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# Finite Rank Operators in Certain Algebras

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*Abstract.* Let $\mathrm{Alg}(\mathcal{L})$ be the algebra of all bounded linear operators on a normed linear space $\mathcal{X}$ leaving invariant each member of the complete lattice of closed subspaces $\mathcal{L}$. We discuss when the subalgebra of finite rank operators in $\mathrm{Alg}(\mathcal{L})$ is non-zero, and give an example which shows this subalgebra may be zero even for finite lattices. We then give a necessary and sufficient lattice condition for decomposing a finite rank operator $F$ into a sum of a rank one operator and an operator whose range is smaller than that of $F$, each of which lies in $\mathrm{Alg}(\mathcal{L})$. This unifies results of Erdos, Longstaff, Lambrou, and Spanoudakis. Finally, we use the existence of finite rank operators in certain algebras to characterize the spectra of Riesz operators (generalizing results of Ringrose and Clauss) and compute the Jacobson radical for closed algebras of Riesz operators and $\mathrm{Alg}(\mathcal{L})$ for various types of lattices.

## 1 Introduction and Preliminaries

Let $\mathcal{L}$ be a complete lattice of closed subspaces of a normed linear space $\mathcal{X}$ (over $\mathbf{C}$) containing both $(0)$ and $\mathcal{X}$. (By complete we mean the lattice is closed under the operations $\cap$ and $\vee$ (closed linear span) on arbitrary collections of its members.) Then $\mathcal{A}_{\mathcal{L}} = \mathrm{Alg}(\mathcal{L})$ is the weakly closed unital algebra of operators which leave invariant each subspace of $\mathcal{L}$. For example, if $\mathcal{A}$ is any subset of $B(\mathcal{X})$, the bounded linear operators on $\mathcal{X}$, then $\mathrm{Lat}(\mathcal{A})$, the lattice of all closed $\mathcal{A}$-invariant subspaces of $\mathcal{X}$ is complete.

Finite rank operators, those operators whose ranges are finite dimensional, play an important role in the study of $\mathcal{A}_{\mathcal{L}}$ for many classes of lattices, and other algebras as well. For example, Ringrose [16] utilized finite rank operators in nest algebras in his elegant characterization of the Jacobson radical of a nest algebra. Longstaff and Lambrou [11] showed that a strongly reflexive lattice $\mathcal{L}$ is characterized by a certain density condition satisfied by the algebra generated by the rank one operators. Barnes and Katavolos [2] demonstrated that when algebras of operators contain enough finite ranks (in a certain sense) the algebras are simultaneously triangularizable.

In Section 3 we investigate the existence of finite rank operators in $\mathcal{A}_{\mathcal{L}}$. Longstaff [14] gives a necessary and sufficient condition for the existence of rank one operators in $\mathcal{A}_{\mathcal{L}}$. No necessary condition is known for $\mathcal{A}_{\mathcal{L}}$ to contain finite rank operators of arbitrary rank. We discuss some situations where the existence of non-nilpotent finite rank operators can be determined. We also provide an example which shows $\mathcal{F}_{\mathcal{L}}$, the subalgebra of finite rank operators in $\mathcal{A}_{\mathcal{L}}$, can be zero even for finite lattices.

The main problem of Section 4 is whether a finite rank operator $F \in \mathcal{A}_{\mathcal{L}}$ is completely decomposable in $\mathcal{A}_{\mathcal{L}}$. Let $F \in \mathcal{F}_{\mathcal{L}}$ and let $n$ be the rank of $F$. We say $F$ is *decomposable* (in $\mathcal{L}$) if $F$ can be written as $F = F_1 + F_2$ where $F_1$ and $F_2$ are in $\mathcal{F}_{\mathcal{L}}$, and $F_1$ has rank one, $F_2$ has rank $n-1$. We say $F$ is *completely decomposable* (in $\mathcal{L}$) if $F$ can be written as $F = \sum_{j=1}^{n} F_j$ where each $F_j$ is a rank one operator in $\mathcal{F}_{\mathcal{L}}$.

