example, with $n = 2$ this gives 3 degrees of freedom for 3 points in the plane, and with $n = 3$ it gives 6 degrees of freedom for 4 points in 3-space, as noted above. So the number of degrees of freedom for $n + 1$ points in n-space is always the same as the number of mutual distances between them. The result for $n = 3$ outlined in Section 2 above is therefore typical: there will not be an identity to be satisfied by the $\frac{1}{2}n(n + 1)$ distances, but there will be one or more inequalities to ensure that distances correspond with configurations of points.

Question: what can be said about the case of four *concyclic* points and their six distances?

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107.41 Langley's Problem: 100 years on

Edward Mann Langley was the founder and first editor of *The Mathematical Gazette*. Among his many contributions is a short note [1]. In just two lines of text, squeezed in at the bottom of a page, he sets a tantalising problem. Figure 1 shows it in diagrammatic form. Given that $AB = AC$, $a = 20°$, $b = 50°$, $c = 60°$ readers were asked to prove that $\theta = 30°$.

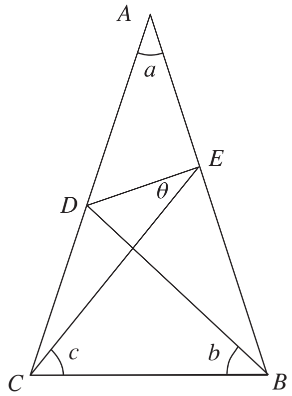

FIGURE 1

Solutions were published in a subsequent issue [2]. Some made use of trigonometry, others only the most elementary geometry.

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Other choices of a, b and c, all multiples of $1°$, can also lead to θ measuring a whole number of degrees. Langley's problem is far from unique in this respect. Ignoring trivial cases where $b = c$ and mirror images obtained by swapping b and c, Colin Tripp [3] conjectured that there are 53 distinct cases. His article prompted a flurry of activity [4], [5], culminating in John Rigby's major contribution [6] which also dropped the restriction that $AB = AC$.

We do not offer new insights. However, the challenge to solve particular cases by truly elementary methods remains. We show how one particular problem can be solved with basic geometrical tools, putting it well within the range of secondary school students.

Our problem, shown in Figure 2, looks somewhat different from Figure 1, but we will show how they are related. First, omit the 'redundant' vertex A from Figure 1 to leave quadrangle $EBCD$. Now make the interior angle at D a reflex angle, so that D lies within triangle EBC, and rename the points.

The problem was shown to one of the authors by a student. It may originally have been intended as an exercise in trigonometry and an exact solution can readily be obtained by the judicious use of the sine rule and trigonometric identities.

We offer two purely geometrical solutions.

Draw equilateral triangles ABD, ACE as shown in Figure 3 and join D and E. Both BC and ED are perpendicular to the line of symmetry of triangle ABC and thus parallel. Hence $\angle BCE$, $\angle CBD$, $\angle$CED and $\angle BDE$ all measure $10°$.

Let F be the midpoint of AE. Then triangles ACF and ECF are congruent and $\angle ACF = \angle EDF$.

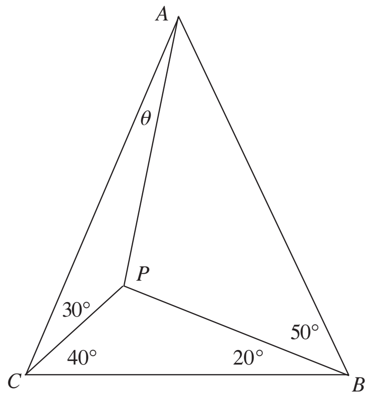

FIGURE 2

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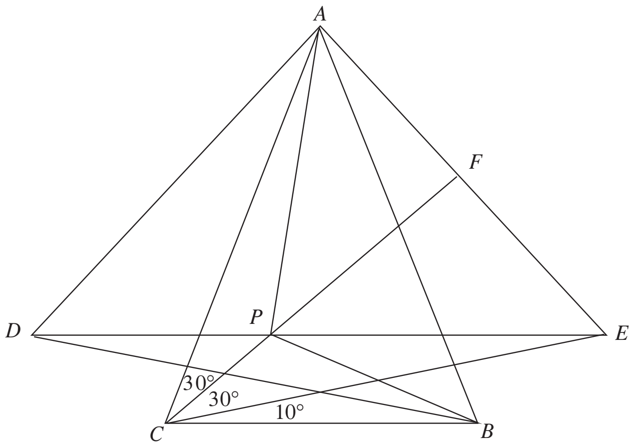

