14

Orthogonal invariance of CAR algebras

Let $(\mathcal{Y}, \nu)$ be a real Hilbert space.

Recall that $O(\mathcal{Y})$ denotes the group of orthogonal operators on $\mathcal{Y}$ and that $o(\mathcal{Y})$ denotes the Lie algebra of bounded anti-self-adjoint operators.

The main aim of this chapter is to discuss the invariance of CAR algebras (mostly C^*- but also W^*-CAR algebras) with respect to the orthogonal group. We will restrict ourselves to the results that are independent of a representation. In particular, they will not involve any Fock representation nor a distinguished Kähler structure. Orthogonal invariance of CAR algebras on Fock spaces will be studied separately in Chap. 16.

To some extent, this chapter can be viewed as an analog of Chap. 10 about the symplectic invariance of CCR in finite dimensions. However, in this chapter we consider the case of an arbitrary dimension, since for the CAR this does not introduce any serious additional difficulties, unlike for the CCR.

14.1 Orthogonal groups

14.1.1 Group $O_1(\mathcal{Y})$

Recall that if $\dim \mathcal{Y}$ is finite, besides the group $O(\mathcal{Y})$ and the Lie algebra $o(\mathcal{Y})$ we have the group $SO(\mathcal{Y}) := \big\{r \in O(\mathcal{Y}) \ : \ \det r = 1\big\}$. If $\dim \mathcal{Y}$ is arbitrary, we still have $O(\mathcal{Y})$ and $o(\mathcal{Y})$, but there seems to be no analog of $SO(\mathcal{Y})$. However, there exists a natural extension of the triple $\big(O(\mathcal{Y}),\ SO(\mathcal{Y}),\ o(\mathcal{Y})\big)$ to infinite dimensions described in the following definition:

Definition 14.1 *Set*

$$\begin{aligned} O_1(\mathcal{Y}) &:= \big\{r \in O(\mathcal{Y}) \ : \quad r - 1\!\!1 \in B^1(\mathcal{Y})\big\}, \\ SO_1(\mathcal{Y}) &:= \big\{r \in O_1(\mathcal{Y}) \ : \ \det r = 1\big\}, \\ o_1(\mathcal{Y}) &:= o(\mathcal{Y}) \cap B^1(\mathcal{Y}). \end{aligned}$$

We equip all of them with the metric given by the trace-class norm.

Proposition 14.2 (1) *$O_1(\mathcal{Y})$ is a group and $SO_1(\mathcal{Y})$ is its subgroup.*
(2) *We have an exact sequence of groups*

$$1 \to SO_1(\mathcal{Y}) \to O_1(\mathcal{Y}) \to \mathbb{Z}_2 \to 1. \tag{14.1}$$

(3) *$o_1(\mathcal{Y})$ is a Lie algebra and if $a \in o_1(\mathcal{Y})$, then $\mathrm{e}^a \in SO_1(\mathcal{Y})$.*

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Proof We use the fact that the determinant is a homomorphism of $O_1(\mathcal{Y})$ onto $\{1,-1\}$. □

We have the following characterization of elements of $SO_1(\mathcal{Y})$:

Theorem 14.3 *Let $r \in O_1(\mathcal{Y})$. Then $\dim \mathrm{Ker}(\mathbb{1}+r)$ is finite. Besides, the following conditions are equivalent:*

(1) $r \in SO_1(\mathcal{Y})$.
(2) $\dim \mathrm{Ker}(\mathbb{1}+r)$ *is even.*
(3) *There exists $a \in o_1(\mathcal{Y})$ such that $\mathrm{e}^a = r$.*

Proof Let us prove (2)⇒(3).
Case 1. Assume $\mathrm{Ker}(\mathbb{1}+r) = \{0\}$. We complexify $\mathcal{Y}$ and consider $r_{\mathbb{C}} \in U(\mathbb{C}\mathcal{Y})$. Since $\mathrm{Ker}(\mathbb{1}+r_{\mathbb{C}}) = \{0\}$ and $r_{\mathbb{C}} - \mathbb{1}$ is compact, we see that

$$\mathrm{spec}\, r_{\mathbb{C}} \subset \left\{\mathrm{e}^{\mathrm{i}\phi} \;:\; \phi \in]-\pi,\pi[\right\}.$$

Take e.g. the principal branch of the logarithm (which maps $\mathbb{C}\backslash]-\infty,0]$ onto $\{-\pi < \mathrm{Im}\, z < \pi\}$) and define $b := \log r_{\mathbb{C}}$. b is an anti-self-adjoint operator, $b \in B^1(\mathbb{C}\mathcal{Y})$ and $r_{\mathbb{C}} = \mathrm{e}^b$. It is real, so there exists $a \in o_1(\mathcal{Y})$ such that $b = a_{\mathbb{C}}$.
Case 2. Assume that $r = -\mathbb{1}$ and $\dim \mathcal{Y}$ is finite and even. We choose an o.n. basis $(e_1,\dots,e_{2n})$, and set $ce_i := e_{n+i}$, $ce_{n+i} := -e_i$. Then $c^2 = -\mathbb{1}$, $c \in o_1(\mathcal{Y})$ and $\mathrm{e}^{tc} = \mathbb{1}\cos t + c \sin t$. Thus $\mathrm{e}^{\pi c} = -\mathbb{1}$.

In the general case we set $\mathcal{Y}_{\mathrm{sg}} := \mathrm{Ker}(\mathbb{1}+r)$ and $\mathcal{Y}_{\mathrm{reg}} := \mathcal{Y}_{\mathrm{sg}}^{\perp}$. These are invariant subspaces of r, so that we can apply case 1 and case 2 to them respectively.

Using that the determinant is continuous in the trace norm topology, we see that $t \mapsto \det \mathrm{e}^{ta}$ is continuous for $a \in o_1(\mathcal{Y})$, which proves (3)⇒(1).

Let us prove (1)⇒(2). Assume that $\dim \mathrm{Ker}(\mathbb{1}+r)$ is odd. Let $y_0 \in \mathrm{Ker}(\mathbb{1}+r)$ be a unit vector and $r_0 := \mathbb{1} - 2|y_0\rangle\langle y_0|$. Then $rr_0 \in O_1(\mathcal{Y})$ and $\mathrm{Ker}(\mathbb{1}+rr_0) = \mathrm{Ker}(\mathbb{1}+r) \ominus \mathbb{R}y_0$. Hence, $\dim \mathrm{Ker}(\mathbb{1}+rr_0)$ is even. Therefore, $\det rr_0 = 1$. Noting that $\det r_0 = -1$, this implies $\det r = -1$. □

14.1.2 Group $O_p(\mathcal{Y})$

There exist other useful extensions of the triple $\big(O(\mathcal{Y}),\ SO(\mathcal{Y}),\ o(\mathcal{Y})\big)$ to infinite dimensions, which we consider in this subsection.

Throughout this subsection, $1 \le p \le \infty$. Recall that $B^p(\mathcal{Y})$ denotes the p-th trace ideal, $B_\infty(\mathcal{Y})$ the ideal of compact operators on $\mathcal{Y}$.

Definition 14.4 *Set*

$$O_p(\mathcal{Y}) := \begin{cases} \big\{r \in O(\mathcal{Y}) \;:\; r - \mathbb{1} \in B^p(\mathcal{Y})\big\}, & 1 \le p < \infty; \\ \big\{r \in O(\mathcal{Y}) \;:\; r - \mathbb{1} \in B_\infty(\mathcal{Y})\big\}, & p = \infty; \end{cases}$$

$$o_p(\mathcal{Y}) := \begin{cases} o(\mathcal{Y}) \cap B^p(\mathcal{Y}), & 1 \le p < \infty; \\ o(\mathcal{Y}) \cap B_\infty(\mathcal{Y}), & p = \infty. \end{cases}$$

We equip all of them with the topology of $B^p(\mathcal{Y})$, resp. $B_\infty(\mathcal{Y})$.

