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The connectedness of space curve invariants

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Abstract. In this paper we will give necessary conditions for a Borel-fixed monomial ideal to be the generic initial ideal of a reduced, irreducible, non-degenerate curve in $\mathbb{P}^3$.

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Key words: Generic Initial Ideal, Space Curve.

0. Introduction

Let $S = \mathbf{C}[x_1, \ldots, x_{n+1}]$ be the ring of polynomials in $n+1$ variables over **C**, corresponding to the homogeneous coordinate ring of $\mathbf{P}^n$. Let $\succ$ be the reverse lexicographical order on the monomials of S. Given a homogeneous idea l $I \subset S$, we can form the monomial ideal of initial terms of I under $\succ$ and, for a generic choice of coordinates, obtain a *Borel-fixed* monomial ideal; this is called the *generic initial ideal* of I and is denoted gin(I). (For more information about generic initial ideal theory see [B], [BM], [BS] or [Gr].)

The generic initial ideal, although it is a monomial ideal and hence basically a combinatorial object, contains quite a bit of the information about the original ideal. For example, it has the same Hilbert function and the same regularity.

The question we would like to answer is: *Which Borel-fixed monomial ideals can arise from geometry?* Here, we will answer a more limited question and give necessary conditions for a Borel-fixed monomial ideal to be the generic initial ideal of a reduced, irreducible, non-degenerate curve in $\mathbf{P}^3$.

Motivation

The simplest examples of generic initial ideals arising from geometry are those of points in the plane. If we let $\Gamma \subset \mathbf{P}^2$ be a set of d points, the generic initial ideal of Γ has the following form

$$\text{gin}(I_\Gamma) = (x_1^s,\ x_1^{s-1}x_2^{\lambda_{s-1}}, \ldots, x_1x_2^{\lambda_1}, x_2^{\lambda_0}),$$

with $\lambda_i \geqslant \lambda_{i+1} + 1$ for all $i < s-1$ and $\sum_{i=0}^{s-1} \lambda_i = d$.

One might ask, given a generic initial ideal as above, what can be said about the geometry of the points? In particular, what can be said about the generic initial ideal of points in uniform position? The answer to this question is given by the following theorem of Gruson and Peskine.

THEOREM 1. (Gruson, Peskine). *Let* $\Gamma \subset \mathbf{P}^2$ *be a set of points in uniform position, with generic initial ideal,* $\mathrm{gin}(I_\Gamma)$, *as above. Then the* invariants, $\{\lambda_i\}_{i=0}^{s-1}$, *satisfy*

$$\lambda_{i+1} + 2 \geqslant \lambda_i \geqslant \lambda_{i+1} + 1 \quad \text{for all } \; i < s - 1,$$

and we say that the invariants of Γ *are* connected.

(The original theorem of Gruson and Peskine considered the invariants $\eta_i = \lambda_i + i$. Then the theorem shows that $\eta_{i+1} + 1 \geqslant \eta_i \geqslant \eta_{i+1}$ for all $i < s - 1$, or that there are no 'gaps' in the sequence $\{\eta_i\}$.)

It is Theorem 1 which motivated the question answered here for space curves. Associated to a space curve there are families of invariants which generalize the invariants of points in $\mathbf{P}^2$. In fact, one of these families is the set of invariants of a generic hyperplane section of the curve, which is a set of points in uniform position. The main aim of this paper is to show that for a reduced, irreducible, non-degenerate space curve each of these families of invariants satisfy a connectedness property.

The organization of the paper is as follows: In Section 1, we will define the invariants of a space curve and state the 'Connectedness' Theorem. In Section 2, we will prove a general result which puts constraints on generators of a generic initial ideal of high degree and 'split' a non-connected (unsaturated) ideal in $\mathbf{C}[x_1, x_2, x_3]$. In Section 3, we will prove the theorem incorporating ideas used in Green's proof of Gruson and Peskine's Theorem ([Gr]) and a more differential approach due to Strano ([S]). In Section 4, we will put some further conditions on the generic initial ideal of a reduced, irreducible, non-degenerate curve in $\mathbf{P}^3$. In particular, we will generalize results of Strano on the effect of sporadic zeros. In Section 5, we will give some examples of the uses of the main theorem.

1. Statement of the Theorem

1.1. A PICTORIAL DESCRIPTION OF MONOMIAL IDEALS

The inspiration for defining the invariants of a space curve and conjecturing what a generalization of connectedness might be, came from considering the generic initial ideal of points in $\mathbf{P}^2$ and of space curves in pictorial way. Thus before stating the theorem for curves, we will rephrase the statement of the theorem of Gruson and Peskine in this new context, where the generalization we intend to prove will become apparent.

The following pictorial represention of the generic initial ideal of a space curve is due to M. Green.

Let C be a curve in $\mathbf{P}^3$. As I_C is saturated, the generators of $I = \mathrm{gin}(I_C)$ will be of the form $x_1^i x_2^j x_3^k$. (See [B] or [Gr] for information regarding saturated ideals.)

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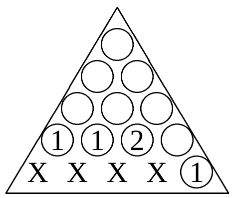
To represent the generic initial ideal of C pictorially, we first draw a triangle such that the (i, j)th position corresponds to the monomial $x_1^i x_2^j$, where $i + j = n$, $0 \leqslant n \leqslant n_0$, and $n_0 \gg 0$. Now let $f(i, j) = \min\{k \mid x_1^i x_2^j x_3^k \in I\}$. For each (i, j), if $f(i, j) = \infty$ (i.e. $x_1^i x_2^j x_3^k \notin I$ for all $k \geq 0$) put a circle in the (i, j) position, if $0 < f(i, j) < \infty$ put an encircled $f(i, j)$ in the (i, j) position, and if $f(i, j) = 0$ put an X in the (i, j) position.

EXAMPLE 1. The Borel-fixed monomial ideal

$$I = (x_1^3 x_3, x_1^2 x_2 x_3, x_1 x_2^2 x_3^2, x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4 x_3, x_2^5)$$

can be represented by the triangle configuration

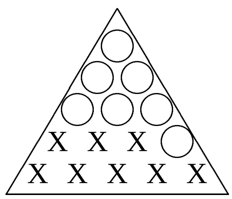


Note. If (i, j) is not in the picture, one may assume that $x_1^i x_2^j \in I$.

One can also represent the generic initial ideal of a set of points in the plane in the same way. For example, the ideal

$$J = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^4)$$

can be represented by the triangle configuration



Notice the following

(1) The number of circles in the ith diagonal of the triangle is λ_i. (In this case 4, 2 and 1.) We will call this number the *length* of the ith diagonal.
(2) Theorem 1 says that the lengths of consecutive diagonals of circles cannot differ by more than 2. (Or that there are no 'big steps'.)
(3) It is a result of Green ([Gr]), that if there were a curve whose generic initial ideal was the one in Example 1, it's generic hyperplane section would have the configuration above.

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We will generalize the idea of the length of a diagonal in note (1) to give a family of invariants. We will then state a ‘Connectedness’ Theorem for these invariants using on the idea of note (2).

1.2. SOME NEW INVARIANTS OF C

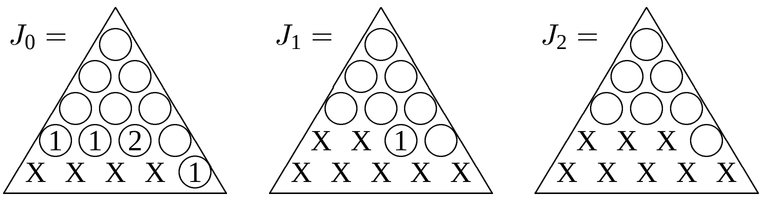
DEFINITION. Let $I = I_C$ be the ideal of C. We define *invariants* $\{\mu_i(k)\}$ of a curve C as follows

Let $J_k = (\mathrm{gin}(I_C)|_{x_4=0}\colon x_3^k)$, then $\mu_i(k)$ is the length of the ith diagonal of circles (including those which contain numbers) in the triangle configuration of J_k.

More formally, let $f(i,j) = \min\{k \mid x_1^i x_2^j x_3^k \in \mathrm{gin}(I)\}$. Let

$$s_k = \min\{i \mid f(i,0) \leqslant k\},$$
$$\mu_i(k) = \min\{j \mid f(i,j) \leqslant k\} \quad \text{for } 0 \leqslant i \leqslant s_k - 1.$$

EXAMPLE 2. The ideal in Example 1 gives the triangle configurations



and invariants

$$\mu_0(0) = 5 \;\; \mu_1(0) = 3 \;\; \mu_2(0) = 2 \;\; \mu_3(0) = 1$$
$$\mu_0(1) = 4 \;\; \mu_1(1) = 3 \;\; \mu_2(1) = 1$$
$$\mu_0(2) = 4 \;\; \mu_1(2) = 2 \;\; \mu_2(2) = 1$$

with $\mu_i(k) = \mu_i(2)$ for $k \geqslant 2$.

