Bull. Aust. Math. Soc. **88** (2013), 51–55
doi:10.1017/S000497271200069X

ELLIPTIC EXTENSIONS IN THE DISK WITH OPERATORS IN DIVERGENCE FORM

ORAZIO ARENA and CRISTINA GIANNOTTI ✉

(Received 27 May 2012; accepted 2 July 2012; first published online 20 August 2012)

Abstract

Let φ_0 and φ_1 be regular functions on the boundary ∂D of the unit disk D in $\mathbb{R}^2$, such that $\int_0^{2\pi} \varphi_1 \, d\theta = 0$ and $\int_0^{2\pi} \sin\theta(\varphi_1 - \varphi_0)\, d\theta = 0$. It is proved that there exist a linear second-order uniformly elliptic operator L in divergence form with bounded measurable coefficients and a function u in $W^{1,p}(D)$, $1 < p < 2$, such that $Lu = 0$ in D and with $u|_{\partial D} = \varphi_0$ and the conormal derivative $\partial u/\partial N|_{\partial D} = \varphi_1$.

2010 *Mathematics subject classification*: primary 35D30; secondary 35J15.
Keywords and phrases: elliptic equations in divergence form, Cauchy data.

1. Introduction

Let D be the unit open disk in $\mathbb{R}^2$, n the outer normal to ∂D and L a linear second-order uniformly elliptic operator, with bounded measurable coefficients in D, of the form

$$L := a^{11}\partial_{11} + 2a^{12}\partial_{12} + a^{22}\partial_{22}. \tag{1.1}$$

In [1], Manselli and the second author proved that, given two arbitrary functions $f^{(0)}$, $f^{(1)}$ on ∂D (with some appropriate regularity assumption, such as $df^{(0)}/d\theta$ and $f^{(1)}$ Hölder continuous with exponent $\eta > \frac{1}{2}$), there exist an operator L of the form (1.1) and a function $u \in W^{2,p}(D)$, $1 < p < 2$, satisfying

$$\begin{cases} Lu = 0 & \text{in } D,\\ u|_{\partial D} = f^{(0)}, & \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = f^{(1)}. & \end{cases}$$

Such a pair (u, L) was called an *elliptic extension* of $f^{(0)}$, $f^{(1)}$ in D.

Here we consider the following similar question for elliptic operators in divergence form. *Given two functions φ_0 and φ_1 on ∂D, do there exist a function u and a linear* *second-order uniformly elliptic operator L in divergence form, such that*

$$\begin{cases} Lu = \partial_i(a^{ij}\partial_j u) = 0 \quad \textit{a.e. in } D, \\ u|_{\partial D} = \varphi_0, \\ \left.\dfrac{\partial u}{\partial N}\right|_{\partial D} = \varphi_1, \end{cases} \tag{1.2}$$

where $\partial u/\partial N|_{\partial\Omega} = a^{ij}u_{x_j}n_i$ *is the conormal derivative of* u*?*

Wolff [3] studied this problem on smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$. He proved that in order to have a solution with u and L smooth, the functions φ_0 and φ_1 must satisfy suitable necessary and sufficient compatibility conditions. He also remarked that, for the case $n = 2$, additional assumptions on φ_0 and φ_1 are required.

Here, using the result in [1], we prove that, if φ_0 and φ_1 are regular functions on ∂D such that

$$\int_{\partial D} \varphi_1 \, ds = 0 \quad \text{and} \quad \int_{\partial D} n_2(\varphi_1 - \varphi_0) \, ds = 0,$$

then there exist an operator L in divergence form, with bounded measurable coefficients, and a function u in $W^{1,p}(D)$, $1 < p < 2$, which satisfy (1.2) in a suitable weak sense.

2. The main result

Consider the problem (1.2), where L is a linear second-order uniformly elliptic operator in divergence form, with bounded measurable coefficients, and u is a function in $W^{1,p}(D)$, $p \geq 1$. Notice that, due to the low regularity of u and L, the conditions of (1.2) have no meaning, unless they are reinterpreted in a weaker sense, which we now specify. Recall that, if Ω is a bounded domain in $\mathbb{R}^2$, a function u in $W^{1,2}(\Omega)$ is considered a solution of the problem

$$\begin{cases} Lu = \partial_i(a^{ij}\partial_j u) = 0 \quad \text{a.e. in } \Omega, \\ \left.\dfrac{\partial u}{\partial N}\right|_{\partial\Omega} = \varphi_1, \end{cases}$$

if it satisfies the identity

$$\int_\Omega a^{ij}\partial_j u \partial_i \eta \, dx = \int_{\partial\Omega} \varphi_1 \eta \, d\sigma \quad \text{for any } \eta \in W^{1,2}(\Omega)$$

(see, for example, [2, p. 161]).