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Clearly, $O_p(\mathcal{Y}) \subset O_q(\mathcal{Y})$ and $o_p(\mathcal{Y}) \subset o_q(\mathcal{Y})$ for $p \le q$.

The determinant is not defined on the whole of $O_p(\mathcal{Y})$ for $p > 1$, which makes the definition of $SO_p(\mathcal{Y})$ harder than that of $SO_1(\mathcal{Y})$. Nevertheless, the following analog of Thm. 14.3 can be shown:

Theorem 14.5 *Let $r \in O_p(\mathcal{Y})$. Set $C(\epsilon) := \{z \in \mathbb{C} \ : \ |z| = 1, \ |z-1| > \epsilon\}$. Then for any $\epsilon > 0$, $\dim \mathbb{1}_{C(\epsilon)}(r)$ is finite. Besides, the following conditions are equivalent:*

(1) *For $\epsilon > 0$, $\dim \mathbb{1}_{C(\epsilon)}(r)$ is even.*
(2) *$\dim \mathrm{Ker}(\mathbb{1} + r)$ is even.*
(3) *There exists $a \in o_p(\mathcal{Y})$ such that $\mathrm{e}^a = r$.*

Proof $r - \mathbb{1}$ is compact, hence $\dim \mathbb{1}_{C(\epsilon)}(r)$ is finite for $\epsilon > 0$.

(1)⇒(2) is obvious. To prove (1)⇐(2) we note that for any $\lambda \in \mathrm{spec}\, r$ we have $\dim \mathbb{1}_{\{\lambda\}}(r) = \dim \mathbb{1}_{\{\overline{\lambda}\}}(r)$.

To show (2)⇔(3) we repeat verbatim arguments of the proof of Thm. 14.3 (2)⇔(3). □

Definition 14.6 *The set of $r \in O_p(\mathcal{Y})$ satisfying the conditions of Thm. 14.5 is denoted by $SO_p(\mathcal{Y})$. We will write $\det r = 1$ for $r \in SO_p(\mathcal{Y})$ and $\det r = -1$ for $r \in O_p(\mathcal{Y}) \backslash SO_p(\mathcal{Y})$, even though, strictly speaking, the determinant is not defined on $SO_p(\mathcal{Y})$.*

Proposition 14.7 (1) *$O_p(\mathcal{Y})$ is a group and $SO_p(\mathcal{Y})$ is its subgroup.*
(2) *We have an exact sequence of groups*

$$1 \to SO_p(\mathcal{Y}) \to O_p(\mathcal{Y}) \to \mathbb{Z}_2 \to 1. \tag{14.2}$$

(3) *$o_p(\mathcal{Y})$ is a Lie algebra, and if $a \in o_p(\mathcal{Y})$, then $\mathrm{e}^a \in SO_p(\mathcal{Y})$.*

Proof Clearly, $SO_1(\mathcal{Y})$ sits inside $SO_p(\mathcal{Y})$. Let us show that $SO_1(\mathcal{Y})^{\mathrm{cl}} = SO_p(\mathcal{Y})$.

First note that the condition (1) of Thm. 14.5 implies that $SO_p(\mathcal{Y})$ is closed inside $O_p(\mathcal{Y})$.

Let $r \in SO_p(\mathcal{Y})$. Using Thm. 14.5 (3), we can write $r = \mathrm{e}^a$ with $a \in o_p(\mathcal{Y})$. Using the spectral decomposition of a, we can approximate it with $a_n \in o_1(\mathcal{Y})$, so that $a_n \to a$. Hence, $\mathrm{e}^{a_n} \to r$ with $\mathrm{e}^{a_n} \in SO_1(\mathcal{Y})$. Hence, the closure of $SO_1(\mathcal{Y})$ contains $SO_p(\mathcal{Y})$.

Similarly, we show that $(O_1 \backslash SO_1(\mathcal{Y}))^{\mathrm{cl}} = O_p(\mathcal{Y}) \backslash SO_p(\mathcal{Y})$. In fact, every $r \in O_p(\mathcal{Y}) \backslash SO_p(\mathcal{Y})$ can be written as $r = \kappa r_0$ with $\kappa = \mathbb{1} - 2|e\rangle\langle e|$ and $r_0 \in SO_p(\mathcal{Y})$. We approximate r_0 with elements of $SO_1(\mathcal{Y})$ as above.

(2) follows then from the corresponding statement in Prop. 14.2. □

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14.2 Quadratic fermionic Hamiltonians

Recall that $\mathrm{Op}(b)$ denotes the anti-symmetric quantization of an anti-symmetric polynomial $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$. In this section we study *quadratic fermionic Hamiltonians*, that is, quantizations of elements from $\mathbb{C}\mathrm{Pol}_{\mathrm{a}}^{2}(\mathcal{Y}^{\#})$. We will also describe some situations where a quadratic fermionic Hamiltonian can be well defined even though its symbol is not of finite rank.

14.2.1 Fermionic harmonic oscillator

Let $e_1, e_2 \in \mathcal{Y}$ be an orthonormal pair of vectors in $\mathcal{Y}$. Consider the following operator in $\mathrm{CAR}^{C^*}(\mathcal{Y})$, which can be viewed as a fermionic analog of the harmonic oscillator:

$$H := \phi(e_1)\phi(e_2).$$

Clearly, $H = \mathrm{Op}(\zeta)$, where $\zeta = e_1 \otimes_{\mathrm{a}} e_2$. If we consider ζ as an element of $L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$, then $\zeta = \frac{1}{2}\big(|e_1\rangle\langle e_2| - |e_2\rangle\langle e_1|\big)$. Straightforward computations yield the following properties of the fermionic harmonic oscillator:

Proposition 14.8 (1) $H^2 = -1\!\mathrm{l}$, $\quad H = -H^*$, $\quad \mathrm{spec}\,(\mathrm{i}H) = \{-1, 1\}$;
(2) $\mathrm{e}^{tH} = \cos t 1\!\mathrm{l} + (\sin t)H$, *in particular,* $\mathrm{e}^{\pm\frac{\pi}{2}H} = \pm H$;
(3) $\mathrm{e}^{tH}\phi(y)\mathrm{e}^{-tH} = \phi(\mathrm{e}^{4t\zeta\nu^{-1}}y)$, $y \in \mathcal{Y}$, *in particular,*

$$H\phi(y)H^{-1} = \phi\left(y - 2e_1\langle e_1|y\rangle - 2e_2\langle e_2|y\rangle\right).$$

Let $y_1, y_2 \in \mathcal{Y}$ be a pair of normalized vectors with $\langle y_1|y_2\rangle = \cos\theta$. Let e_1, e_2 be any o.n. basis of $\mathrm{Span}(y_1, y_2)$ with the same orientation as that of y_1, y_2. Then

$$\begin{aligned}\phi(y_1)\phi(y_2) &= \cos\theta 1\!\mathrm{l} + \sin\theta\phi(e_1)\phi(e_2)\\ &= \mathrm{e}^{\theta\phi(e_1)\phi(e_2)} = \mathrm{Op}(\cos\theta + \sin\theta e_1 \cdot e_2).\end{aligned}$$

14.2.2 Commutation properties of quadratic fermionic Hamiltonians

The following theorem can be viewed as the fermionic analog of Thm. 10.13 (1).