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Erdos [5] credited Ringrose with proving each finite rank operator in $\mathcal{A}_{\mathcal{L}}$ is completely decomposable when $\mathcal{L}$ is a totally ordered lattice (nest) of subspaces of a Hilbert space, and Spanoudakis [18] extended this result to nests on arbitrary normed linear spaces. Longstaff showed this result holds for atomic Boolean subspace lattices (ABSLs) over a Hilbert space, and Lambrou [11] generalized to normed linear spaces. In the same paper he asked whether complete decomposability might hold for the class of strongly reflexive lattices, a class which contains both nests and ABSLs. Hopenwasser and Moore [9] answered this question in the negative. Spanoudakis [19] shows the result to be false even for finite distributive lattices.

We provide a necessary and sufficient lattice condition for determining if finite rank operators in $\mathcal{A}_{\mathcal{L}}$ are decomposable. This condition can be used to prove (easily) the results stated above, thus unifying the theory. We also show, in the case where $\mathcal{L}$ is $\vee$-distributive, a partial decomposition result is possible for non-nilpotent operators in $\mathcal{F}_{\mathcal{L}}$.

In Section 5 we make use of finite rank operators to characterize the spectra of Riesz operators in terms of a subspace lattice, and to find the Jacobson radical for various algebras. These include $\mathcal{A}_{\mathcal{L}}$ for various types of lattices, and closed algebras of Riesz operators. (An operator $T \in B(\mathcal{X})$ is said to be a *Riesz operator* if its image in the Calkin algebra, $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$, under the canonical quotient map has zero spectrum. So, for example, all compact operators are Riesz operators.) The work concerning Riesz operators generalizes results of Ringrose [17] and Clauss [3]. Ringrose determined when a single compact operator $T$ is quasinilpotent in terms of $\operatorname{Lat}(T)$, and Clauss extended Ringrose's result to closed algebras of compact operators which leave invariant a certain type of nest which he called elementary.

## 2 Notation and Definitions

For a finite set of vectors $x_1, \dots, x_n$, we denote their linear span by $\langle x_1, \dots, x_n \rangle$. We reserve the symbol $\subset$ for proper containment between subspaces, and use the symbol $\subseteq$ otherwise. Given a subspace $M \in \mathcal{L}$, let $M^\perp$ be the collection of all linear functionals that annihilate $M$. Define $M_- = \bigvee\{L \in \mathcal{L} : M \nsubseteq L\}$. A gap in $\mathcal{L}$ is a pair of subspaces $(N, M)$ such that no other subspace in $\mathcal{L}$ lies properly between $N$ and $M$. That is, if $K \in \mathcal{L}$ and $N \subset K \subseteq M$ then $K = M$. Given a nest $\mathcal{N}$ (a totally ordered lattice) which is a sublattice of $\mathcal{L}$, we call $\mathcal{N}$ an $\mathcal{L}$-*maximal nest* if there is no other nest in $\mathcal{L}$ which properly contains $\mathcal{N}$. When $(N, M)$ is a gap in $\mathcal{L}$ and $\mathcal{N}$ is an $\mathcal{L}$-maximal nest containing both $N$ and $M$ we say $N$ is the predecessor of $M$ in $\mathcal{L}$, and write $N = \operatorname{pred}_{\mathcal{N}}(M)$.

Let $\mathcal{F}_{\mathcal{L}}$ be the collection of all finite rank operators in $\mathcal{A}_{\mathcal{L}}$. We shall use the notation $\alpha \otimes x$ (where $\alpha$ is a linear functional and $x \in \mathcal{X}$) to denote the rank one operator whose range is the one-dimensional subspace spanned by $x$. Given any operator $T$ in $B(\mathcal{X})$ we use $\mathcal{R}(T)$ to denote the range of $T$, and $\sigma(T)$ for the spectrum of $T$, the set of all complex numbers $\lambda$ such that $\lambda - T$ is not invertible.