Note that due to the work of Green ([Gr] Proposition 2.21), for $k \gg 0$, we have $J_k = \mathrm{gin}(I_\Gamma)$, where Γ is a generic hyperplane section of C. Thus these invariants generalize the invariants of a generic hyperplane section of C

1.3. STATEMENT OF THE THEOREM

THEOREM 2 (The Connectedness of Curve Invariants). *If C is a reduced, irreducible, non-degenerate curve in* $\mathbf{P}^3$. *Then the invariants, $\{\mu_i(k)\}$, of C are such that for each k*

$$\mu_{i+1}(k) + 2 \geqslant \mu_i(k) \geqslant \mu_{i+1}(k) + 1 \quad \textit{for } \; 0 \leqslant i < s_k - 1,$$

and we say that $\{\mu_i(k)\}$ is connected. *Furthermore, if $s_k < s_0$, then $\mu_{s_k-1}(k) \leqslant 2$.*

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Note. The invariants of Example 2 above *are* connected.

2. Splitting a non-connected ideal

In this section we will assume J is a homogeneous (not necessarily saturated) ideal in $\tilde{S} = \mathbf{C}[x_1, x_2, x_3]$. The generators of $\text{gin}(J)$ are still of the form $x_1^a x_2^b x_3^c$ and so we may define the invariants $\mu_i(j)$. We will assume the invariants of J are such that

$$\mu_{i+1}(0) + 2 < \mu_i(0)$$

for some $0 \leqslant i < s_0 - 1$.

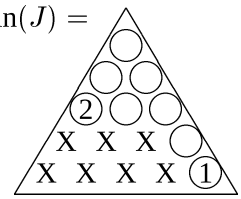
We will show that there exists a polynomial X of degree $i + 1$ such that, after a generic choice of coordinates

$$\text{in}(X) \cap \text{in}(J) = \text{in}(X \cap J).$$

This means that if $x^M \in \text{in}(J)$ is such that $x_1^{i+1} \mid x^M$ then $x^M = \text{in}(f)$ for some $f = Xh \in J$. In terms of the pictorial representation of $\text{gin}(J)$ this means that every monomial of $\text{in}(J)$ corresponding to point in the triangle to the left of the ith diagonal corresponds to an element of J divisible by some polynomial X. So the pictorial representation may be 'split' along the ith diagonal. We will call this construction the *splitting* of the ideal J.

EXAMPLE 3. The triangle configuration below corresponds to a monomial ideal with disconnected invariants $\mu_0(0) = 5$, $\mu_1(0) = 2$.

$\text{gin}(J) =$



Notice the triangle on the right has a big 'step' between the Oth and 1st diagonals. We would like to write elements in J corresponding to elements to the left of the step as a multiple of a some polynomial.

2.1. GENERATORS OF $\text{gin}(I)$ IN HIGH DEGREE

First we need to prove a general result which we will need later. Let I be a homogeneous ideal in $S = \mathbf{C}[x_1, \ldots, x_{n+1}]$, we will put conditions on the generators of a generic initial ideal, $\text{gin}(I)$, whose degree is larger than that of the generators of I.

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DEFINITION. An *elementary move* e_k for $1 \leqslant k \leqslant n$ is defined by $e_k(x^J) = x^{\hat{J}}$, where $\hat{J} = (j_1, \dots j_{k-1}, j_k + 1, j_{k+1} - 1, j_{k+2}, \dots, j_{n+1})$ and we adopt the convention that $x^J = 0$ if $j_m < 0$ for some m. (Note, we are using the multi-index notation; $x^J = x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$.)

One can show that a monomial ideal, I, is Borel-fixed if and only if for all $x^J \in I$ and for every elementary move e_k, $e_k(x^J) \in I$. Thus it is Borel-fixedness which gives the right-hand inequality of Theorem 2 and the step-like look in the triangle configuration.

THEOREM 3 (Syzygy Configuration). *Let I be a Borel-fixed monomial ideal with generators $x^{J_1}, \dots, x^{J_N}$. Then the first syzygies of I is generated by*

$$\{x_i \otimes x^{J_j} - x^{L_{ij}} \otimes x^{J_{l_{ij}}} \mid 1 \leqslant j \leqslant N,\ \ 0 < i < \max(J_j),$$
$$\min(L_{ij}) \geqslant \max(J_{l_{ij}})\},$$

where $\max(J) = \max\{i \mid j_i > 0\}$ *and* $\min(J) = \min\{i \mid j_i > 0\}$.

(The $\otimes$ is a place holder.)

This theorem is due to Eliahou and Kervaire ([EK]) and a proof may be found in [Gr] (Theorem 1.31).

THEOREM 4. *Let I be a homogeneous ideal generated in degree $\leqslant r$, with generators $x^{J_1}, \dots, x^{J_N}$ of* $\mathrm{gin}(I)$ *in degree $\leqslant r$. Then any generator P of* $\mathrm{gin}(I)$ *of degree $r+1$ is such that $P \prec x_i x^{J_j}$, for some J_j such that $|J_j| = r$ and* $i < \max(J_j)$.

Proof. $(x^{J_1}, \dots, x^{J_N})$ is a Borel-fixed monomial ideal. So, by Theorem 3, the first syzygies among the x^{J_j} are generated by syzygies of the form

$$\{\, x_k \otimes x^{J_j} - x^{L_{kj}} \otimes x^{J_{l_{kj}}} \mid 1 \leqslant j \leqslant N,\ \ 1 \leqslant k < \max(J_j)\}.$$

We will first use the syzygies of $(x^{J_1}, \dots, x^{J_N})$ to obtain some new generators of $\mathrm{gin}(I)_{r+1}$ which satisfy the condition stated in the theorem. By Galligo's Theorem ([Ga]) we may assume, after a generic change of basis, that $\mathrm{gin}(I) = \mathrm{in}(I)$. Let $g_i \in I$ be monic polynomials such that $\mathrm{in}(g_i) = x^{J_i}$ for $i = 1, \dots, N$. Given a syzygy $x_k \otimes x^{J_j} - x^{L_{kj}} \otimes x^{J_{l_{kj}}}$, let

$$h_1 = x_k g_j - x^{L_{kj}} g_{l_{kj}}.$$

As the leading terms of $x_k g_j$ and $x^{L_{kj}} g_{l_{kj}}$ will cancel, the initial term of h_1, $\mathrm{in}(h_1) \prec x_k x^{J_j}$.

Given h_i, if $\mathrm{in}(h_i) = x^{K_{i+1}} x^{J_{j_{i+1}}}$, let

$$h_{i+1} = h_i - a_{i+1} x^{K_{i+1}} g_{j_{i+1}},$$

where a_{i+1} is the leading coefficient of h_i. Then $\mathrm{in}(h_{i+1}) \prec \mathrm{in}(h_i) \prec x_k x^{J_j}$.

This process must terminate, so for i sufficiently large either x^{J_k} does not divide $\mathrm{in}(h_i)$ for any $1 \leq k \leq N$, in which case $\mathrm{in}(h_i)$ is a new generator of $\mathrm{gin}(I)$ satisfying the conditions stated in the theorem or $h_i = 0$. (In particular, the latter will occur if $\deg(h_i) \leqslant r$.)

Now let $P = \mathrm{in}(h)$ be a generator of $\mathrm{in}(I)$ of degree $r+1$. As I is generated in degree $\leqslant r$, $\{g_i\}_{i=1}^N$ generates I and we may write $h = \sum f_i g_i$.

Let i_0 be such that

(i) $\mathrm{in}(f_{i_0} g_{i_0})$ is maximal,
(ii) if $\mathrm{in}(f_{i_0} g_{i_0}) = \mathrm{in}(f_i g_i)$ then g_{i_0} has maximal degree,
(iii) if $\mathrm{in}(f_{i_0} g_{i_0}) = \mathrm{in}(f_i g_i)$ and $\deg(g_{i_0}) = \deg(g_i)$ then $\mathrm{in}(g_{i_0})$ is minimal.

As $P \neq \mathrm{in}(f_{i_0} g_{i_0})$, there exists i_1 such that $\mathrm{in}(f_{i_1} g_{i_1}) = \mathrm{in}(f_{i_0} g_{i_0})$. We have picked i_0 in such a way that either $\deg(g_{i_0}) > \deg(g_{i_1})$, or $\deg(g_{i_0}) = \deg(g_{i_1})$ and $\mathrm{in}(g_{i_0}) \prec \mathrm{in}(g_{i_1})$.

We want to show that $x_i \mid \mathrm{in}(f_{i_0})$ for some $i < \max(J_{i_0})$.