Theorem 14.9 *Let* $\chi \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}^{2}(\mathcal{Y}^{\#})$ *and* $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$. *Then*

$$[\mathrm{Op}(\chi), \mathrm{Op}(b)] = 2\mathrm{Op}\big((\nabla\chi)\cdot\nu\nabla b\big); \tag{14.3}$$

$$\frac{1}{2}\big(\mathrm{Op}(\chi)\mathrm{Op}(b) + \mathrm{Op}(b)\mathrm{Op}(\chi)\big) = \mathrm{Op}\big(\chi\cdot b + \nabla_v\cdot\nu(\nabla^{(2)}\chi)\nu\nabla_v b\big). \tag{14.4}$$

(In the above expression, $\nabla^{(2)}\chi$ is considered as an element of $L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$ and $\nabla_v\cdot\nu(\nabla^{(2)}\chi)\nu\nabla_v$ is a differential operator acting on the anti-symmetric polynomial b.)

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Proof Let us use the "functional notation" for anti-symmetric polynomials. Thus v_1, v_2 are "generic variables" in $\mathcal{Y}$.

If $\deg b_1$ or $\deg b_2$ is equal to 2, then

$$\begin{aligned}\mathrm{e}^{\nabla_{v_2}\cdot\nu\nabla_{v_1}} b_1(v_1)b_2(v_2) = b_1(v_1)b_2(v_2) &+ \nabla_{v_2}\cdot\nu\nabla_{v_1} b_1(v_1)b_2(v_2)\\ &+ \frac{1}{2}(\nabla_{v_2}\cdot\nu\nabla_{v_1})^2 b_1(v_1)b_2(v_2).\end{aligned}$$

We insert $v_1 = v_2 = v$, switch to the coordinate notation as in the proof of Prop. 12.42, and use the summation convention. We obtain that the symbol of $\mathrm{Op}(b_1)\mathrm{Op}(b_2)$ equals

$$\begin{aligned}&b_1 b_2 + (-1)^{\deg b_1 - 1}\nu_{ij}\,(\nabla_{v^j} b_1)\,\nabla_{v^i} b_2\\ &-\frac{1}{2}\nu_{i,i'}\nu_{j,j'}\,(\nabla_{v^{i'}}\nabla_{v^{j'}} b_1)\,\nabla_{v^i}\nabla_{v^j} b_2.\end{aligned} \tag{14.5}$$

The formula for $\mathrm{Op}(b_2)\mathrm{Op}(b_1)$ coincides with (14.5), except that the second term changes sign. Then we replace b_1, b_2 with b, χ. □

In what follows it will be convenient to change slightly the parametrization of quadratic fermionic Hamiltonians.

Definition 14.10 $B_{\mathrm{a}}^{\mathrm{fd}}(\mathcal{Y}^{\#}, \mathcal{Y})$ *will denote the space of finite rank anti-symmetric operators, that is,* $B_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y}) \cap B^{\mathrm{fd}}(\mathcal{Y}^{\#}, \mathcal{Y})$.

Every $\chi \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}^2(\mathcal{Y}^{\#})$ (a complex homogeneous anti-symmetric quadratic polynomial on $\mathcal{Y}^{\#}$) can be represented as

$$\mathcal{Y}^{\#} \times \mathcal{Y}^{\#} \ni (v, w) \mapsto \chi(v, w) = v\cdot\zeta w, \tag{14.6}$$

for $\zeta \in \mathbb{C}B_{\mathrm{a}}^{\mathrm{fd}}(\mathcal{Y}^{\#}, \mathcal{Y})$. Therefore, we have an identification $\mathbb{C}\mathrm{Pol}_{\mathrm{a}}^2(\mathcal{Y}^{\#}) \simeq \mathbb{C}B_{\mathrm{a}}^{\mathrm{fd}}(\mathcal{Y}^{\#}, \mathcal{Y})$. Note that $\nabla\chi(v) = 2\zeta v$ and $\nabla^{(2)}\chi = 2\zeta$.

Definition 14.11 *We will write Op*(ζ) *for the anti-symmetric quantization of (14.6).*

Clearly, if we choose orthonormal coordinates in $\mathcal{Y}^{\#}$, then (14.6) equals

$$v\cdot\zeta w = \sum_{1\le i,j\le m} \zeta_{ij} v_i w_j,$$

where $[\zeta_{ij}]$ is an anti-symmetric matrix and its quantization equals

$$\mathrm{Op}(\zeta) = \sum_{i,j=1}^{m} \phi_i \zeta_{ij} \phi_j. \tag{14.7}$$

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14.2.3 Quadratic Hamiltonians in C^*-CAR algebras

It is natural to extend the definition of quadratic fermionic Hamiltonians to symbols that are not finite rank. In this subsection, we will consider quadratic Hamiltonians inside the algebra $\mathrm{CAR}^{C^*}(\mathcal{Y})$.

Definition 14.12 *$B^1_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$ will denote the space of trace-class anti-symmetric operators, that is, $B_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})\cap B^1(\mathcal{Y}^{\#},\mathcal{Y})$.*

Theorem 14.13 (1) *The map $\mathbb{C}B^{\mathrm{fd}}_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})\ni\zeta\mapsto\mathrm{Op}(\zeta)\in\mathrm{CAR}^{C^*}(\mathcal{Y})$ extends by continuity to $\zeta\in\mathbb{C}B^1_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$.*

(2) *Let $\zeta\in B^1_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$. Then $\mathrm{Op}(\zeta)$ is self-adjoint,*

$$\|\mathrm{Op}(\zeta)\|=\mathrm{Tr}|\zeta\nu|,\quad \inf\mathrm{Op}(\zeta)=-\mathrm{Tr}|\zeta\nu|,\quad \sup\mathrm{Op}(\zeta)=\mathrm{Tr}|\zeta\nu|. \tag{14.8}$$

(3) *If $\zeta_1,\zeta_2\in\mathbb{C}B^1_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$, then*

$$[\mathrm{Op}(\zeta_1),\mathrm{Op}(\zeta_2)]=4\mathrm{Op}(\zeta_1\nu\zeta_2-\zeta_2\nu\zeta_1). \tag{14.9}$$

Thus

$$o_1(\mathcal{Y})\ni a\mapsto\frac{1}{4}\mathrm{Op}(a\nu^{-1})\in\mathrm{CAR}^{C^*}(\mathcal{Y})$$

is a homomorphism of Lie algebras, where $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is equipped with the commutator.

Proof Assume first that $\mathcal{Y}$ is of finite dimension. Let $\zeta\in B_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$. By Corollary 2.85, we can find an orthonormal system $\{e_{i,\pm}\}_{i\in I}$ and positive real numbers $\{\lambda_i\}_{i\in I}$ such that

$$\zeta\nu=\sum_{i=1}^{m}\lambda_i\big(|e_{i,-}\rangle\langle e_{i,+}|-|e_{i,+}\rangle\langle e_{i,-}|\big). \tag{14.10}$$

Then

$$\mathrm{Op}(\zeta):=\sum_{i\in I}2\lambda_i\phi(e_{i,-})\phi(e_{i,+}). \tag{14.11}$$

Using the Jordan–Wigner representation adapted to the above o.n. basis, we see that

$$\mathrm{spec}\,\mathrm{Op}(\zeta)=\Big\{\sum\nolimits_{i\in I}\lambda_i\epsilon_i,\quad \epsilon_i=\pm1,\quad i\in I\Big\}.$$

Note that

$$|\zeta\nu|=\sum_{i\in I}\lambda_i\big(|e_{i,-}\rangle\langle e_{i,-}|+|e_{i,+}\rangle\langle e_{i,+}|\big).$$

Therefore, $\mathrm{Tr}|\zeta\nu|=\sum_{i\in I}\lambda_i$. This implies (14.8) in the finite-dimensional case.