FIGURE 3

Let P denote the intersection of CF with DE. Triangles ACP, ECP are congruent and it follows that $\angle CAP = \angle CEP = 10°$. Since $\angle BAC = 40°$ we see that $\angle DAC = 20°$ and $\angle DAP$ and $\angle BAP$ both measure 30°. Thus triangles DAP and BAP are congruent and $PD = PB$. Triangle PBD is isosceles with $\angle PBD = \angle PDB = 10°$. It follows that $\angle PBC = 20°$.

The point P meets the conditions of Figure 2 and $\theta = 10°$.

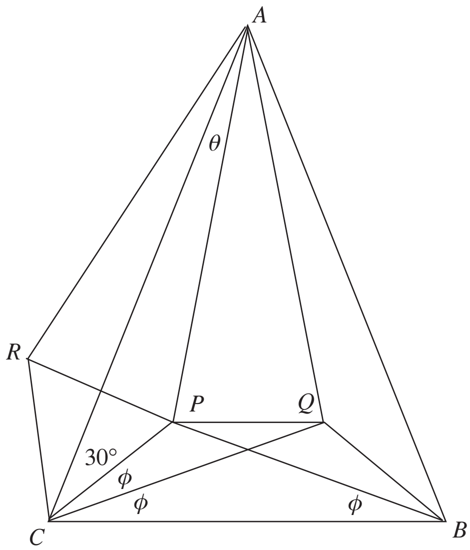

FIGURE 4

Our second solution allows a generalisation. We keep $AB = AC$ and replace 20° and 40° by ϕ and 2ϕ respectively, as shown in Figure 4. Let Q

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be the reflection of P in the line of symmetry of triangle ABC. Join Q to A, B, C and P. Since PQ and CB are both perpendicular to the line of symmetry they are parallel and $\angle PQC = \angle QCB = \phi$. Thus triangle PQC is isosceles with $PQ = PC$.

Reflect triangle ACP in AC. Join P to its image R. Then $PC = RC$ and $\angle PCR = 60°$, so triangle PCR is equilateral and $PR = PC$.

Since $AR = AP = AQ$ and $PR = PQ$, triangles APR and APQ are congruent. Noting that $\angle RAC = \angle CAP = \angle BAQ = \theta$, we find that $\angle PAQ = 2\theta$ and $\angle BAC = 4\theta$.

The interior angle sum of triangle $\angle ABC = 4\theta + 2(30° + 2\phi) = 180°$ and it follows that $\theta + \phi = 30°$. In our special case $\phi = 20°$, so $\theta = 10°$.

References

1. E. M. Langley, Note 644, *Math. Gaz.* **11** (October 1922) p. 173.
2. Problems and Solutions, *Math. Gaz.* **11** (May 1923) pp. 321-323.
3. C. Tripp, Adventitious Angles, *Math. Gaz.* **59** (June 1975) pp. 98-106.
4. D. A. Quadling, The adventitious angles problem: a progress report, *Math. Gaz.* **61** (March 1977) pp. 55-58.
5. J. F. Rigby, Last words on adventitious angles, *Math. Gaz.* **62** (October 1978) pp. 174-183.
6. J. F. Rigby, Adventitious quadrangles: a geometrical approach, *Math. Gaz.* **62** (October 1978) pp. 183-191.

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107.42 On the Lemoine line for the triangle

For obtuse triangles, the circumcentre and orthocentre are external points so the centroid G is the unique point on the Euler line which lies within every triangle ABC.

Now the isogonal conjugate of G is the Lemoine or symmedian point K and we will use areal coordinates to find the largest segment on the Lemoine line GK produced which lies within every triangle.