Let $\mathrm{in}(g_{i_0}) = x^A$, $\mathrm{in}(g_{i_1}) = x^B$. x^A and x^B are generating monomials of a Borel-fixed monomial ideal and $x^M x^A = x^N x^B$ for some monomials x^M and x^N.

Case 1. $\deg(A) > \deg(B)$.

Let $A = (a_1, \dots, a_s, 0, \dots, 0)$ with $s = \max(A)$ and $B = (b_1, \dots, b_{n+1})$. Suppose $b_i \leq a_i$ for all $i \leq s-1$, then we may apply elementary moves to B to get $\hat{B}$ such that $x^{\hat{B}} \in \mathrm{in}(I)$ with $\hat{b}_i \leqslant a_i$ for all I. As $\deg(\hat{B}) = \deg(B) < \deg(A)$, this would imply x^A is not a generator. Therefore there exists $b_i > a_i$ for some $i \leqslant s-1$.

Case 2. $\deg(A) = \deg(B)$ and $x^A \prec x^B$.

Let $A = (a_1, a_2, \dots, a_{n+1})$ and $B = (b_1, b_2, \dots, b_{n+1})$. Then there exists s such that $a_k = b_k$ for all $k > s$ and $a_s > b_s$. As the degrees are the same, there must exist $a_i < b_i$ for some $i < s$.

In either case there exists an i such that $x_i \mid \mathrm{in}(f_{i_0}) = x^M$ for some $i < \max(J_{i_0})$.

Consider the syzygy

$$x_i \otimes x^{J_{i_0}} - x^{L_{ii_0}} \otimes x^{J_{l_{ii_0}}}.$$

Let h^* be the element of I constructed formally, as in the first part of the proof, from this syzygy.

$$h^* = x_i g_{i_0} - x^{L_{ii_0}} g_{l_{ii_0}} - \sum a_i x^{K_i} g_i,$$

where $\mathrm{in}(x^{K_i} g_i) \prec \mathrm{in}(x_i g_{i_0})$. Let $P^* = \mathrm{in}(h^*) \prec x_i x^{J_{i_0}}$.

Notice that $h^* = 0$ if $\deg(g_{i_0}) < r$.

Let

$$h^1 = h - e_{i_0} h^*,$$

where $e_{i_0} = a_{i_0}\mathrm{in}(f_{i_0})/x_i$ and a_{i_0} is the leading coefficient of f_{i_0}. Let $P^1 = \mathrm{in}(h^1)$.
Then as above we can find i_0^1 for h^1 and we have either

(i) $\mathrm{in}(f_{i_0^1} g_{i_0^1}) \prec \mathrm{in}(f_{i_0} g_{i_0})$,
(ii) $\mathrm{in}(f_{i_0^1} g_{i_0^1}) = \mathrm{in}(f_{i_0} g_{i_0})$ and $\deg(g_{i_0^1}) > \deg(g_{i_0})$,
(iii) $\mathrm{in}(f_{i_0^1} g_{i_0^1}) = \mathrm{in}(f_{i_0} g_{i_0})$ and $\deg(g_{i_0^1}) = \deg(g_{i_0})$ and $\mathrm{in}(g_{i_0^1}) \prec \mathrm{in}(g_{i_0})$.

Thus we have two cases, either $P \preceq P^* \prec x_i x^{J_{i_0}}$ and we are done, or $P \succ P^*$ in which case $P = P^1$ and we may proceed by induction.

COROLLARY 5. *If I is an ideal generated in degree $\leqslant r$ and $\mathrm{gin}(I)$ has no generators in degree $r+1$, then $\mathrm{gin}(I)$ is generated in degree $\leqslant r$.*

Proof. If there are no generators in degree $r+1$, then by the construction of generators of $\mathrm{gin}(I)$ of degree $r+2$ in Theorem 4 there can be no generators in degree $r+2$. Continuing one one can show there are no generators in degree $\geq r+i$ for all $i \geqslant 1$.

2.2. SPLITTING A NON-CONNECTED IDEAL

Let J is a homogeneous ideal in $\tilde{S} = \mathbf{C}[x_1, x_2, x_3]$ with invariants $\mu_{i+1}(0) + 2 < \mu_i(0)$ for some $0 \leq i < s_0 - 1$. Let K be the ideal generated by elements of degree $\leq i + \mu_{i+1}(0) + 2$ in J. We want to show that there exists an ideal $K \subseteq \hat{K} \subseteq J$ such that $\mathrm{gin}(\hat{K}) = (x_1^{i+1}) \cap \mathrm{gin}(J)$.

LEMMA 6. *All elements of* $\mathrm{gin}(K)$ *are divisible by* x_1^{i+1}.

Proof. Let $x_1^a x_2^b x_3^c \in \mathrm{gin}(K)_d$ for $d \leqslant i + \mu_{i+1}(0) + 2$. If $a \leq i$, then by Borel-fixedness $x_1^i x_2^{\mu_{i+1}(0)+2} \in \mathrm{gin}(K) \subset \mathrm{gin}(J)$, but $x_1^i x_2^{\mu_i(0)}$ is a generator of $\mathrm{gin}(J)$ and so $\mu_{i+1}(0) + 2 \geq \mu_i(0)$. But $\mu_i(0) > \mu_{i+1}(0) + 2$, hence $a > i$.

Suppose all elements of $\mathrm{gin}(K)_d$ are divisible by x_1^{i+1} for some $d \geqslant i + \mu_{i+1}(0) + 2$.

CLAIM. If $d \geqslant i + \mu_{i+1}(0) + 2$, then any generator of $\mathrm{gin}(K)_d$ has an x_3 term.

Proof of Claim. Let $x_1^a x_2^b x_3^c$ be a generator of $\mathrm{gin}(K)_d$ and suppose $c = 0$. By assumption $a \geqslant i+1$. Let $a = i+1+j$. Then $x_1^a x_2^b x_3^c = x_1^{i+1+j} x_2^{\mu_{i+1+j}(0)} \in \mathrm{gin}(J)$. $x_1^{i+1} x_2^{\mu_{i+1}(0)}$ is also in $\mathrm{gin}(J)$ and so by Borel-fixedness $d = i+1+j+\mu_{i+1+j}(0) \leqslant i+1+\mu_{i+1}(0) < d$, but this is a contradiction. Therefore $c > 0$.

Now let $P = x_1^a x_2^b x_3^c$ be a generator of $\mathrm{gin}(K)_{d+1}$. If $a \leqslant i$, then by Borel-fixedness $x_1^i x_2^{d+1-i} \in \mathrm{gin}(K)_{d+1}$, and as x_1^{i+1} divides all elements of degree $\leqslant d$, $x_1^i x_2^{d+1-i}$ is a generator of $\mathrm{gin}(K)_{d+1}$, and hence by Theorem 4, $x_1^i x_2^{d+1-i} \prec x_k x^M$ for x^M some generator of $\mathrm{gin}(K)_d$ and $k < \max(M)$. However, by the claim x^M

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has an x_3 term, which implies $x_1^i x_2^{d+1-i} \succ x_k x^M$; a contradiction. Hence by induction every generator of $\mathrm{gin}(K)$ is divisible by x_1^{i+1}.

LEMMA 7. *Let K be an ideal in $\mathbf{C}[x_1, x_2, x_3]$ such that $\mathrm{gin}(K) \subseteq (x_1^k)$, with $k \geqslant 1$ and k maximal. Then, after a possible change of basis, there exists a homogeneous polynomial X such that $\mathrm{in}(X) = x_1^k$ and $K = XK'$.*

Proof. $K \subseteq K^{\mathrm{sat}}$, therefore $\mathrm{gin}(K) \subseteq \mathrm{gin}(K^{\mathrm{sat}})$ and $\mathrm{gin}(K^{\mathrm{sat}})$ is generated by

$$\{x_1^a x_2^b \mid x_1^a x_2^b x_3^c \in \mathrm{gin}(K) \ \text{ for some } \ c \geqslant 0\}.$$

Therefore $\mathrm{gin}(K^{\mathrm{sat}}) \subseteq (x_1^k)$. We may assume K is saturated.

Let $V = V(K)$ be the variety in $\mathbf{P}^2$ associated to K. Considering the Hilbert function of V and the fact that $\mathrm{gin}(K) \subseteq (x_1^k)$ we find that V contains a plane curve $Z = \{F = 0\}$. Hence every element of K is a multiple of F.

Let $K = FK_1$ then either $K_1 \not\subset (x_1)$ in which case we are done with $F = X$ and $K_1 = K'$, or $K_1 \subset (x_1^{k_1})$ with $k_1 < k$ and we may proceed by induction on k.

(Note that after a change of basis, we may assume $\mathrm{gin}(K) = \mathrm{in}(K)$ which would automatically imply that $\mathrm{in}(X) = x_1^k$.)