In the case of $\zeta\in B^1(\mathcal{Y}^{\#},\mathcal{Y})$ in arbitrary dimension, we can still use Corollary 2.85 to find an orthonormal system $\{e_{i,\pm}\}_{i\in I}$ and positive real numbers $\{\lambda_i\}_{i\in I}$

such that (14.10) is true. Note that the sum in (14.11) is convergent, which allows us to define $\mathrm{Op}(\zeta)$. An obvious approximation argument extends (14.8) to the infinite-dimensional case.

Let us prove (14.9). By (14.3) applied to $\chi_i \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$, $i = 1, 2$,

$$[\mathrm{Op}(\chi_1), \mathrm{Op}(\chi_2)] = 2\mathrm{Op}\big((\nabla\chi_1)\cdot\nu\nabla\chi_2\big). \tag{14.12}$$

Let us compute the symbol on the r.h.s. of (14.12):

$$\begin{aligned}&\big((\nabla\chi_1)\cdot\nu\nabla\chi_2\big)(v,w)\\ =\ &\frac{1}{2}\big(\nabla\chi_1(v)\cdot\nu\nabla\chi_2(w) - \nabla\chi_2(v)\cdot\nu\nabla\chi_1(w)\big),\quad v,w\in\mathcal{Y}^{\#}.\end{aligned}$$

Then we use $\nabla\chi_i(v) = 2\zeta_i v$, obtaining (14.9). □

14.2.4 Quadratic Hamiltonians in W^*-CAR algebras

Let us now consider quadratic fermionic Hamiltonians in the setting given by the algebra $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

Definition 14.14 *Let $B^2_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$ denote the set of Hilbert–Schmidt anti-symmetric operators from $\mathcal{Y}^{\#}$ to $\mathcal{Y}$, that is, $B^2_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y}) := B_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})\cap B^2(\mathcal{Y}^{\#},\mathcal{Y})$.*

For simplicity, let us assume that $\mathcal{Y}$ is infinite-dimensional separable. Let $\zeta\in B^2_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$. By diagonalizing $\zeta\nu$, we can bring it to a diagonal form:

$$\zeta\nu = \sum_{i=1}^{\infty}\lambda_i\big(|e_{i,-}\rangle\langle e_{i,+}| - |e_{i,+}\rangle\langle e_{i,-}|\big). \tag{14.13}$$

Set

$$H_n := \sum_{i=1}^{n} 2\lambda_i\phi(e_{i,-})\phi(e_{i,+}).$$

Proposition 14.15 *For any $t\in\mathbb{R}$, there exists the strong limit*

$$\mathrm{s}-\lim_{n\to\infty}\mathrm{e}^{\mathrm{i}tH_n}. \tag{14.14}$$

The limit (14.14) defines a one-parameter strongly continuous unitary group. It can be written as $\mathrm{e}^{\mathrm{i}tH}$, where H is a certain self-adjoint operator, possibly unbounded. We denote H by $\mathrm{Op}(\zeta)$.

If $\zeta\in B^1_{\mathrm{a}}(\mathcal{Y},\mathcal{Y}^{\#})$, then the above defined $\mathrm{Op}(\zeta)$ coincides with that defined in Thm. 14.13. Furthermore, the definition does not depend on the choice of an ordered o.n. basis diagonalizing ζ. Moreover, $\mathrm{Op}(\zeta)$ is affiliated to $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

Proof It is enough to suppose that $(e_{i,-}, e_{i,+}\ :\ i = 1,2,\dots)$ is an o.n. basis. We use the inductive limit of the representation described in Subsect. 12.4.3. Thus $\mathrm{CAR}^{W^*}(\mathcal{Y})$ is represented on the infinite tensor product of grounded Hilbert

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spaces

$$\mathop{\otimes}_{i=1}^{\infty}\left(B^2(\mathbb{C}^2),\ \frac{1}{\sqrt{2}}\mathbb{1}\right).$$

The operator $\mathrm{e}^{\mathrm{i}tH_n}$ acts in this representation as the multiplication from the right by

$$\mathop{\otimes}_{i=1}^{n}\begin{bmatrix}\mathrm{e}^{\mathrm{i}t\lambda_i} & 0\\ 0 & \mathrm{e}^{\mathrm{i}t\lambda_i}\end{bmatrix}\otimes\mathop{\otimes}_{i=n+1}^{\infty}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Set

$$\Omega := \mathop{\otimes}_{i=1}^{\infty}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Clearly,

$$(\Omega|\mathrm{e}^{\mathrm{i}tH_n}\Omega) = \prod_{i=1}^{n}\cos t\lambda_i. \tag{14.15}$$

(14.15) converges as $n\to\infty$ iff $\sum_{i=1}^{\infty}\lambda_i^2<\infty$. But by Thm. 3.16, the convergence of (14.15) is equivalent to the $*$-strong convergence of $\mathrm{e}^{\mathrm{i}tH_n}$. □

14.3 Pin^{c} and Pin groups

We keep the same notation as in the rest of the chapter. In particular, $(\mathcal{Y},\nu)$ is a real Hilbert space.

The groups $Pin^{\mathrm{c}}(\mathcal{Y})$ and $Pin(\mathcal{Y})$ are well known in finite dimensions. It is convenient to consider them as subgroups of the $*$-algebra $\mathrm{CAR}(\mathcal{Y})$, resp. its real sub-algebra $\mathrm{Cliff}(\mathcal{Y})$.

Recall that these algebras are equipped with the parity automorphism α. As usual, the set of even, resp. odd elements of $\mathrm{CAR}(\mathcal{Y})$ is denoted $\mathrm{CAR}_0(\mathcal{Y})$, resp. $\mathrm{CAR}_1(\mathcal{Y})$.

In this section we concentrate on generalizing the groups $Pin^{\mathrm{c}}(\mathcal{Y})$ and $Pin(\mathcal{Y})$ to infinite dimensions. The first generalization will involve subgroups of the algebra $\mathrm{CAR}^{C^*}(\mathcal{Y})$, and the second those of $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

14.3.1 Pin^{c} and Pin groups in finite dimensions

In this subsection we assume in addition that the dimension of $\mathcal{Y}$ is finite. Let us describe the well-known results about Pin^{c} and Pin groups in finite dimensions. We do not give their proofs, which are well known. Besides, they will follow from the more general results about the infinite-dimensional case to be described later on.

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Definition 14.16 *We define* $Pin^{\mathrm{c}}(\mathcal{Y})$ *as the set of all unitary elements* U *in* $\mathrm{CAR}(\mathcal{Y})$ *such that*

$$\big\{U\phi(y)U^* \ :\ y \in \mathcal{Y}\big\} = \big\{\phi(y)\ :\ y \in \mathcal{Y}\big\}.$$

We set

$$\begin{aligned} Spin^{\mathrm{c}}(\mathcal{Y}) &:= Pin^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{CAR}_0(\mathcal{Y}),\\ Pin(\mathcal{Y}) &:= Pin^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{Cliff}(\mathcal{Y}),\\ Spin(\mathcal{Y}) &:= Spin^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{Cliff}(\mathcal{Y}). \end{aligned}$$

The following theorem is immediate:

Theorem 14.17 *Let* $U \in Pin^{\mathrm{c}}(\mathcal{Y})$*. Then there exists a unique* $r \in O(\mathcal{Y})$ *such that*

$$U\phi(y)U^* = \det(r)\phi(ry), \quad y \in \mathcal{Y}. \tag{14.16}$$

The map $Pin^{\mathrm{c}}(\mathcal{Y}) \to O(\mathcal{Y})$ *obtained this way is a homomorphism of groups.*

Definition 14.18 *If (14.16) is satisfied, we say that* U det-implements r.