We call a lattice $\mathcal{L}$ *reflexive* if $\mathcal{L} = \operatorname{Lat}(\mathcal{A}_{\mathcal{L}})$. The class of reflexive lattices includes nests, finite distributive lattices, ABSLs, and $\operatorname{Lat}(\mathcal{A})$ as defined above. An atom of $\mathcal{L}$ is a subspace $A \in \mathcal{L}$ such that $(0) \subseteq L \subset A \Rightarrow L = (0)$. A lattice is called *atomic* if every subspace is generated by the atoms it contains. Nests, ABSLs and finite distributive lattices are all members of the class of strongly reflexive lattices. A lattice is said to be *strongly reflexive* if it is both complete and completely distributive. Such lattices have been studied extensively by Longstaff [14], Lambrou [11], and others. The following characterization of a strongly reflexive lattice will prove useful (see [11]). A lattice $\mathcal{L}$ is strongly reflexive if and only if given $x \in \mathcal{X}$ and $\varepsilon > 0$, there exists a finite rank operator $F \in \mathcal{A}_{\mathcal{L}}$ (depending on $x$) such that $\|Fx - x\| < \varepsilon$.

Finally, we denote the Jacobson radical of an algebra $\mathcal{A}$, *i.e.*, the intersection of the kernels of all algebraically irreducible representations of $\mathcal{A}$, by $\mathrm{J}(\mathcal{A}_{\mathcal{L}})$. A characterization of $\mathrm{J}(\mathcal{A}_{\mathcal{L}})$ we shall find useful is the following: $\mathrm{J}(\mathcal{A}_{\mathcal{L}})$ is the smallest left (right, or two-sided) ideal containing all quasinilpotent left (right, or two-sided) ideals. (See, for example, [15].)

## 3 Existence of Finite Rank Operators in $\mathcal{A}_{\mathcal{L}}$

The following lemma (due to Longstaff [14]) gives a necessary and sufficient lattice-theoretic condition for the existence of rank one operators in $\mathcal{A}_{\mathcal{L}}$.

***Lemma 3.1*** *The rank one operator $\alpha \otimes x$ is in $\mathcal{A}_{\mathcal{L}}$ if and only if there exists $M \in \mathcal{L}$ such that $x \in M$ and $\alpha \in (M_-)^{\perp}$.*

The membership of a rank one operator in the Jacobson radical of $\mathcal{A}_{\mathcal{L}}$ can also be characterized in terms of the lattice. The following is an improvement of a result of Katavolos and Katsoulis [10] (since they require $\mathcal{L}$ to be reflexive).

***Lemma 3.2*** *Let $\alpha \otimes x \in \mathcal{A}_{\mathcal{L}}$. Then $\alpha \otimes x \in \mathrm{J}(\mathcal{A}_{\mathcal{L}})$ if and only if there exists $M \in \mathcal{L}$ such that $x \in M$ and $\alpha \in M^{\perp}$.*

**Proof** First, suppose whenever $M \in \mathcal{L}$ and $x \in M$ that $\alpha \notin M^{\perp}$. Then there exists $M$ such that $x \in M$, $\alpha \in (M_-)^{\perp}$ (by Lemma 3.1, since $\alpha \otimes x \in \mathcal{A}_{\mathcal{L}}$), and $x \notin M_-$. Since $\alpha \in (M_-)^{\perp} \setminus M^{\perp}$, $M \nsubseteq M_-$. Choose $y \in M$ such that $\alpha(y) = 1$, and choose $\beta \in (M_-)^{\perp}$ such that $\beta(x) = 1$. Again by Lemma 3.1, $\beta \otimes y \in \mathcal{A}_{\mathcal{L}}$. Moreover, $\alpha \otimes y = (\beta \otimes y)(\alpha \otimes x) \in \mathcal{A}_{\mathcal{L}}$. But $(\alpha \otimes y)^2 = \alpha \otimes y$. So $(\beta \otimes y)(\alpha \otimes x)$ is not quasinilpotent, whence $\alpha \otimes x$ is not in $\mathrm{J}(\mathcal{A}_{\mathcal{L}})$.