Let K be an ideal contained in J, maximal with respect to the properties

(1) $(K)_d = (J)_d$ for $d \leqslant i + \mu_{i+1}(0) + 2$.
(2) $K = XK'$ with $\deg(X) = i + 1$.

By maximality $K = X \cap J$. We would like to show that every monomial in $\mathrm{gin}(J)$ divisible by x_1^{i+1} can arise from an polynomial in K.

LEMMA 8. $\mathrm{gin}(X) \cap \mathrm{gin}(J) = \mathrm{gin}(X \cap J)$

Proof. As we may make a generic choice of coordinates, by Galligo's Theorem ([Ga]) it is sufficient to prove that

$$\mathrm{in}(X) \cap \mathrm{in}(J) \subseteq \mathrm{in}(X \cap J),$$

where $\mathrm{in}(X) = x_1^{i+1}$.

Let $M = x_1^a x_2^b x_3^c \in \mathrm{in}(X) \cap \mathrm{in}(J)$, then $a \geqslant i + 1$, and we may write

$$M = x_1^{a-\alpha} x_2^{b-\beta} x_3^{c-\gamma} (x_1^\alpha x_2^\beta x_3^\gamma),$$

where $A = x_1^\alpha x_2^\beta x_3^\gamma$ is a generator of $\mathrm{in}(J)$.

If $\deg A \leqslant i + \mu_{i+1}(0) + 2$, then $A \in \mathrm{gin}(X \cap J)$ and hence $M \in \mathrm{gin}(X \cap J)$.

Suppose $\deg A > i + \mu_{i+1}(0) + 2$, and α is maximal.

If $\alpha < a$ and β or $\gamma \geqslant 1$, then either $x_1^{\alpha+1} x_2^{\beta-1} x_3^\gamma$ or $x_1^{\alpha+1} x_2^\beta x_3^{\gamma-1} \in \mathrm{gin}(J)$. Then there would exist $B = x_1^{\alpha+1} x_2^{\beta'} x_3^{\gamma'}$, a generator of $\mathrm{gin}(J)$, such that $B \mid M$. Then either $\deg(B) \leqslant i + \mu_{i+1}(0) + 2$ in which case $M \in \mathrm{gin}(X \cap J)$ for degree reasons as above or we contradict the maximality of α.

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If $\alpha < a$ and $\beta = \gamma = 0$, then x_1^α is a generator of $\mathrm{in}(J)$. However $x_1^{i+1}x_2^{\mu_{i+1}(0)} \in \mathrm{in}(J)$ and hence by Borel-fixedness we must have $\alpha \leqslant i + \mu_{i+1}(0) + 2$, which again is a contradiction. Therefore we may assume $\alpha = a \geqslant i + 1$.

If $\gamma \leq 1$, then either $x_1^\alpha x_2^\beta$ or $x_1^\alpha x_2^{\beta+1}$ is a generator of $\mathrm{in}(J)$. As $x_1^{i+1}x_2^{\mu_{i+1}(0)} \in \mathrm{in}(J)$ and $\mathrm{in}(J)$ is Borel-fixed, $x_1^\alpha x_2^{\mu_{i+1}(0)+(i+1)-\alpha} \in \mathrm{in}(J)$. In either case we have $\deg(A) \leqslant i + \mu_{i+1}(0) + 1$. Which again is a contradiction. Thus, we may assume $\gamma \geqslant 2$.

So we are reduced to the situation

$$\begin{aligned}&\deg(A) = \alpha + \beta + \gamma > i + \mu_{i+1}(0) + 2,\\ &\alpha \geqslant i + 1,\\ &\gamma \geqslant 2.\end{aligned}$$

Suppose $A \notin \mathrm{in}(X \cap J) = \mathrm{in}(K)$. Pick A satisfying the conditions above of minimal degree m, and among those of minimal degree, let A be maximal. Pick $f \in J$ with $\mathrm{in}(f) = A$. Let $L = (K, f)$ be the ideal generated by K and f. Then $\mathrm{in}(L)_d = \mathrm{in}(K)_d$ for $d < m$ and $\mathrm{in}(L)_m = \mathrm{in}(K)_m + A$. As A is maximal, the generators of $\mathrm{in}(L)$ in degree $\leqslant m$ form a Borel-fixed monomial ideal and as in the claim in Lemma 6 every generator of degree m has an x_3 term and all elements of $\mathrm{in}(L)$ must be divisible by x_1^{i+1}. This however contradicts the maximality of K. Therefore $A \in \mathrm{in}(X \cap J)$ and $\mathrm{in}(X) \cap \mathrm{in}(J) \subset \mathrm{in}(X \cap J)$.

If I_C is the homogeneous ideal of a space curve which is disconnected, we will use Lemma 8 to give invariants of a hyperplane section of C. This will be used in the final part of the proof of Theorem 2.

3. Proof of the connectedness theorem

We will prove the theorem in two steps. In Section 3.1, we will use the results of Section 2 to show that if I is an ideal in $S = \mathbf{C}[x_1, \ldots, x_4]$ with invariants $\{\mu_i(j)\}$ such that for some i and j, with $0 \leqslant i < s_j - 1$, $\mu_{i+1}(j) + 2 < \mu_i(j)$. Then, for a general linear form h, the ideal $J = (I|_h : x_3^j)$ is an ideal in $\tilde{S} = \mathbf{C}[x_1, x_2, x_3]$ with invariants $\{\nu_i(j)\}$ such that $\nu_{i+1}(0) + 2 < \nu_i(0)$. Hence there exists a polynomial X of degree $i+1$ such that, after a general choice of coordinates, $\mathrm{in}(X) \cap \mathrm{in}(J) = \mathrm{in}(X \cap J)$. I.e. we can *split* the ideal J. In Section 3.2, we will show, that if I is the ideal of a reduced, irreducible, non-degenerate curve $C \in \mathbf{P}^3$, such an X would give rise to a contradiction.

3.1. SPLITTING AN IDEAL RELATED TO A NON-CONNECTED I

If I is an ideal in $S = \mathbf{C}[x_1, \ldots, x_{n+1}]$, let $I|_{x_{n+1}} \subseteq \tilde{S} = \mathbf{C}[x_1, \ldots, x_n]$ be the ideal generated by

$$\{f \mid f + x_{n+1}f' \in I,\ f \in \mathbf{C}[x_1, \ldots, x_n]\}.$$

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Furthermore, if V be the set of linear forms in S and $h \in V$ is generic, we may assume the x_{n+1} coordinate of h is nonzero and define $I|_h$ to be the ideal in $\mathbf{C}[x_1, \ldots, x_n]$ generated by

$$\{f \mid f + hf' \in I, \;\; f \in \mathbf{C}[x_1, \ldots, x_n]\}.$$

Let $\phi_h \in \mathrm{GL}(V)$ be defined by $\phi_h(x_i) = x_i$ for $i < n+1$, $\phi_h(x_{n+1}) = h$. Note, on one hand as $\phi_h^{-1}|_{\tilde{S}}$ is the identity operator and so $\phi_h^{-1}(I|_h) = I|_h$, on the other hand one can show that $\phi_h^{-1}(I|_h) = \phi_h^{-1}(I)|_{x_{n+1}}$.

PROPOSITION 9. *Let I be a homogeneous ideal in S. Then for a general choice of coordinates and a generic choice of $h \in V$*

$$\mathrm{in}(I|_h : x_n^k) = (\mathrm{gin}(I)|_{x_{n+1}} : x_n^k).$$

Proof. Without loss of generality we may assume $\mathrm{gin}(I) = \mathrm{in}(I)$.

CLAIM 1.

$$\mathrm{in}(I)|_{x_{n+1}} = \mathrm{in}(I|_{x_{n+1}})$$

Proof. Let $g \in \mathrm{in}(I)|_{x_{n+1}}$, then $g + x_{n+1}h \in \mathrm{in}(I)$ and hence $g = \mathrm{in}(f)$ for some $f \in I$. Then $f|_{x_{n+1}=0} \in I|_{x_{n+1}}$, and $\mathrm{in}(f|_{x_{n+1}=0}) = \mathrm{in}(f) = g \in \mathrm{in}(I|_{x_{n+1}})$.

Conversely, if $g \in \mathrm{in}(I|_{x_{n+1}})$, then $g = \mathrm{in}(f)$ for some f such that $f + x_{n+1}f' \in I$. Then $\mathrm{in}(f + x_{n+1}f') = \mathrm{in}(f) = g \in \mathrm{in}(I)$ and so $g \in \mathrm{in}(I)|_{x_{n+1}=0}$.