Note that in the context of CAR and Clifford algebras, the concept of det-implementation turns out to be more natural than that of implementation.

Theorem 14.19 *Let* $r \in O(\mathcal{Y})$.

(1) *The set of elements of* $Pin(\mathcal{Y})$ *det-implementing* r *consists of a pair of operators differing by sign,* $\pm U_r = \{U_r, -U_r\}$.
(2) *The set of elements of* $Pin^{\mathrm{c}}(\mathcal{Y})$ *det-implementing* r *consists of operators of the form* μU_r *with* $|\mu| = 1$.
(3) $r \in SO(\mathcal{Y})$ *iff* U_r *is even;* $r \in O(\mathcal{Y})\backslash SO(\mathcal{Y})$ *iff* U_r *is odd.*
(4) *If* $r_1, r_2 \in O(\mathcal{Y})$*, then* $U_{r_1}U_{r_2} = \pm U_{r_1 r_2}$.

The above statements can be summarized by the following commuting diagrams of Lie groups and their continuous homomorphisms, where all vertical and horizontal sequences are exact:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & U(1) & \to & U(1) & \to & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Spin^{\mathrm{c}}(\mathcal{Y}) & \to & Pin^{\mathrm{c}}(\mathcal{Y}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & SO(\mathcal{Y}) & \to & O(\mathcal{Y}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{14.17}$$

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$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & \mathbb{Z}_2 & \rightarrow & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Spin(\mathcal{Y}) & \rightarrow & Pin(\mathcal{Y}) & \rightarrow & \mathbb{Z}_2 & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & SO(\mathcal{Y}) & \rightarrow & O(\mathcal{Y}) & \rightarrow & \mathbb{Z}_2 & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array} \tag{14.18}$$

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & U(1) & \rightarrow & U(1) & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Pin(\mathcal{Y}) & \rightarrow & Pin^{\mathrm{c}}(\mathcal{Y}) & \rightarrow & U(1) & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & O(\mathcal{Y}) & \rightarrow & O(\mathcal{Y}) & \rightarrow & 1 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{14.19}$$

It is well known that $SO(\mathcal{Y})$ is connected and its fundamental group $\pi_1(SO(\mathcal{Y}))$ equals $\mathbb{Z}$ if $\dim\mathcal{Y} = 2$ and $\mathbb{Z}_2$ if $\dim\mathcal{Y} > 2$. Thus $SO(\mathcal{Y})$ possesses a unique two-fold covering group, equal to its universal covering if $\dim\mathcal{Y} > 2$. This two-fold covering is isomorphic to $Spin(\mathcal{Y})$.

14.3.2 Pin_1^{c} *and* Pin_1 *groups*

In this subsection we allow $\dim\mathcal{Y}$ to be arbitrary.

Definition 14.20 *Define* $Pin_1^{\mathrm{c}}(\mathcal{Y})$ *as the set of unitary operators* U *in* $\mathrm{CAR}^{C^*}(\mathcal{Y})$ *such that*

$$\{U\phi(y)U^* \ : \ y \in \mathcal{Y}\} = \{\phi(y) \ : \ y \in \mathcal{Y}\}.$$

Set

$$\begin{aligned} Spin_1^{\mathrm{c}}(\mathcal{Y}) &:= Pin_1^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{CAR}_0^{C^*}(\mathcal{Y}), \\ Pin_1(\mathcal{Y}) &:= Pin_1^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{Cliff}^{C^*}(\mathcal{Y}), \\ Spin_1(\mathcal{Y}) &:= Pin_1(\mathcal{Y}) \cap \mathrm{Cliff}_0^{C^*}(\mathcal{Y}). \end{aligned}$$

We equip all these groups with the metric given by the operator norm.

The concept of implementability has an obvious definition:

Definition 14.21 *Let* $U \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ *and* $r \in O(\mathcal{Y})$.

(1) *We say that* U intertwines r *if*

$$U\phi(y) = \phi(ry)U, \quad y \in \mathcal{Y}. \tag{14.20}$$

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(2) *If in addition U is unitary, then we also say that U* implements r.
(3) *If there exists $U \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ that implements r, then we say that r is* implementable in $\mathrm{CAR}^{C^*}(\mathcal{Y})$.

A more useful concept is given in the definition below.

Definition 14.22 *Let $r \in O(\mathcal{Y})$.*

(1) *We say that $A \in \mathrm{CAR}^{C^*}(\mathcal{Y})$* α-intertwines *$r \in O(\mathcal{Y})$ if*

$$\alpha(A)\phi(y) = \phi(ry)A,$$
$$\text{or, equivalently,}\quad A\phi(y) = \phi(ry)\alpha(A), \quad y \in \mathcal{Y}. \tag{14.21}$$

(2) *If in addition A is unitary, then we also say that A* α-implements r.
(3) *If there exists $U \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ that α-implements r, then we say that r is* α-implementable in $\mathrm{CAR}^{C^*}(\mathcal{Y})$.

We will see later that if there exists an invertible A α-intertwining r, then necessarily $r \in O_\infty(\mathcal{Y})$ (actually, $r \in O_1(\mathcal{Y})$). Therefore, $\det r$ is well defined by Def. 14.6, and we can introduce the following definition, essentially equivalent to α-implementability.

Definition 14.23 *Let $r \in O_\infty(\mathcal{Y})$.*

(1) *We say that $A \in \mathrm{CAR}^{C^*}(\mathcal{Y})$* det-intertwines *$r$ if*

$$A\phi(y) = \det r\, \phi(ry)A, \quad y \in \mathcal{Y}. \tag{14.22}$$

(2) *If in addition A is unitary then we also say that A* det-implements r.
(3) *If there exists $U \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ that det-implements r, then we say that r is* det-implementable in $\mathrm{CAR}^{C^*}(\mathcal{Y})$.

The following two theorems are the main results of this subsection.

Theorem 14.24 (1) *Let $r \in O(\mathcal{Y})$. Then r is det-implementable in $\mathrm{CAR}^{C^*}(\mathcal{Y})$ iff r is α-implementable in $\mathrm{CAR}^{C^*}(\mathcal{Y})$ iff $r \in O_1(\mathcal{Y})$.*
(2) *Let $U \in Pin_1^{\mathrm{c}}(\mathcal{Y})$. Then there exists a unique $r \in O_1(\mathcal{Y})$ such that r is det-implemented and α-implemented by U in $\mathrm{CAR}^{C^*}(\mathcal{Y})$. The map $Pin_1^{\mathrm{c}}(\mathcal{Y}) \to O_1(\mathcal{Y})$ obtained this way is a homomorphism of groups.*

Theorem 14.25 *All the statements of Thm. 14.19 are true if we replace $O(\mathcal{Y})$, $SO(\mathcal{Y})$, $Pin^{\mathrm{c}}(\mathcal{Y})$, $Spin^{\mathrm{c}}(\mathcal{Y})$, $Pin(\mathcal{Y})$, $Spin(\mathcal{Y})$ with $O_1(\mathcal{Y})$, $SO_1(\mathcal{Y})$, $Pin_1^{\mathrm{c}}(\mathcal{Y})$, $Spin_1^{\mathrm{c}}(\mathcal{Y})$, $Pin_1(\mathcal{Y})$, $Spin_1(\mathcal{Y})$.*

Before we prove Thms. 14.24 and 14.25, let us show the following lemma.