Conversely, suppose $x \in M \in \mathcal{L}$ and $\alpha \in M^{\perp}$. Let $T \in \mathcal{A}_{\mathcal{L}}$. Then $T(\alpha \otimes x) = \alpha \otimes Tx$, and for all $S \in \mathcal{A}_{\mathcal{L}}$, $(\alpha \otimes Tx)(\alpha \otimes Sx) = 0$. Thus the left ideal of $\mathcal{A}_{\mathcal{L}}$ generated by $\alpha \otimes x$ is nilpotent, whence $\alpha \otimes x$ is contained in $\mathrm{J}(\mathcal{A}_{\mathcal{L}})$. ∎

We introduce some terminology. Let $(N, M)$ be a gap in $\mathcal{L}$. If $M \nsubseteq M_-$ (*i.e.*, $M \cap M_- = N \subset M$) then we call $(N, M)$ a *strong gap*. We call $M$ a *branch point* for $\mathcal{L}$ if there are two distinct subspaces, $N_1$ and $N_2$, such that each $(N_j, M)$ is a gap in $\mathcal{L}$. Note that a strong gap cannot occur at a branch point of $\mathcal{L}$, but if $M$ is not a branch point and $(N, M)$ is a gap, then $(N, M)$ need not be a strong gap. This can be seen by considering a five element double-triangle lattice (described in Proposition 5.9 below).

The next two results point out some connections between the existence of finite rank operators (in this case idempotents) and gaps in $\mathcal{L}$. We mention these results do not require reflexivity of $\mathcal{L}$.

***Proposition 3.3*** *If there is a non-zero idempotent in $\mathcal{F}_{\mathcal{L}}$, then there is a gap in $\mathcal{L}$. Moreover, this gap occurs at an $M \in \mathcal{L}$ that is not a branch point.*

**Proof** The first statement is an easy generalization of the proof of Clauss' Lemma 1.2 [3]. Let $0 \neq E = E^2 \in \mathcal{F}_{\mathcal{L}}$. Let $\mathcal{N}$ be any $\mathcal{L}$-maximal nest and let

$$M = \bigcap \{K \in \mathcal{N} : \mathcal{R}(E) \cap K \neq (0)\}.$$

It is easy to check that $\mathcal{R}(E) \cap M \neq (0)$ and that $\mathcal{R}(E) \cap \mathrm{pred}_{\mathcal{N}}(M) = (0)$, since if $L \subset M$ then $EL = (0)$. Thus $\mathrm{pred}_{\mathcal{N}}(M) \neq M$ and so $\big(\mathrm{pred}_{\mathcal{N}}(M), M\big)$ is a gap in $\mathcal{L}$.

Now suppose $M \in \mathcal{L}$ is a branch point of $\mathcal{L}$, *i.e.*, suppose there are two $\mathcal{L}$-maximal nests, $\mathcal{N}_1$ and $\mathcal{N}_2$ with $M_j = \mathrm{pred}_{\mathcal{N}_j}(M)$ $(j = 1, 2)$, and $M_1 \neq M_2$. Then $E(M_j) = 0$ by the way we chose $M$. Since $M_1 \subset M_1 \vee M_2 \subseteq M$, we must have $M_1 \vee M_2 = M$ ($\mathcal{N}_1$ is $\mathcal{L}$-maximal). But then $E(M) = E(M_1 \vee M_2) = 0$, a contradiction. ∎

***Proposition 3.4*** *Let $\mathcal{L}$ be $\vee$-distributive. Then there exists a non-zero finite rank idempotent in $\mathcal{A}_{\mathcal{L}}$ if and only if there exists a strong gap in $\mathcal{L}$.*

