Proc. London Math. Soc. (3) 93 (2006) 82–84 © 2006 London Mathematical Society
doi:10.1112/S0024611506015759

CORRIGENDUM

FAMILIES OF D-MINIMAL MODELS AND APPLICATIONS TO 3-FOLD DIVISORIAL CONTRACTIONS

(*Proc. London Math. Soc.* (3) 90 (2005) 345–370)

NIKOLAOS TZIOLAS

A gap in the proofs of Lemma 2.3 and Theorem 2.4 in [**3**] is corrected. Lemma 2.3 is easy to fix but since it is part of a much stronger result proved by Kawamata [**1**], we refer to this statement instead. I also want to mention that Theorem 2.4 is independent of the rest of the material discussed in [**3**].

THEOREM 2.3 [**1**]. *Let $f\colon X \longrightarrow T$ be a flat morphism from a germ of an algebraic variety to a germ of a smooth curve. Assume that the central fiber $X_0 = f^{-1}(P)$ has only canonical singularities. Then so has the total space X as well as any fiber X_t of f. Moreover, the pair (X, X_0) is canonical.*

It is also known [**2**] that if X_0 is terminal, then so are X and X_t. An immediate consequence of the above theorem is that if $g\colon W \longrightarrow X$ is a resolution of X, then there is no g-exceptional crepant divisor with center in X_0. Hence all crepant exceptional divisors of X dominate T.

THEOREM 2.4. *Let $X \xrightarrow{\sigma} T$ be a proper family of canonical 3-folds over a smooth curve T. Let D in X be a family of divisors in X over T and let $0 \in T$ be a closed point. Then*

$$e(X_0) \geqslant e(X_t)$$

for t in a small neighborhood of $0 \in T$. Moreover:

(i) *if $e(X_0) = e(X_t)$ for all t, then, after a finite base change, there is a morphism $Y \xrightarrow{g} X$, with Y $\mathbb{Q}$-factorial and terminal, such that $Y_t \xrightarrow{g_t} X_t$ is a terminalization of X_t, for all $t \in T$;*

(ii) *if $e(X_0; D_0) = e(X_t; D_t)$ for all t, then there is a morphism $Y \xrightarrow{g} X$ such that $Y_t \xrightarrow{g_t} X_t$ is the D_t-minimal model of X_t.*

Proof. The claimed inequality between the number of crepant divisors of the central and general fibers is independent of base change and so we will perform one when necessary. First I claim that up to a finite base change we can assume that $e(X_t) = e(X)$, for general $t \in T$. Indeed, let $g\colon W \longrightarrow X$ be a log resolution of X such that all the g-crepant divisors are smooth. Let E be a g-crepant divisor. From the previous theorem, it follows that the center of E is not contained in X_0 and, since there are finitely many crepant divisors, after removing finitely many points

Received 27 April 2005.
2000 *Mathematics Subject Classification* 14E30 (primary), 14E35 (secondary).

https://doi.org/10.1017/S0024611506015759 Published online by Cambridge University Press

from T we can assume that all the g-crepant divisors dominate T. By generic smoothness, E_t and W_t are smooth, for general t, and hence $g_t\colon W_t \longrightarrow X_t$ is a resolution of X_t. Then by adjunction it follows that E_t is g_t-crepant. The problem here is that E_t may have more than one connected component and so in general we only get the fact that $e(X_t) \geqslant e(X)$. However, I claim that after a finite base change, E_t is irreducible and smooth, for all irreducible crepant exceptional divisors E of X, and therefore $e(X_t) = e(X)$. So, let E be an irreducible g-crepant divisor of X and let $E \xrightarrow{h} D \xrightarrow{\tau} T$ be the Stein factorization of $f = \tau \circ g\colon E \longrightarrow T$, where $D = \operatorname{Spec}(f_*\mathcal{O}_E)$. Then h has connected fibers and τ is finite. We now make a base change with $D \longrightarrow T$. So, let $X_D = X \times_T D$, $W_D = W \times_T D$ and $E_D = E \times_T D$. Note that by the previous theorem, X_D is also canonical since all the fibers $X_{D,d}$ are canonical. Also $W_{D,d}$ is smooth for general $d \in D$. Moreover, by the universal property of fiber products, we see that there is an embedding $E \subset E_d$, and by construction, $E \longrightarrow D$ has connected fibers. Hence E_d is irreducible, smooth and crepant for $X_{D,d} = X_t$, for general $t = \tau(d)$. Since D may not be normal, make another base change with its normalization $\overline{D}$. Repeat this process for any crepant divisor F of $W_{\overline{D}}$ such that F_d is not irreducible. Since the crepant divisors are at most $e(X_t)$, this process ends with a family $X' \longrightarrow T'$, with T' smooth, such that there is a log resolution $f'\colon W' \longrightarrow X'$ such that if E is any f'-exceptional crepant divisor, then E_t is smooth and irreducible and hence $e(X'_t) = e(X')$, for general t.