Lemma 14.26 *Let $r \in O(\mathcal{Y})$. Then the following is true:*

(1) *If A α-intertwines r, then A is either even or odd.*

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(2) *If there exists an invertible A α-intertwining r, then $r \in O_\infty(\mathcal{Y})$.*
(3) *If $A \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ α-intertwines r, then A det-intertwines r.*

Proof Let $U = U_0 + U_1 \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ α-intertwine r with U_0 even and U_1 odd. Then

$$(U_0 + U_1)\phi(y) = \phi(ry)(U_0 - U_1), \quad y \in \mathcal{Y}. \tag{14.23}$$

Comparing even and odd terms in (14.23), we obtain

$$U_0\phi(y) = \phi(ry)U_0, \quad U_1\phi(y) = -\phi(ry)U_1, \quad y \in \mathcal{Y}. \tag{14.24}$$

Hence, $U_0^*U_0$ and $U_1^*U_1$ commute with $\phi(y)$, $y \in \mathcal{Y}$. Clearly, they are even. Hence, by Prop. 12.61, they are proportional to identity. Hence, the operators U_i are proportional to a unitary operator.

(14.24) implies also that $U_1^*U_0$ anti-commutes with $\phi(y)$, $y \in \mathcal{Y}$. By Prop. 12.61 this implies that $U_1^*U_0$ is even. But $U_1^*U_0$ is odd. Hence, $U_1^*U_0 = 0$. Thus one of the U_i is zero. This proves (1).

Let us now prove (2). Let an invertible $U \in \mathrm{CAR}^{C^*}(\mathcal{Y})$ α-intertwine r. Assume that $r \notin O_\infty(\mathcal{Y})$. Then there exists a sequence $y_n \in \mathcal{Y}$ with $\mathrm{w}-\lim y_n = 0$ and $y_n - ry_n \not\to 0$. It follows that $U\phi(y_n) \mp \phi(y_n)U \to 0$ in norm, if U is even, resp. odd. Hence, $\phi(ry_n - y_n)U$, and consequently $\phi(ry_n - y_n)$ tend to 0 in norm, which is a contradiction.

Now set $\mathcal{Y}_{\mathrm{sg}} = \mathrm{Ker}(\mathbb{1} + r)$. Let E_{sg} be the associated conditional expectation. Then for $y \in \mathcal{Y}_{\mathrm{sg}}$ we have

$$U_{\mathrm{sg}}\phi(y) = \mp\phi(y)U_{\mathrm{sg}},$$

if U is even, resp. odd and $U_{\mathrm{sg}} = E_{\mathrm{sg}}(U) \in \mathrm{CAR}(\mathcal{Y}_{\mathrm{sg}})$. By Prop. 12.36, this implies that $\dim \mathrm{Ker}(\mathbb{1} + r)$ is even, resp. odd, i.e. $\det r = \pm 1$. Therefore, U also det-intertwines r. □

The following proposition gives another possible equivalent definition of the Spin group. It follows easily from the commutation properties of quadratic Hamiltonians.

Proposition 14.27 *$Spin_1(\mathcal{Y})$ consists of operators of the form $\mathrm{e}^{\mathrm{Op}(\zeta)}$ where $\zeta \in B^1(\mathcal{Y}^\#, \mathcal{Y})$. More precisely, let $r \in SO_1(\mathcal{Y})$. By Thm. 14.3, there exists $a \in o_1(\mathcal{Y})$ such that $r = \mathrm{e}^a$. Then*

$$\pm U_r = \pm \mathrm{e}^{\frac{1}{4}\mathrm{Op}(a\nu^{-1})} \in \mathrm{Cliff}_0^{C^*}(\mathcal{Y}) \tag{14.25}$$

intertwines r.

Proof of Thm. 14.24. Let $r \in SO_1(\mathcal{Y})$. Then r is det-implementable by Prop. 14.27. Since U_r in (14.25) is even, r is also α-implementable. Thus $Spin_1(\mathcal{Y}) \to SO_1(\mathcal{Y})$ is onto.

If $r \in O_1(\mathcal{Y}) \backslash SO_1(\mathcal{Y})$, choose any $e \in \mathcal{Y}$ of norm 1. Set $\kappa_e := \mathbb{1} - 2|e\rangle\langle e|$. Clearly, $\phi(e)$ implements $-\kappa_e$. Hence, $\kappa_e r \in SO_1(\mathcal{Y})$ and r is det-implemented by

$\pm\phi(e)U_{\kappa_e r}$. Since $\phi(e)U_{\kappa_e r}$ is odd, r is also α-implementable. Thus $Pin_1(\mathcal{Y}) \to O_1(\mathcal{Y})$ is onto.

Now let $r \in O(\mathcal{Y})$ be α-implementable. By Prop. 14.26, $r \in O_\infty(\mathcal{Y})$ and r is also det-implementable. It remains to prove that $r \in O_1(\mathcal{Y})$. Without loss of generality we may assume that $\mathcal{Y}$ is separable.

Assume first that $r \in SO_\infty(\mathcal{Y})$. By Thm. 14.5, there exists $a \in o_\infty(\mathcal{Y})$ such that $r = \mathrm{e}^a$. By Corollary 2.85, there exists an o.n. basis $(e_{i\pm})_{i\in\mathbb{N}}$ and real numbers $\lambda_i \geq 0$ such that

$$a = \sum_{i\in\mathbb{N}} \lambda_i \big(|e_{i-}\rangle\langle e_{i+}| - |e_{i+}\rangle\langle e_{i-}|\big).$$

We set $\mathcal{Y}_n = \mathrm{Span}\{e_{i\pm},\ 1 \leq i \leq n\}$. Let E_n be the conditional expectation associated with $\mathcal{Y}_n$. Set $U_n = E_n(U)$, where $U \in \mathrm{CAR}_0^{C^*}(\mathcal{Y})$ implements r. Also set

$$V_n = \exp\left(\sum_{i=1}^n \frac{\lambda_i}{2}\phi(e_{i+})\phi(e_{i-})\right).$$

Applying Prop. 14.8 and Prop. 6.83, we obtain

$$V_n\phi(y) = \phi(ry)V_n,\ U_n\phi(y) = \phi(ry)U_n,\ y \in \mathcal{Y}_n.$$

Hence, by Prop. 12.61, $U_n = \lambda_n V_n$, $\lambda_n \in \mathbb{C}$. Clearly, $E_{n-1}(U_n) = U_{n-1}$, and computing in the real-wave representation we see that $E_{n-1}(V_n) = V_{n-1}$, hence λ_n does not depend on n.

Since by (12.34) $U_n \to U$ in norm, it follows that V_n converges in norm.

