilbert space goes through for $p = 1$ in any Banach space, with some minor modifications and by replacing the Cauchy-Schwarz inequality (wherever it occurs) by the corresponding Hölder's inequality for $p = 1$ and $q = \infty$.

Further, by (3.1), we have

(3.3) $$\lim_{t \to \infty} \frac{\log \| e^{tB} \|}{t} \leqq a < 0.$$

Therefore, by Theorem 11, p. 622, Dunford-Schwartz [2], the whole imaginary axis $\{i\lambda\}_{\lambda \in J}$ is contained in the resolvent set $\rho(B)$ of B. Consequently, by Theorem 1, $\sigma(v(t)) = \sigma(g(t))$.

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References

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[2] N. Dunford, and J. T. Schwartz, *Linear operators, part I*, (Interscience Publishers, Inc., New York (1958)).

[3] S. Zaidman, 'An existence result for Stepanoff almost periodic differential equations', *Canad. Math. Bull.* 14 (1971), 551–554.

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