CLAIM 2. For any ideal J in $\tilde{S}$,

$$(\mathrm{in}(J) : x_n^k) = \mathrm{in}(J : x_n^k).$$

Proof. Let $g \in (\mathrm{in}(J) : x_n^k)$ then $x_n^k g = \mathrm{in}(f)$ for some $f \in J$. As we are using the reverse lexicographical ordering, $x_n^k \mid f$ and $f = x_n^k h$ for some $h \in (J : x_n^k)$. Then $g = \mathrm{in}(h) \in \mathrm{in}(J : x_n^k)$.

Conversely if $g \in \mathrm{in}(J : x_n^k)$. Then $g = \mathrm{in}(f)$ where $x_n^k f \in J$. $\mathrm{in}(x_n^k f) = x_n^k g \in \mathrm{in}(J)$ and so $g \in (\mathrm{in}(J) : x_n^k)$.

Putting the two claims together we have, for a general choice of coordinates,

$$(\mathrm{gin}(I)|_{x_{n+1}} : x_n^k) = \mathrm{in}(I|_{x_{n+1}} : x_n^k).$$

Now, as h is generic we may assume the x_{n+1} coordinate of h is nonzero. Let $\phi_h \in \mathrm{GL}(V)$ be defined as above, then $\mathrm{in}(\phi_h^{-1}(I)|_{x_{n+1}} : x_n^k) = \mathrm{in}(I|_h : x_n^k)$ and this monomial ideal is constant for a generic choice of h. As we may choose a general choice of coordinates, we may assume x_{n+1} is generic and hence for generic h, $\mathrm{in}(I|_{x_{n+1}} : x_n^k) = \mathrm{in}(I|_h : x_n^k)$.

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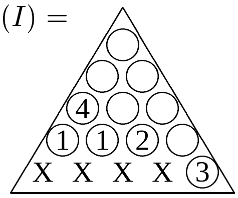
Let I be an ideal in $S = \mathbf{C}[x_1, \dots, x_4]$ with disconnected invariants $\{\mu_i(j)\}$. There exist i and j, with $0 \leqslant i < s_j - 1$ and $\mu_{i+1}(j) + 2 < \mu_i(j)$. Then for a generic linear form h, the ideal $J = (I|_h : x_3^j)$ is an ideal in $\tilde{S} = \mathbf{C}[x_1, x_2, x_3]$ such that $\mathrm{in}(J) = (\mathrm{gin}(I)|_{x_4} : x_3^j)$ and J has invariants $\nu_i(k) = \mu_i(k+j)$, and in particular

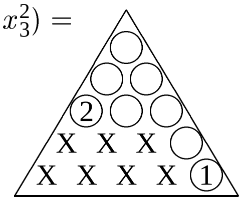
$$\nu_{i+1}(0) + 2 < \nu_i(0).$$

Lemma 8 implies that there exists a homogeneous polynomial $X \in \tilde{S}$ such that after a general choice of coordinates

$$\mathrm{in}(X \cap J) = \mathrm{in}(X) \cap \mathrm{in}(J).$$

EXAMPLE 4. If I is an ideal giving rise to the triangle configuration on the left with disconnected invariants $\mu_0(2) = 5$, $\mu_1(2) = 2$. Then $J = (I|_h : x_3^2)$ will give the configuration on the right

$\mathrm{gin}(I) =$ 

$\mathrm{gin}(J) = (\mathrm{gin}(I)|_{x_4} : x_3^2) =$ 

Lemma 8 allows us to find four elements $f_1, f_2, f_3, f_4 \in J$ such that after a general change of basis $\mathrm{in}(f_1) = x_1^3$, $\mathrm{in}(f_2) = x_1^2 x_2$, $\mathrm{in}(f_3) = x_1 x_2^2$ and $\mathrm{in}(f_4) = x_1^2 x_3$ and there exists an $X \in \tilde{S}$ such that $f_i = X g_i$ for $i = 1, 2, 3, 4$.

3.2. THE FINAL STEP OF THE PROOF

So far, we have shown that for a generic linear form h, the ideal $J_h = (I|_h : x_3^j)$ is such that there exists a polynomial X_h of degree $i + 1$ and an ideal $K_h \subseteq J_h$ such that $K_h = X_h K_h' \subset J_h$ and $\mathrm{gin}(J_h) \cap (x_1^{i+1}) = \mathrm{gin}(K_h)$. As $\tilde{S} \subset S$, we may view $\{X_h\}$ as a family of polynomials in $S = \mathbf{C}[x_1, \dots, x_4]$. We will would like to show that this family $\{X_h\}$ is, in some sense, independent of h. We will first prove a more general result.

Fix coordinates $x_1, \dots, x_{n+1}$ of $\mathbf{P}^n$, (we will need $n \geqslant 3$) let $t_1, \dots, t_{n+1}$ be the dual coordinates of $\mathbf{P}^{n*}$. Let $\{X_h\}$ be a family of homogeneous polynomials in the polynomial ring $S = \mathbf{C}[x_1, \dots, x_{n+1}]$, parametrized by generic hyperplanes $H = \{\sum t_i x_i = h = 0\}$.

The family $\{X_h\}$ corresponds to a function F which is a homogeneous polynomial in the $\{x_i\}$ with coefficients which are rational functions in the $\{t_i\}$. The field of rational functions in $\{t_i\}$ has derivations $\partial/\partial t_i$ and we may extend these derivations to act on the family $\{X_h\}$.

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PROPOSITION 10. *Suppose* $\{X_h\}$ *is a family of homogeneous polynomials varying with* $h = \sum t_i x_i$ *such that*

$$x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j} \in (X_h) \operatorname{mod}(h).$$

Then, for a generic h*, we may write* $X_h = \alpha X + hY_h$*, where* α *is a rational function in the* $\{t_i\}$ *and* $\{Y_h\}$ *is a family of homogeneous polynomials varying with* h*. (In this case we will say that, for generic* h*,* X_h *is* projectively constant *up to a multiple of* h*.)*

Proof. (Green). Let

$$Y = X_h|_h. \left(x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j} \right)\bigg|_h \in (Y),$$

so

$$\left(x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j} \right)\bigg|_h = l_{ij} Y.$$

$$x_k \left(x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j} \right) - x_j \left(x_k \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_k} \right)$$

$$+ x_i \left(x_k \frac{\partial X_h}{\partial t_j} - x_j \frac{\partial X_h}{\partial t_k} \right) = 0,$$

and hence $(x_k l_{ij} - x_j l_{ik} + x_i l_{jk})Y = 0$. Assuming $Y \neq 0$ generically (otherwise we are done), we have $x_k l_{ij} - x_j l_{ik} + x_i l_{jk} = 0$, and hence

$$l_{ij} = \alpha_i x_j - \alpha_j x_i, \qquad l_{ik} = \alpha_i x_k - \alpha_k x_i, \qquad l_{jk} = \alpha_j x_k - \alpha_k x_j,$$

up to a multiple of h, for some α_i, α_j and α_k. (Note that as $n \geqslant 3$ and h is generic, for distinct i, j and k, the linear forms x_i, x_j, x_k and h form a regular sequence.)

Therefore

$$\left(x_j \left(\frac{\partial X_h}{\partial t_i} - \alpha_i X_h \right) - x_i \left(\frac{\partial X_h}{\partial t_j} - \alpha_j X_h \right) \right)\bigg|_h = 0,$$

and

$$\frac{\partial X_h}{\partial t_i} - \alpha_i X_h = x_i U \operatorname{mod}(h), \qquad \frac{\partial X_h}{\partial t_j} - \alpha_j X_h = x_j U \operatorname{mod}(h),$$

similarly

$$\frac{\partial X_h}{\partial t_k} - \alpha_k X_h = x_k U \operatorname{mod}(h).$$

Therefore $(\partial X_h/\partial t_i) - x_i U = \alpha_i X_h \bmod(h)$. We may change X_h by adding a multiple of h without changing the hypothesis or claims of the Proposition. Letting $X'_h = X_h - hU$ we get

$$\begin{aligned}\frac{\partial X'_h}{\partial t_i} &= \frac{\partial X_h}{\partial t_i} - x_i U - h\frac{\partial U}{\partial t_i}\\ &= \alpha_i X_h - h\frac{\partial U}{\partial t_i} \bmod(h) = \alpha_i X'_h \bmod(h).\end{aligned}$$

Therefore we may assume that X_h is such that

$$\frac{\partial X_h}{\partial t_i} = \alpha_i X_h + hU_i,$$

where α_i is a function of $\{t_j\}$.