Now set $A_i = \phi(e_{i+})\phi(e_{i-})$, so that, by Prop. 14.8,

$$V_n = \prod_{i=1}^n \mathrm{e}^{\frac{\lambda_i}{2}A_i} = \prod_{i=1}^n \Big(\cos(\lambda_i/2)\mathbb{1} + \sin(\lambda_i/2)A_i\Big).$$

Computing in the real-wave representation, we check that

$$(\Omega|V_n\Omega) = \prod_{i=1}^n \cos(\lambda_i/2). \tag{14.26}$$

Therefore, the infinite product $\prod\limits_{i\in\mathbb{N}}\cos(\lambda_i/2)$ converges, and hence $\prod\limits_{i\in\mathbb{N}}(\mathbb{1} + \tan(\lambda_i/2)A_i)$ converges in norm. Since the A_i commute, this implies that the product $\prod\limits_{i\in\mathbb{N}}\|\mathbb{1} + \tan(\lambda_i/2)A_j\|$ converges. Since $\|\mathbb{1} + \tan(\lambda_i/2)A_i\| = 1 + \tan(\lambda_i/2)$ for i large enough, this implies that the series $\sum\limits_{i\in\mathbb{N}}\lambda_i$ is convergent. Hence, $a \in o_1(\mathcal{Y})$ and $r \in SO_1(\mathcal{Y})$.

Assume now that $r \in O_\infty(\mathcal{Y})\backslash SO_\infty(\mathcal{Y})$. Let $U \in \mathrm{CAR}_1^{C^*}(\mathcal{Y})$ α-intertwine r. Then, as above, $\phi(e)U \in \mathrm{CAR}_0^{C^*}(\mathcal{Y})$ implements $r\kappa_e \in SO_\infty(\mathcal{Y})$. Hence, $r\kappa_e \in SO_1(\mathcal{Y})$ and $r \in O_1(\mathcal{Y})$. □

Proof of Thm. 14.25. We deduce the theorem from Thm. 14.19, reducing ourselves to the finite-dimensional case by the same argument as in Prop. 12.61. □

The implementability of Bogoliubov rotations can be easily deduced from the results about the det-implementability.

Corollary 14.28 *$r \in O(\mathcal{Y})$ is implementable in $\mathrm{CAR}^{C^*}(\mathcal{Y})$ iff*

(1) $r \in SO_1(\mathcal{Y})$
or
(2) $-r \in O_1(\mathcal{Y}) \backslash SO_1(\mathcal{Y})$.

14.3.3 Pin_2^{c} and Pin_2 groups

In this subsection we again allow $\dim \mathcal{Y}$ to be arbitrary.

Definition 14.29 *Define $Pin_2^{\mathrm{c}}(\mathcal{Y})$ as the set of unitary operators U in $\mathrm{CAR}^{W^*}(\mathcal{Y})$ such that*

$$\{U\phi(y)U^* \ : \ y \in \mathcal{Y}\} = \{\phi(y) \ : \ y \in \mathcal{Y}\}.$$

Set

$$\begin{aligned} Spin_2^{\mathrm{c}}(\mathcal{Y}) &:= Pin_2^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{CAR}_0^{W^*}(\mathcal{Y}), \\ Pin_2(\mathcal{Y}) &:= Pin_2^{\mathrm{c}}(\mathcal{Y}) \cap \mathrm{Cliff}^{W^*}(\mathcal{Y}), \\ Spin_2(\mathcal{Y}) &:= Pin_2(\mathcal{Y}) \cap \mathrm{Cliff}_0^{W^*}(\mathcal{Y}). \end{aligned}$$

We equip all these groups with the σ-weak topology.

We also have the obvious analogs of Defs. 14.21, 14.22 and 14.23, with $\mathrm{CAR}^{C^*}(\mathcal{Y})$ replaced with $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

Theorem 14.30 (1) *Let $r \in O(\mathcal{Y})$. Then r is det-implementable in $\mathrm{CAR}^{W^*}(\mathcal{Y})$ iff r is α-implementable in $\mathrm{CAR}^{W^*}(\mathcal{Y})$ iff $r \in O_2(\mathcal{Y})$.*
(2) *Let $U \in Pin_2^{\mathrm{c}}(\mathcal{Y})$. Then there exists a unique $r \in O_2(\mathcal{Y})$ such that r is det-implemented by U. The map $Pin_2^{\mathrm{c}}(\mathcal{Y}) \to O_2(\mathcal{Y})$ obtained this way is a homomorphism of groups.*

Proof We can follow closely the proofs in Subsect. 14.3.2, with some modifications. Instead of Prop. 12.61 we use Prop. 12.62.

First we show that if $r \in O_2(\mathcal{Y})$, then r is α-implementable and det-implementable, following the proof of the C^* case, using Prop. 14.15 instead of Thm. 14.13.

It remains to prove that if r is α-implementable, then $r \in O_2(\mathcal{Y})$. $r \in O_\infty(\mathcal{Y})$ is proved as in the proof of Lemma 14.26, replacing the norm convergence by the σ-weak convergence.

We then follow the proof of Thm. 14.24, and obtain that V_n converges in the σ-weak topology. We are left to prove that $\sum \lambda_i^2$ is convergent. But this follows from (14.26). □

Theorem 14.31 *All the statements of Thm. 14.19 are true if we replace $O(\mathcal{Y})$, $SO(\mathcal{Y})$, $Pin^{\rm c}(\mathcal{Y})$, $Spin^{\rm c}(\mathcal{Y})$, $Pin(\mathcal{Y})$, $Spin(\mathcal{Y})$ with $O_2(\mathcal{Y})$, $SO_2(\mathcal{Y})$, $Pin_2^{\rm c}(\mathcal{Y})$, $Spin_2^{\rm c}(\mathcal{Y})$, $Pin_2(\mathcal{Y})$, $Spin_2(\mathcal{Y})$.*

Again, the implementability of Bogoliubov rotations can easily be deduced from the results about the det-implementability.

Corollary 14.32 *$r \in O(\mathcal{Y})$ is implementable in $\mathrm{CAR}^{W^*}(\mathcal{Y})$ iff*

(1) $r \in SO_2(\mathcal{Y})$
or
(2) $-r \in O_2(\mathcal{Y})\backslash SO_2(\mathcal{Y})$.

14.3.4 Symbol of elements of $Spin(\mathcal{Y})$

We again assume that $\mathcal{Y}$ is finite-dimensional, although the results of this subsection are easily generalized to an arbitrary dimension. In this subsection we study the anti-symmetric symbol of elements of the Spin group.

Proposition 14.33 *Let $a \in o(\mathcal{Y})$. Then*

$$\mathrm{e}^{\frac{1}{4}\mathrm{Op}(a\nu^{-1})} = \mathrm{Op}\Big((\det\cosh(2a))^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\tanh(2a)\nu^{-1}} \Big). \tag{14.27}$$

Proof By Corollary 2.85, there exists an orthonormal system $(e_{i,\pm})_{i=1,\dots,n}$ and positive numbers $(\lambda_i)_{i=1,\dots,n}$ such that

$$a = \sum_{i=1}^{n} a_i, \quad a_i = \frac{\lambda_i}{2}\big(|e_{i,-}\rangle\langle e_{i,+}| - \lambda_i |e_{i,+}\rangle\langle e_{i,-}|\big).$$

Note that $[a_i, a_j] = 0$ and $\big[\mathrm{Op}(a_i\nu^{-1}), \mathrm{Op}(a_j\nu^{-1})\big] = 0$.