By analogy with such interpretation, in the following *a function* $u \in W^{1,p}(D)$*,* $p \geq 1$*, will be considered a solution of (1.2) if*

$$\begin{cases} \displaystyle\int_D a^{ij}\partial_j u \partial_i \eta \, dx = \int_{\partial D} \varphi_1 \eta \, d\sigma \quad \text{for any } \eta \in C^1(\overline{D}), \\ \text{trace of } u \text{ on } \partial D = \varphi_0. \end{cases} \tag{2.1}$$

Our result is the following.

https://doi.org/10.1017/S000497271200069X Published online by Cambridge University Press

Theorem 2.1. *Let $1 < p < 2$ and φ_0, φ_1 be of class C^∞ on ∂D and such that:*

(i) $\int_{\partial D} \varphi_1 \, ds = 0$;

(ii) $\int_{\partial D} n_2(\varphi_1 - \varphi_0) \, ds = 0$.

Then there exist a linear second-order uniformly elliptic operator L in divergence form, with bounded measurable coefficients, and a function $u \in W^{1,p}(D)$, which is a solution of (1.2) (that is, satisfies (2.1)).

Proof. Let (ψ_0, ψ_1) be the solution on ∂D of the system

$$\begin{cases} n_1\psi_1 - n_2 \dfrac{d\psi_0}{d\theta} = \varphi_0, \\ \dfrac{d}{d\theta}\Big(n_2\psi_1 + n_1 \dfrac{d\psi_0}{d\theta}\Big) = -\varphi_1. \end{cases} \tag{2.2}$$

The functions ψ_0 and ψ_1 exist and are regular on ∂D by the hypotheses on φ_0, φ_1. In particular, condition (ii) is equivalent to the condition $\int_{\partial D}(d\psi_0/d\theta)\, ds = 0$ by integration by parts.

By [1, Theorem 3.3], there exist $v \in W^{2,p}(D)$ and a second-order uniformly elliptic operator in nondivergence form and with bounded measurable coefficients, $\widetilde{L} := \widetilde{a}^{11}\partial_{xx} + 2\widetilde{a}^{12}\partial_{xy} + \widetilde{a}^{22}\partial_{yy}$, such that

$$\begin{cases} \widetilde{L}v = 0 & \text{in } D, \\ v|_{\partial D} = \psi_0, \\ \dfrac{\partial v}{\partial n}\Big|_{\partial D} = \psi_1. \end{cases}$$

Let $u = v_x \in W^{1,p}(D)$. The equation $\widetilde{L}v = 0$ can be written as

$$\frac{\widetilde{a}^{11}}{\widetilde{a}^{22}} v_{xx} + 2\frac{\widetilde{a}^{12}}{\widetilde{a}^{22}} v_{xy} + v_{yy} = 0 \quad \text{a.e. in } D,$$

and, by formally differentiating with respect to x,

$$Lu = (a^{11}u_x)_x + (a^{12}u_y)_x + u_{yy} = 0, \tag{2.3}$$

where $a^{11} = \widetilde{a}^{11}/\widetilde{a}^{22}$ and $a^{12} = 2\widetilde{a}^{12}/\widetilde{a}^{22}$. According to our definition, $u \in W^{1,p}$ is a solution to (1.2) for the operator L defined in (2.3) if and only if the trace of u on ∂D is equal to φ_0 and

$$\int_D (a^{11}u_x\eta_x + a^{12}u_y\eta_x + u_y\eta_y)\, dx\, dy = \int_{\partial D} \varphi_1\eta\, ds$$

for any $\eta \in C^1(\overline{D})$. On the other hand,

$$\begin{aligned} &\int_D (a^{11}u_x\eta_x + a^{12}u_y\eta_x + u_y\eta_y)\, dx\, dy \\ &\quad = \int_D \frac{\tilde{a}^{11}v_{xx}\eta_x + 2\tilde{a}^{12}v_{xy}\eta_x + \tilde{a}^{22}v_{xy}\eta_y}{\tilde{a}^{22}}\, dx\, dy \end{aligned}$$