We may also assume that X is $\mathbb{Q}$-factorial. If this is not the case, then let $f\colon Y \longrightarrow X$ be a $\mathbb{Q}$-factorialization, which exists by the Minimal Model Program (MMP) in dimension 4. Then $e(X) = e(Y)$. Moreover, $Y_t \longrightarrow X_t$ is an isomorphism in codimension 1 for general t and hence $e(Y_t) = e(X_t) = e(X) = e(Y)$. The central fiber contraction $Y_0 \longrightarrow X_0$ may be divisorial, but in any case it is crepant and hence Y_0 is normal and canonical and $e(Y_0) \leqslant e(X_0)$. Hence, in addition to our hypothesis, we may also assume that X is $\mathbb{Q}$-factorial and $e(X_t) = e(X)$.

Let $n = e(X)$ and let $E_1, \dots, E_n$ be the crepant exceptional divisors of X. Then by standard MMP arguments, we may extract them from X with a series of crepant morphisms

$$Y = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{f_1} X \tag{2.1}$$

where $f_i\colon X_i \longrightarrow X_{i-1}$ is crepant and its exceptional set is E_i. Then by Theorem 2.3, the centers of these divisors are not contained in X_0. Hence $X_{i,0}$ is irreducible and $E_i \cap X_{i,0}$ is a divisor. Moreover, I claim that $X_{i,0}$ is normal and canonical. Indeed, inductively it easily follows that $K_{X_i} + X_{i,0} = f_i^*(K_{X_{i-1}} + X_{i-1,0})$ and hence since (X, X_0) is canonical, $(X_i, X_{i,0})$ is canonical as well, and therefore $X_{i,0}$ is normal. Moreover, by adjunction it follows that $K_{X_{i,0}} = f_i^* K_{X_{i-1,0}}$ and hence $X_{i,0}$ is canonical and $E_i \cdot X_{i,0}$ is a crepant divisor for $X_{i-1,0}$ (which may be reducible). Therefore $e(X_0) \geqslant n = e(X_t)$.

Suppose now that $e(X_0) = e(X_t)$. Let $f\colon Y \longrightarrow X$ be the composition of the maps f_i in (2.1) above. Then by its construction, Y is a $\mathbb{Q}$-factorial terminal 4-fold and since $e(Y_t) = 0$ for general t, Y_t is terminal as well for general t. Moreover, by the above discussion, it follows that Y_0 is irreducible and $e(Y_0) = 0$, and hence Y_0 is terminal too. Now $Y \longrightarrow X$ satisfies all the conditions of Theorem 2.4(i).

Now suppose that $D \subset X$ is a family of divisors such that $e(X_0; D_0) = e(X_t; D_t)$, for all t. Let $Z \xrightarrow{g} X$ be the D-minimal model of X, which exists by the MMP in dimension 4. Then $Z_t \longrightarrow X_t$ is an isomorphism in codimension 1 for general t

https://doi.org/10.1017/S0024611506015759 Published online by Cambridge University Press

and therefore $e(Z_t) = e(X_t)$. I now claim that $Z_0 \longrightarrow X_0$ is also an isomorphism in codimension 1 and hence it is also the D_0-minimal model of X_0. Suppose that this is not so. Let $D' = f_*^{-1}D$. Then by the definition of D-minimal models, $-D'$ is g-ample and hence if $Z_0 \longrightarrow X_0$ is divisorial, then the center of any g_0-exceptional divisor is contained in D_0. Therefore, $e(D'_0; Z_0) < e(D_0; X_0) = e(D_t; X_t) = e(D'_t; Z_t)$, which is impossible from the first part of the proof. □

REMARK. The condition $e(X_0) = e(X_t)$ is not sufficient for the existence of a morphism $g\colon Z \longrightarrow X$ such that $g_t\colon Z_t \longrightarrow X_t$ is a $\mathbb{Q}$-factorialization of X_t, for all t, as was mistakenly claimed in [**3**]. The reason is that there may be divisors in X_0 that do not deform with X_0, that is, they do not extend to a divisor in X. For example, let $X = (xy - zu + t = 0) \subset \mathbb{C}^4$. Then $X_0 = (t = 0)$ is the ordinary double point $xy - zu = 0$, and X_t for $t \neq 0$ is smooth. Thus X_0 is not $\mathbb{Q}$-factorial and there is no morphism $g\colon Y \longrightarrow X$ such that $g_0\colon Y_0 \longrightarrow X_0$ is a $\mathbb{Q}$-factorialization of X_0 because if there was such a morphism g, then g would be an isomorphism in codimension 1 which is impossible since X itself is smooth.

References

1. Y. Kawamata, 'Deformations of canonical singularities', *J. Amer. Math. Soc.* 12 (1999) 85–92.

2. N. Nakayama, 'Invariance of plurigenera of algebraic varieties', Preprint 1191, Research Institute for Mathematical Sciences, Kyoto, 1998.

3. N. Tziolas, 'Families of D-minimal models and applications to 3-fold divisorial contractions', *Proc. London Math. Soc.* (3) 90 (2005) 345–370.

Nikolaos Tziolas
Department of Mathematics
University of Cyprus
P.O. Box 20537
1678 Nicosia
Cyprus

tziolas@ucy.ac.cy

https://doi.org/10.1017/S0024611506015759 Published online by Cambridge University Press