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ANOTHER LAW FOR 3-METABELIAN GROUPS

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Abstract. We show that $[z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x] = 1$ is another defining law for the variety of 3-metabelian groups.

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A group G is defined to be metabelian if $[G', G']$ is the trivial subgroup and is defined to be 3-metabelian if all of its three generator subgroups are metabelian. In 1956, Neumann [**7**] gave an example of a group that is 3-metabelian but is not metabelian. In 1961, Macdonald [**4**], among other results, obtained information about the structure of 3-metabelian groups and observed that such groups satisfy the law $[x, y\,;\, x, z] = 1$. In 1962, Macdonald [**5**] proved as a special case of Theorem 7 in his paper that any group that satisfies $[x, y\,;\, x, z] = 1$ is 3-metabelian, and hence this law defines the variety of 3-metabelian groups. Of related interest, in 1964, Bachmuth and Lewin [**1**] proved that the law $[x, y, z][y, z, x][z, x, y] = 1$ also defines the variety of 3-metabelian groups. Macdonald [**6**] was aware of this last result and proved, also in 1964, that the law $[x, y\,;\, y, z][y, z\,;\, z, x][z, x\,;\, x, y] = 1$ is another law that defines the variety of 3-metabelian groups. We will use Macdonald's results to prove our result. The reader will find a discussion of these results and definitions for unexplained notation and terminology in Neumann's book [**8**].

The notation $W(x, y, z)$ for $[z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x]$ was introduced by Jackson, Gaglione and Spellman for expository convenience in [**2**] and used more extensively in [**3**]. In those papers, the following three properties of $W(x, y, z)$ were used: for G any group and x, y, z any elements of G,

$$[z, y, x] = ([y, x, z]^{-1})^{[z,x][z,y]} W(x, y, z)[z, x, y]^{[y,x]},$$
$$W(x, y, z) = [z, y\,;\, z, x][z, y\,;\, y, x]^{[z,x]}[z, x\,;\, y, x] \text{ and}$$
$$W(x, y, z) = [z, x\,;\, y, x]^{[z,y]}[z, y\,;\, y, x][z, y\,;\, z, x]^{[y,x]}.$$

Jackson et al. were also aware of other identities, such as $(W(y, x, z))^{[y,x]} = (W(x, y, z))^{-1}$, $W(x, y, z) = (W(y, z, x))^{[z,x][y,x]}$ and $W(x, x, z) = 1$, but did not use or publish these.

THEOREM. *The variety of groups defined by the law $W(x, y, z) = 1$ is the variety of 3-metabelian groups.*

Proof. Permuting variable names when necessary, and using the law $[x, y\,;\, x, z] = 1$ from Macdonald's 1962 paper [**5**], the commutators $[x, y]$, $[x, z]$ and $[y, z]$ commute with one another in any 3-metabelian group. Since $W(x, y, z)$ is defined to be $[z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x]$, it is easy to see that $W(x, y, z) = 1$ for any elements x, y and z of a 3-metabelian group.

To see that any group that satisfies the law $W(x, y, z) = 1$ is 3-metabelian, we will use a result from Macdonald's 1964 paper [**6**]. It is proved there that the law $[x, y\,;\, y, z][y, z\,;\, z, x][z, x\,;\, x, y] = 1$ defines the variety of 3-metabelian groups. We will show for any group G and arbitrary elements x, y and z in G that $[x, y\,;\, y, z][y, z\,;\, z, x][z, x\,;\, x, y] = 1$ if $W(x, y, z) = 1$.

Using $W(x, y, z) = 1$, we see that

$$[z, y]^{-1}[z, x]^{-1}[y, x]^{-1} = [y, x]^{-1}[z, x]^{-1}[z, y]^{-1}.$$

Using this and the commutator identity $[a, b] = [b, a]^{-1}$, we obtain

$$[y, z][z, x]^{-1}[x, y] = [x, y][z, x]^{-1}[y, z]. \tag{1}$$

We next observe that $[x, y\,;\, y, z][y, z\,;\, z, x][z, x\,;\, x, y]$ first expands by obvious substitutions to

$$\left([x, y]^{-1}[y, z]^{-1}[x, y][y, z]\right)\left([y, z]^{-1}[z, x]^{-1}[y, z][z, x]\right)\left([z, x]^{-1}[x, y]^{-1}[z, x][x, y]\right),$$

which reduces with obvious cancellations to

$$[x, y]^{-1}[y, z]^{-1}[x, y][z, x]^{-1}[y, z][x, y]^{-1}[z, x][x, y]. \tag{2}$$

We then use equation (1) to substitute $[y, z][z, x]^{-1}[x, y]$ for the product of the third, fourth and fifth commutator factors in equation (2). We obtain

$$\begin{aligned}
&[x, y\,;\, y, z][y, z\,;\, z, x][z, x\,;\, x, y] \\
&\quad = [x, y]^{-1}[y, z]^{-1}\left([x, y][z, x]^{-1}[y, z]\right)[x, y]^{-1}[z, x][x, y] \\
&\quad = [x, y]^{-1}[y, z]^{-1}\left([y, z][z, x]^{-1}[x, y]\right)[x, y]^{-1}[z, x][x, y],
\end{aligned}$$

which then easily reduces to 1. □

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