Differentiating twice we get

$$\begin{aligned}\frac{\partial^2 X_h}{\partial t_j \partial t_i} &= \frac{\partial \alpha_i}{\partial t_j} X_h + \alpha_i \frac{\partial X_h}{\partial t_j} + x_j U_i + h\frac{\partial U_i}{\partial t_j}\\ &= \frac{\partial \alpha_i}{\partial t_j} X_h + \alpha_i(\alpha_j X_h + hU_j) + x_j U_i + h\frac{\partial U_i}{\partial t_j}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 X_h}{\partial t_i \partial t_j} &= \frac{\partial \alpha_j}{\partial t_i} X_h + \alpha_j \frac{\partial X_h}{\partial t_i} + x_i U_j + h\frac{\partial U_j}{\partial t_i}\\ &= \frac{\partial \alpha_j}{\partial t_i} X_h + \alpha_j(\alpha_i X_h + hU_i) + x_i U_j + h\frac{\partial U_j}{\partial t_i}.\end{aligned}$$

Thus

$$x_j U_i - x_i U_j = \left(\frac{\partial \alpha_j}{\partial t_i} - \frac{\partial \alpha_i}{\partial t_j}\right) X_h + h\left(\alpha_j U_i - \alpha_i U_j + \frac{\partial U_j}{\partial t_i} - \frac{\partial U_i}{\partial t_j}\right)$$

and

$$\begin{aligned}&\left(x_k\left(\frac{\partial \alpha_j}{\partial t_i} - \frac{\partial \alpha_i}{\partial t_j}\right) - x_j\left(\frac{\partial \alpha_k}{\partial t_i} - \frac{\partial \alpha_i}{\partial t_k}\right)\right.\\ &\quad\left. + x_i\left(\frac{\partial \alpha_k}{\partial t_j} - \frac{\partial \alpha_j}{\partial t_k}\right)\right) X_h|_h = 0.\end{aligned}$$

Therefore $\partial \alpha_j/\partial t_i = \partial \alpha_i/\partial t_j$ for all i, j.

Assume, for the purposes of induction, that

$$\frac{\partial X_h}{\partial t_i} = \alpha_i X_h + h^k U_i$$

for some $k \geqslant 1$ and $\partial\alpha_j/\partial t_i = \partial\alpha_i/\partial t_j$ for all i, j. Let M be a multi-index such that $|M| = k$, then

$$\frac{\partial^{k+1} X_h}{\partial t^M \partial t_i} = \beta_{M+v_i} X_h + k! x^M U_i \operatorname{mod}(h),$$

where β_{M+v_i} is a sum of products of differentials of the $\{\alpha_i\}$ depending only on the index $M + v_i$. If $M + v_i = M' + v_j$, then $\partial^{k+1} X_h/\partial t^M \partial t_i = \partial^{k+1} X_h/\partial t^{M'} \partial t_j$ and so $(x^{M'} U_j - x^M U_i)|_h = 0$. Now, $x^{M'} = x_i x^N$ and $x^M = x_j x^N$ and so $(x_i U_j - x_j U_i)|_h = 0$, therefore $U_i = x_i V + h V_i$ and $\partial X_h/\partial t_i = \alpha_i X_h + h^k(x_i V + h V_i)$. Let $X_h' = X_h - (1/(k+1)) h^{k+1} V$, then

$$\begin{aligned}
\frac{\partial X_h'}{\partial t_i} &= \frac{\partial X_h}{\partial t_i} - h^k x_i V - \frac{1}{k+1} h^{k+1} \frac{\partial V}{\partial t_i} \\
&= \alpha_i X_h + h^{k+1} V_i - \frac{1}{k+1} h^{k+1} \frac{\partial V}{\partial t_i} \\
&= \alpha_i X_h + h^{k+1} W_i.
\end{aligned}$$

Therefore we may assume $\partial X_h/\partial t_i = \alpha_i X_h$.

As we may multiply the family $\{X_h\}$ by rational functions in the $\{t_i\}$ without changing the hypothesis or results, we may assume that the family $\{X_h\}$ corresponds to a bihomogeneous polynomial F of degree (a, b) and $\partial F/\partial t_i = \alpha_i F$. If γ is a bihomogeneous polynomial of degree $(a', 0)$ and $F = \gamma G$ then G will also satisfy $\partial G/\partial t_i = \alpha_i' G$. Therefore we may assume a is minimal. If $a = 0$ we are done. Otherwise $F = \sum f_j M_j$ where M_j is a monomial in the $\{x_i\}$ and f_j is a polynomial in the $\{t_i\}$. $\partial f_j/\partial t_i = \alpha_i f_j$ for all j, α_i is a rational function of degree -1, therefore we may write $\alpha_i = \beta_i/\gamma_i$ and thus $(\partial f_j/\partial t_i)\gamma_i = \beta_i f_j$ for all j and thus $\gamma_i \mid f_j$ for all j. But this contradicts the minimality of a. Therefore X_h is projectively constant up to a multiple of h.

We will know restrict our attention to the family $\{X_h\}$ we obtained at the beginning of this section using the disconnectedness of $\mathrm{gin}(I_C)$. For a generic linear form h, we found that there exists a polynomial X_h of degree $i + 1$ and an ideal $K_h \subseteq J_h = (I|_h : x_3^j)$ such that $(K_h)_d = (J_h)_d$ for $d \leqslant i + 2 + \mu_{i+1}(j)$, $K_h = X_h K_h'$ and $\mathrm{in}(J_h) \cap (x_1^{i+1}) = \mathrm{in}(K_h)$.

COROLLARY 11. *X_h is projectively constant up to a multiple of h.*

Proof. Pick $p_h \in K_h'$ coprime to X_h such that $\deg(p_h X_h) = m \leqslant i+1+\mu_{i+1}(j)$. (As $x_1^{i+1} x_2^{\mu_{i+1}(j)} \in \mathrm{gin}(J_h)_{i+1+\mu_{i+1}(j)} = \mathrm{gin}(K_h)_{i+1+\mu_{i+1}(j)}$, there exists $p_h \in$

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K'_h such that $\mathrm{in}(p_h) = x_2^{\mu_{i+1}(j)}$. As $\mathrm{in}(X_h) = x_1^{i+1}$, p_h and X_h cannot have a common factor.)

Then

$$x_3^j p_h X_h + hA_h \in I \quad \text{for some } \ A_h.$$

Letting $h = \sum t_i x_i$ vary and differentiating with respect to the t_i we get

$$x_3^j \left(\frac{\partial p_h}{\partial t_i} X_h + p_h \frac{\partial X_h}{\partial t_i} \right) + x_i A_h \in I|_{h=0}$$

and so

$$\begin{aligned} &x_k \left(\frac{\partial p_h}{\partial t_i} X_h + p_h \frac{\partial X_h}{\partial t_i} \right) - x_i \left(\frac{\partial p_h}{\partial t_k} X_h + p_h \frac{\partial X_h}{\partial t_k} \right) \in (I|_h : x_3^j)_{m+1} \\ &\quad = (J_h)_{m+1}. \end{aligned}$$

$m+1 \leqslant i+2+\mu_{i+1}(j)$, hence $(J_h)_{m+1} = (K_h)_{m+1} \subseteq (X_h)$. Therefore

$$p_h \left(x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j} \right) \in (X_h), \quad (p_h, X_h) = 1$$

and so

$$x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j} \in (X_h).$$

Therefore, for generic h, $\{X_h\}$ satisfies the hypothesis of Proposition 10 and hence X_h is constant up to a multiple of h.

Proof of Theorem 2. Let $X_h = \alpha X + hY_h$ and $Y = x_3^j X$. For a generic hyperplane $H = \{h = 0\}$ and all $p_h \in K'_h$ there exists A_h such that

$$p_h Y + hA_h \in I.$$

Let $\Gamma_H = H \cap C$, then $\Gamma_H \subset V(p_h Y) = V(Y) \cup V(p_h)$. If for a generic H, there exists q_H a point in Γ_H such that $q_H \in V(Y)$, then $S = \{q_H \mid q_H \in V(Y)\}$ will be a 1-dimensional space and $S \subset C$, C is reduced and irreducible therefore S is dense in C and hence $C \subset V(Y)$. However $V(Y) = V(x_3^j) \cup V(X)$ and as C is nondegenerate, this would imply $C \subset V(X)$. However, $i < s_j - 1 \leqslant s_0 - 1$ and so $i+1 < s_0$. But s_0 is the smallest degree of elements of I, therefore $C \not\subset V(X)$. Therefore for generic H, $\Gamma_H \subset V(K'_h)$. However the invariants of $(K'_h)^{\mathrm{sat}}$ are $\lambda_{i+1} > \cdots > \lambda_{s-1}$ and as $i+1 > 0$, $\Gamma_H \not\subset V(K'_h)$. This concludes the proof of the main part of Theorem 2.