https://doi.org/10.1017/S000497271200069X Published online by Cambridge University Press

$$= \int_D \Big(\frac{\tilde{a}^{11} v_{xx} + 2\tilde{a}^{12} v_{xy} + \tilde{a}^{22} v_{yy}}{\tilde{a}^{22}} \eta_x + (v_{xy}\eta_y - v_{yy}\eta_x) \Big)\, dx\, dy$$
$$= \int_D (v_{xy}\eta_y - v_{yy}\eta_x)\, dx\, dy.$$

If we consider a sequence $v_n \in C^3(\overline{D})$ converging to v in $W^{2,p}(D)$, then

$$\int_D (v_{xy}\eta_y - v_{yy}\eta_x)\, dx\, dy = \lim_{n\to\infty} \int_D ((v_n)_{xy}\eta_y - (v_n)_{yy}\eta_x)\, dx\, dy.$$

However,

$$\begin{aligned}\int_D ((v_n)_{xy}\eta_y - (v_n)_{yy}\eta_x)\, dx\, dy &= \int_D (((v_n)_{xy}\eta)_y - ((v_n)_{yy}\eta)_x)\, dx\, dy \\ &= \int_{\partial D} (n_2 (v_n)_{xy} - n_1 (v_n)_{yy})\eta\, ds \\ &= -\int_{\partial D} \Big(\frac{d(v_n)_y}{d\theta}\Big)\eta\, ds = \int_{\partial D} (v_n)_y \frac{d\eta}{d\theta}\, ds\end{aligned}$$

and

$$\lim_{n\to\infty} \int_{\partial D} (v_n)_y \frac{d\eta}{d\theta}\, ds = \int_{\partial D} v_y \frac{d\eta}{d\theta}\, ds.$$

Moreover, since

$$v|_{\partial D} = \psi_0 \in C^\infty(\partial D) \quad \text{and} \quad \frac{\partial v}{\partial n}\Big|_{\partial D} = \psi_1 \in C^\infty(\partial D),$$

we have also that

$$v_y|_{\partial D} = n_2 v_r|_{\partial D} + n_1 v_\theta|_{\partial D} \in C^\infty(\partial D)$$

and

$$\int_{\partial D} v_y \frac{d\eta}{d\theta}\, ds = -\int_{\partial D} \frac{dv_y}{d\theta}\eta\, ds.$$

This implies that

$$\int_D (a^{11} u_x \eta_x + a^{12} u_y \eta_x + u_y \eta_y)\, dx\, dy = -\int_{\partial D} \frac{dv_y}{d\theta}\eta\, ds,$$

which means that u solves

$$\begin{cases} Lu = 0 & \text{a.e. in } D, \\ \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = -\dfrac{dv_y}{d\theta}. \end{cases}$$

Since, from (2.2),

$$u|_{\partial D} = v_x|_{\partial D} = n_1\psi_1 - n_2 \frac{d}{d\theta}\psi_0 = \varphi_0,$$
$$\frac{\partial u}{\partial N}\Big|_{\partial D} = -\frac{dv_y}{d\theta} = -\frac{d}{d\theta}\Big(n_2\psi_1 + n_1\frac{d\psi_0}{d\theta}\Big) = \varphi_1,$$

it follows immediately that u is a solution to (1.2) with L defined in (2.3). □

https://doi.org/10.1017/S000497271200069X Published online by Cambridge University Press

References

[1] C. Giannotti and P. Manselli, 'On elliptic extensions in the disk', *Potential Anal.* **33** (2010), 249–262.

[2] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations* (Academic Press, New York, 1968).

[3] T. H. Wolff, 'Some constructions with solutions of variable coefficient elliptic equations', *J. Geom. Anal.* **3** (1993), 423–511.

ORAZIO ARENA, Dipartimento di Costruzioni e Restauro, Università di Firenze, Piazza Brunelleschi 6, I-50121 Firenze, Italy
e-mail: arena@unifi.it

CRISTINA GIANNOTTI, Scuola di Scienze e Tecnologie, Università di Camerino, Via Madonna delle Carceri, I- 62032 Camerino (Macerata), Italy
e-mail: cristina.giannotti@unicam.it