Therefore,

$$\mathrm{e}^{a} = \prod_{i=1}^{n} \mathrm{e}^{a_i}, \quad \mathrm{e}^{\frac{1}{4}\mathrm{Op}(a\nu^{-1})} = \prod_{i=1}^{n} \mathrm{e}^{\frac{1}{4}\mathrm{Op}(a_i\nu^{-1})},$$

and we can assume without loss of generality that

$$\dim\mathcal{Y} = 2, \quad a = \frac{\lambda}{2}\big(|e_1\rangle\langle e_2| - |e_2\rangle\langle e_1|\big), \quad \mathrm{Op}(a\nu^{-1}) = \lambda\phi(e_1)\phi(e_2).$$

By Prop. 14.8, we know that

$$\begin{aligned}\mathrm{e}^{\mathrm{Op}(a\nu^{-1})} &= \cos\lambda + (\sin\lambda)\phi(e_1)\phi(e_2) \\ &= \mathrm{Op}\Big(\cos\lambda\left(\mathbb{1} + \lambda^{-1}(\tan\lambda)a\nu^{-1}\right)\Big).\end{aligned}$$

Thus the anti-symmetric symbol of $\mathrm{e}^{\frac{1}{4}\mathrm{Op}(a\nu^{-1})}$ equals

$$\begin{aligned}\cos\lambda\left(\mathbb{1} + \lambda^{-1}(\tan\lambda)a\nu^{-1}\right) &= (\cos^2\lambda)^{\frac{1}{2}} \mathrm{e}^{\lambda^{-1}(\tan\lambda)a\nu^{-1}} \\ &= \big(\det\cosh(2a)\big)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\tanh(2a)\nu^{-1}},\end{aligned}$$

where we have used $\cos\lambda\mathbb{1} = \cosh(2a)$ and $\lambda^{-1}(\tan\lambda)a = \frac{1}{2}\tanh(2a)$. □

Definition 14.34 *We say that $r \in O(\mathcal{Y})$ is* regular *if* $\operatorname{Ker}(r + \mathbb{1}) = \{0\}$.

Proposition 14.35 *Let $r \in O(\mathcal{Y})$ be regular. Let $\gamma \in o(\mathcal{Y})$ be its Cayley transform, that is, $\gamma = \frac{\mathbb{1}-r}{\mathbb{1}+r}$ (see Subsect. 1.4.6).*

Then

$$U_r = \pm \mathrm{Op}\big(\det(\mathbb{1}-\gamma)^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\gamma\nu^{-1}}\big). \tag{14.28}$$

Proof We can assume that $r = \mathrm{e}^a$ with $a \in o(\mathcal{Y})$. Moreover, by Prop. 14.8 we have

$$\mathrm{e}^{\frac{1}{4}\mathrm{Op}(a\nu^{-1})}\phi(y)\mathrm{e}^{-\frac{1}{4}\mathrm{Op}(a\nu^{-1})} = \phi(\mathrm{e}^a y).$$

Next we note that $\tanh(\frac{1}{2}a) = \frac{r-\mathbb{1}}{r+\mathbb{1}} = \gamma$, $\cosh(\frac{1}{2}a) = \mathrm{e}^{-\frac{1}{2}a}(\mathbb{1}-\gamma)^{-1}$. Since $\det \mathrm{e}^{\frac{1}{2}a} = 1$, this proves (14.28). □

Let $r_1, r_2 \in SO(\mathcal{Y})$, $r = r_1 r_2$. We know that

$$U_{r_1} U_{r_2} = \pm U_r. \tag{14.29}$$

It is instructive to prove this fact for regular r_1, r_2, r by a direct calculation involving the Berezin calculus.

Let $\gamma_1, \gamma_2, \gamma$ be the Cayley transforms of r_1, r_2, r. By Prop. 12.42, $U_{r_1} U_{r_2}$ has the anti-symmetric symbol

$$\begin{aligned}
&\det(\mathbb{1}-\gamma_1)^{-\frac{1}{2}} \det(\mathbb{1}-\gamma_2)^{-\frac{1}{2}} \\
&\times \iint \mathrm{e}^{(v-v_1)\cdot\nu^{-1}(v-v_2)} \mathrm{e}^{\frac{1}{2}v_1\cdot\gamma_1\nu^{-1}v_1} \mathrm{e}^{\frac{1}{2}v_2\cdot\gamma_2\nu^{-1}v_2} \mathrm{d}v_2 \mathrm{d}v_1 \\
= {}&\det(\mathbb{1}-\gamma_1)^{-\frac{1}{2}} \det(\mathbb{1}-\gamma_2)^{-\frac{1}{2}} \\
&\times \iint \mathrm{e}^{\theta\cdot(v_1,v_2)+(v_1,v_2)\cdot\sigma(v_1,v_2)} \mathrm{d}v_2 \mathrm{d}v_1,
\end{aligned} \tag{14.30}$$

where

$$\theta := (-\nu^{-1}v, \nu^{-1}v), \quad \sigma := \begin{bmatrix} \gamma_1\nu^{-1} & \nu^{-1} \\ -\nu^{-1} & \gamma_2\nu^{-1} \end{bmatrix}.$$

By Prop. 7.19, (14.30) equals

$$\det(\mathbb{1}-\gamma_1)^{-\frac{1}{2}} \det(\mathbb{1}-\gamma_2)^{-\frac{1}{2}} \mathrm{Pf}(\sigma) \exp(\frac{1}{2}\theta\cdot\sigma^{-1}\theta). \tag{14.31}$$

Next $\mathrm{Pf}(\sigma) = \pm \det(\sigma)^{\frac{1}{2}}$. Since the Pfaffian and the determinant above are computed w.r.t. a volume form compatible with the Euclidean structure ν, we have

$$\det(\sigma) = \det \begin{bmatrix} \gamma_1 & \mathbb{1} \\ -\mathbb{1} & \gamma_2 \end{bmatrix} = \det(\mathbb{1} + \gamma_1\gamma_2),$$

using (1.4). By (1.49), we know that

$$(\mathbb{1} - \gamma) = (\mathbb{1} - \gamma_2)(\mathbb{1} + \gamma_1\gamma_2)^{-1}(\mathbb{1} - \gamma_1). \tag{14.32}$$

This implies that the first line of (14.31) equals $\det(\mathbb{1} - \gamma)^{-\frac{1}{2}}$.

By (1.3),

$$\sigma^{-1} = \begin{bmatrix} \nu\gamma_2(\gamma_1\gamma_2 + \mathbb{1})^{-1} & -\nu(\gamma_2\gamma_1 + \mathbb{1})^{-1} \\ \nu(\gamma_1\gamma_2 + \mathbb{1})^{-1} & \nu\gamma_1(\gamma_2\gamma_1 + \mathbb{1})^{-1} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} &\theta\cdot\sigma^{-1}\theta \\ &= v\cdot \left(\gamma_2(\gamma_1\gamma_2 + \mathbb{1})^{-1} + \gamma_1(\gamma_2\gamma_1 + \mathbb{1})^{-1} + (\gamma_2\gamma_1 + \mathbb{1})^{-1} - (\gamma_1\gamma_2 + \mathbb{1})^{-1}\right)\nu^{-1}v \\ &= v\cdot \left(\mathbb{1} - (\mathbb{1} - \gamma_2)(\mathbb{1} + \gamma_1\gamma_2)^{-1}(\mathbb{1} - \gamma_1)\right)\nu^{-1}v \\ &= v\cdot\gamma\nu^{-1}v. \end{aligned}$$

14.4 Notes

The so-called spinor representations of orthogonal groups were studied by Cartan (1938) and Brauer–Weyl (1935).

The first famous non-trivial application of the orthogonal invariance to quantum physics seems to be the version of the BCS theory due to Bogoliubov, described e.g. in Fetter–Walecka (1971).

A very comprehensive article devoted to CAR C^*-algebras was written by Araki (1987). More literature references to the subject of this chapter can be found in the notes to Chap. 16.