Now suppose $s_k < s_0$, and $\mu_{s_k-1}(k) \geqslant 3$. For a generic hyperplane H, $(I|_h\colon x_3^k)_{s_k} = (X_h)_{s_k}$, and $(I|_h\colon x_3^k)_{s_k+1} = (X_h)_{s_k+1}$, and so there exists a homogeneous polynomial A_h such that

$$x_3^k X_h + hA_h \in I.$$

Allowing $h = \sum t_i x_i$ to vary and differentiating with respect to $\{t_j\}$, we get

$$x_3^k \left(\frac{\partial X_h}{\partial t_i}\right) + x_i A_h \in I|_h$$

and so

$$\left(x_j \frac{\partial X_h}{\partial t_i} - x_i \frac{\partial X_h}{\partial t_j}\right) \in (I|_h\colon x_3^k)_{s_k+1} = (X_h)_{s_k+1}.$$

Thus by Proposition 10, $X_h = \alpha X + hY_h$ is constant up to a multiple of h.

Let $\Gamma_H = H \cap C$ be a generic hyperplane section of C. As $x_3^k X + hA_h \in I$, $\Gamma_H \subset V(x_3) \cup V(X)$. But as the points of Γ_H are in general position, there must exist at least one point of $\Gamma_H \in V(X)$. But varying h as above would, again, imply that $C \subset V(X)$, and hence that $s_k \geqslant s_0$, which is a contradiction. This completes the proof of Theorem 2.

4. Further restrictions on the generic initial ideal of a curve

4.1. Generalized Strano

This result generalizes a result of Strano ([S])

DEFINITION. If $x_1^i x_2^j x_3^{f(i,j)}$ is a generator of $\mathrm{gin}(I_C)$ with $f(i,j) > 0$, then $x_1^i x_2^j x_3^k$ is a *sporadic zero* for all $0 \leqslant k < f(i,j)$.

THEOREM 12 (Strano). *If C is a reduced irreducible curve and has a sporadic zero in degree m, then I_Γ has a syzygy in degree $\leqslant m+2$.*

THEOREM 13 (Generalized Strano). *Let C be a reduced irreducible curve with a sporadic zero $x_1^i x_2^j x_3^{k-a}$ of degree m, such that $x_1^i x_2^j x_3^k$ is a generator of $\mathrm{gin}(I_C)$. Then, for a generic linear form h, $J = (I_C|_h\colon x_3^a)$ has a syzygy in degree $\leqslant m+2$.*

Proof. $x_1^i x_2^j x_3^{k-a} \in \mathrm{gin}(J)_m$, therefore there exists $F_h \in (I_C|_{h=0}\colon x_3^a)_m = (J)_m$ varying with h, and hence for generic H there exists A_h, such that

$$x_3^a F_h + hA_h \in I_C.$$

The families $\{F_h\}$ and $\{A_h\}$ correspond to homogeneous polynomials F and A in the $\{x_i\}$ whose coefficients are rational functions in the $\{t_i\}$, where $h = \sum t_i x_i$.

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On clearing denominators we may assume F is a bihomogeneous polynomial in t_i and x_i. Choose F so that the degree of F with respect to t_i is minimal.

Letting h vary and differentiating with respect to $\{t_j\}$, we get

$$x_3^a \frac{\partial F}{\partial t_j} + x_j A \in I_C|_{h=0}$$

and so

$$x_j \frac{\partial F}{\partial t_i} - x_i \frac{\partial F}{\partial t_j} \in (J)_{m+1}.$$

Hence

$$x_k \left(x_j \frac{\partial F}{\partial t_i} - x_i \frac{\partial F}{\partial t_j} \right) - x_j \left(x_k \frac{\partial F}{\partial t_i} - x_i \frac{\partial F}{\partial t_k} \right)$$
$$+ x_i \left(x_k \frac{\partial F}{\partial t_j} - x_j \frac{\partial F}{\partial t_k} \right) = 0$$

is a syzygy of J in degree $m+2$.

Suppose J does not have a syzygy in degree $\leqslant m+2$, then

$$x_j \frac{\partial F}{\partial t_i} - x_i \frac{\partial F}{\partial t_j} = x_j U_i - x_i U_j \quad \text{where } U_i \in (J)_m.$$

Rewriting, we get

$$x_j \left(\frac{\partial F}{\partial t_i} - U_i \right) - x_i \left(\frac{\partial F}{\partial t_j} - U_j \right) = 0$$

and so

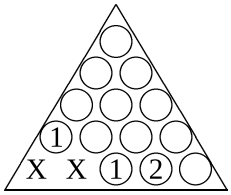
$$\frac{\partial F}{\partial t_i} = U_i + x_i R.$$

Letting $F' = F - hR$ we get

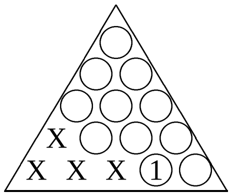
$$\frac{\partial F'}{\partial t_i} = \frac{\partial F}{\partial t_i} - x_i R - h \frac{\partial R}{\partial t_i} = U_i \operatorname{mod}(h) \in (J)_m.$$

As we have assumed the degree of F is minimal with respect to t_i we get that F is constant up to a multiple of h. Hence, by an argument similar to that of Theorem 2, $F \in I_C$. This, however, is a contradiction.

EXAMPLE 5. The following diagram can not correspond to a generic initial ideal of a reduced irreducible curve, even though it is connected.

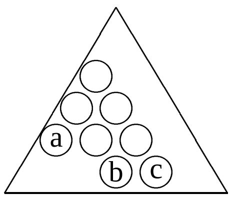


The ideal has a sporadic zero in degree 3, and so by the Theorem 12, $J = (I_C|_{h=0} : x_3)$ has a syzygy in degree $\leqslant 5$. The diagram of $\text{gin}(J)$ is



Let $K \subseteq J$ be the ideal generated by elements of J in degree $\leqslant 4$. Every new generator of $\text{gin}(K)$ in degree 5 arises from syzygies of elements of $\text{gin}(K)$ in degree 4 as constructed in Theorem 4. J has only two generators in degree $\leqslant 4$, corresponding to x_1^3 and $x_1^2 x_2^2$ and so if J has a syzygy in degree $\leqslant 5$ this would imply that $\text{gin}(K)$ has no generators in degree 5 and $\text{gin}(K) = (x_1^3, x_1^2 x_2)$. Hence we may 'split' the ideal J as in the proof of Lemma 8 and obtain a contradiction as in Theorem 2.

More generally, suppose we have a triangle configuration as below.



Let $s = \min\{i \mid f(i, 0) \leqslant k\}$ for $k \gg 0$. By connectedness $b \leqslant a$. Let $J = (I_C|_h : x_3^a)$ and $K \subset J$ be the ideal generated by elements of J in degree $\leqslant s+1$, then by Theorem 12, J has a syzygy in degree $\leqslant s+2$. If $c > a$, we find, as above, that $\text{gin}(K)$ is generated in degree $\leqslant s+1$. But again this would imply that we could *split* the ideal J and obtain a contradiction as in Theorem 2.

4.2. Complete intersections and almost complete intersections

The result in this section is inspired by the work of Ellia ([E]) and again generalizes a result of Strano ([S]).

DEFINITION. If Γ is a set of d points in general position with invariants $\lambda_0 > \cdots > \lambda_{k-1} > 0$ such that $\lambda_i = \lambda_0 - 2i$ for all i. Then Γ is a *complete intersection* of type $(k, d/k)$. (See [Gr]).

THEOREM 14 (Strano). *If C is a reduced irreducible curve whose generic hyperplane section has the Hilbert function of a complete intersection of type (m, n), where $n \geqslant m > 2$, then C is a complete intersection of type (m, n).*

Proof. Let $J = (I_C|_h : x_3^k)$ for $k \gg 0$ so that $J = I_\Gamma$ where Γ is a general hyperplane section of C and hence the only syzygy of J is in degree $m + n$.

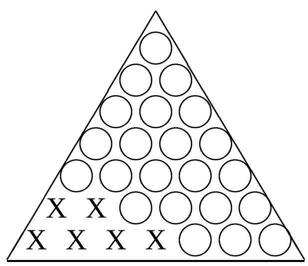
Suppose C is not a complete intersection of type (m, n), then gin(I_C) has a sporadic zero, M. If the degree of $M = m$, Theorem 12 implies there is a syzygy of J in degree $\leqslant m + 2$. But $m + n > m + 2$ and we obtain a contradiction. If the degree of $M > m$, connectedness implies there is a sporadic zero of degree n, but again this implies there is a syzygy of degree $\leqslant n + 2$ and we obtain a contradiction.

PROPOSITION 15. *Let C be a reduced, irreducible, non-degenerate curve in $\mathbf{P}^3$, let $\Gamma = C \cap H$ be a generic hyperplane section with invariants $\{\lambda_i\}_{i=0}^{s-1}$. If $\lambda_{s-i} = \lambda_{s-1} + 2(i-1)$ for $1 \leqslant i \leqslant k$, where $k \geqslant 3$, then $f(i, j) > 0$ only if $i < s - k$.*

Proof. Let $J = (I|_h : x_3^j)$ for $j \gg 0$, so that gin$(J) =$ gin(I_Γ). Let f correspond to x_1^s and let g correspond to $x_1^{s-1} x_2^{\lambda_{s-1}}$, where f and g are in J. If f and g have a syzygy in degree $d \leqslant \lambda_{s-k} + (s-k)$, then generators of gin$(J)$ in degree d correspond to generators of J and thus we may split the ideal J and obtain a contradiction as in the proof of Theorem 2. Therefore f and g have no syzygy in degree $\leqslant \lambda_{s-k} + (s-k)$. By Theorem 12 or Theorem 13, this means that there can be no sporadic zeroes in degree $\leqslant \lambda_{s-k} + (s-k) - 2$.

If there is a sporadic zero in degree $\lambda_{s-k} + (s-k) - 1 = \lambda_{s-(k-1)} + s - (k-1)$, then $\mu_{s-(k-1)}(0) > \lambda_{s-(k-1)}$ and $\mu_{s-(k-2)}(0) = \lambda_{s-(k-2)} = \lambda_{s-(k-1)} - 2$, which contradicts the connectedness of the $\{\mu_i(0)\}$. Similarly if $f(s-k, \lambda_{s-k}) > 0$ then $\mu_{s-k}(0) > \lambda_{s-k} = \lambda_{s-(k-1)} + 2 = \mu_{s-(k-1)}(0) + 2$ which again contradicts connectedness.

Thus for the following configuration of a hyperplane section, the only possibly spots for sporadic zeros are in the (1,6), (0,7) or (0,8) positions.



5. Some curves of low degree

As an application to the theorems in the previous sections, we will discuss the possible Borel-fixed monomial ideals occuring for some curves of low degree. The theorems seem to work particularly well for curves of high genus with respect to their degree. From the possible Borel-fixed monomial ideals one can then find all the possible Hilbert functions. Furthermore, the minimal resolution of the generic initial ideal of a curve gives us an upper bound on the degrees of all the syzygies of the actual curve (see [Gr]), thus we also get some ideas as to what the degrees of syzygies of the curves can be. As an illustration we have chosen to discuss curves of degree 7, genus 2 and curves of degree 8, genus 4.

First we will give a list of some known results which we will find useful in eliminating more Borel-fixed monomial ideals.

5.1. Some known results

1. Genus. Let C be a curve whose generic hyperplane section has invariants $\lambda_0, \ldots, \lambda_{s-1}$. The (arithmetic) genus of C is

$$g(C) = 1 + \sum_{i=0}^{s-1} \left((i-1)\lambda_i + \binom{\lambda_i}{2} \right) - \sum_{f(i_1,i_2)<\infty} f(i_1, i_2).$$

This equation is may be found by considering the Hilbert polynomial of C and that associated to $\mathrm{gin}(I_C)$. For details see [Gr] (Proposition 4.19).

2. Regularity. The following is due to Gruson, Lazarsfeld and Peskine ([GLP]).

THEOREM 16. *Let C be a reduced irreducible non-degenerate curve of degree d in $\mathbf{P}^n$. Then C is $(d+2-n)$-regular. Furthermore, if $d > n+1$ and C is $(d+1-n)$-irregular, then C is smooth and rational, with a $(d+2-n)$-secant line.*

As the regularity is the same as the maximal degree of the minimal generators of $\mathrm{gin}(I_C)$ (see [BS]), it follows that $\mathrm{gin}(I_C)$ is generated in degree $\leqslant d-1$. Furthermore, if degree$(C) > 4$, and genus$(C) > 0$, then $\mathrm{gin}(I_C)$ is generated of degree $\leqslant d-2$.

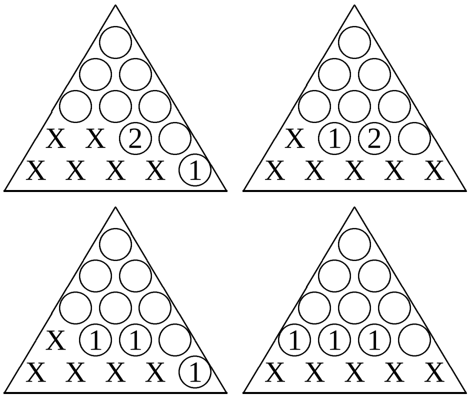
3. Liaison. (a) If C is linked via a complete intersection to an Arithmetically Cohen–Macaulay curve, then C is Arithmetically Cohen–Macaulay. This is equivalent to C having no sporadic zeros. (For more information on Liaison Theory see for example the work of Rao [R]).

(b) If C is linked via a complete intersection to a curve of degree 2, then C can have at most one sporadic zero in each degree. (This paraphrases some of the work of Juan Migliore, [M]).

5.2. CURVES OF DEGREE 7, GENUS 2

If the degree of C is 7, then the invariants $\{\lambda_i\}_{i=0}^{s-1}$ of a generic hyperplane section of C are such that $\sum \lambda_i = 7$ and $\lambda_{i+1} < \lambda_i \leqslant \lambda_{i+1} + 2$ for $i = 0, \ldots s - 2$. Hence either $s = 2$ and $\lambda_0 = 4, \lambda_1 = 3$, or $s = 3$ and $\lambda_0 = 4, \lambda_1 = 2, \lambda_2 = 1$.

In the former case C must lie on a quadric. But the only curves of degree 7 lying on a quadric are of genus 6, 4 or 0. Therefore we must have $s = 3$ and $\lambda_0 = 4, \lambda_1 = 2, \lambda_2 = 1$. In this case C has 3 sporadic zeros and by Theorem 16 $\mathrm{gin}(I)$ is generated in degree $\leqslant 5$. The only possible connected configurations are



Using Liaison Theory we may also eliminate the first possibility as this corresponds to a curve linked to a curve of degree 2. Hence by 3(b) above, it can not have two sporadic zeros in degree 4.

Thus we are left with 3 possible generic initial ideals. Considering the resolutions of these ideals we get two possible Hilbert functions for curves of degree 7, genus 2. (The second and third configurations give the same Hilbert function.) Furthermore, we find there are nine possible minimal resolutions of these curves.

5.3. CURVES OF DEGREE 8, GENUS 4

If the degree of C is 8 then either $s = 2$ and $\lambda_0 = 5, \lambda_1 = 3$ or $s = 3$ and $\lambda_0 = 4, \lambda_1 = 3, \lambda_2 = 1$. As there are no curves of degree 8, genus 4 lying on a quadric, we must have $s = 3$ and $\lambda_0 = 4, \lambda_1 = 3, \lambda_2 = 1$. In this case the number of sporadic zeros is 3 and $\mathrm{gin}(I)$ is generated in degree $\leqslant 6$.

We may also eliminate configurations of the form

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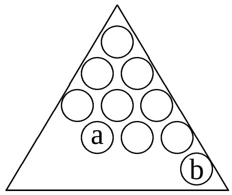
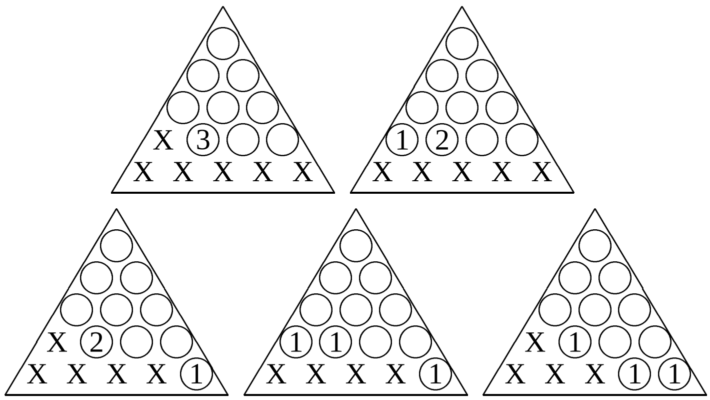


where $a < b$, as then $J = (I|_h : (x_3)^a)$ has a syzygy in degree 5 (by Theorem 13). x_1^3 and $x_1^2 x_2$ correspond to the generators of J in degree 3, and $x_1 x_2^3$ cannot represent a new generator of J, otherwise we would be able to *split* the ideal along the x_1^2 line. As J has a syzygy in degree 5, this means $x_1 \otimes x_1 x_2^3$ corresponds to a real syzygy of J and thus we can *split* J along the x_1 line and obtain a contradiction.

The possible connected configurations of gin(I) are



Thus we have 5 possible generic initial ideals. Considering the resolutions of these ideals we get 3 possible Hilbert functions for curves of degree 8, genus 4. (The second and fourth and the third and fifth configurations give the same Hilbert function.) Furthermore, we find there are 18 possibilities for the degrees of the syzygies of the minimal resolutions of these curves. (Notice that the first possibility is a curve with a secant line of order 6).

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