**Proof** Let $0 \neq E = E^2 \in \mathcal{F}_{\mathcal{L}}$ and choose $M \in \mathcal{L}$ such that $\dim\big(M \cap \mathcal{R}(E)\big)$ is non-zero and minimal. By the proof of Proposition 3.3, there is a gap $(N, M)$ at $M$. We claim $M \nsubseteq M_-$, whence $(N, M)$ is a strong gap. So assume $M \subseteq M_-$. Then there is a collection $\{K_\alpha\}$ of subspaces in $\mathcal{L}$ with $M \subseteq \vee K_\alpha$ and $M$ is not contained in any individual $K_\alpha$. Then $K_\alpha \cap M \subseteq N$ for each $\alpha$, and

$$M = \Big(\bigvee K_\alpha\Big) \cap M = \bigvee (K_\alpha \cap M) \subseteq N \subset M,$$

a contradiction.

Conversely, suppose there is a strong gap at $M$. Choose $x \in M \setminus M_-$ and $\alpha \in (M_-)^\perp$ such that $\alpha(x) = 1$. By Lemma 3.1, $\alpha \otimes x \in \mathcal{A}_{\mathcal{L}}$, and $(\alpha \otimes x)^2 = \alpha \otimes x$. ∎

The following example shows even when $\mathcal{L}$ is a finite lattice, $\mathcal{F}_{\mathcal{L}}$ can be zero. (Such an $\mathcal{L}$ can not be distributive, since a finite distributive lattice is strongly reflexive. In this case there are plenty of finite rank operators. See, for example, Lambrou [11].)

***Example 3.5*** Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space, and let $A$ be an injective, non-surjective operator on $\mathcal{H}$ with dense range. By a theorem of von Neumann (see [6]), there exists an injective, non-surjective operator $B$ on $\mathcal{H}$ with dense range, satisfying $\mathcal{R}(A) \cap \mathcal{R}(B) = (0)$. Define six subspaces of $\mathcal{X} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ as follows:

$$\begin{aligned} M_1 &= \{(x, y, 0) : x, y \in \mathcal{H}\} & L_1 &= \{(0, 0, x) : x \in \mathcal{H}\} \\ M_2 &= \{(0, x, y) : x, y \in \mathcal{H}\} & L_2 &= \{(0, x, Ax) : x \in \mathcal{H}\} \\ M_3 &= \{(x, 0, y) : x, y \in \mathcal{H}\} & L_3 &= \{(x, 0, Bx) : x \in \mathcal{H}\}. \end{aligned}$$

One easily checks the lattice $\mathcal{L}$ consisting of these subspaces, together with $L_2 \vee L_3$, $M_1 \cap M_3$, $M_1 \cap M_2$, $(0)$, and $\mathcal{X}$, is complete.

Now, suppose $F \in \mathcal{F}_{\mathcal{L}}$. Since $\overline{M_1 + L_2} = M_1 \vee L_2 = \mathcal{X}$,

$$F(\mathcal{X}) = F(\overline{M_1 + L_2}) \subseteq \overline{F(M_1 + L_2)} = F(M_1 + L_2) \subseteq M_1 + L_2.$$

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Likewise, $F(\mathcal{X}) \subseteq M_1 + L_3$. Thus $F(L_2) \subseteq L_2 \cap (M_1 + L_3)$, and $F(L_3) \subseteq L_3 \cap (M_1 + L_2)$.

We claim $L_2 \cap (M_1 + L_3) = (0)$ (and similarly, $L_3 \cap (M_1 + L_2) = (0)$). For if $(0, x, Ax) \in L_2 \cap (M_1 + L_3)$ then there must exist $(w, z, 0) \in M_1$ and $(y, 0, By) \in L_3$ such that $(0, x, Ax) = (w, z, 0) + (y, 0, By)$. Then $Ax = By \in \mathcal{R}(A) \cap \mathcal{R}(B) = (0)$, and so $Ax = x = 0$, since $A$ is injective.

We have shown $F(L_2) = F(L_3) = (0)$. Since $\mathcal{X} = M_1 \vee (L_2 \vee L_3)$, $F(\mathcal{X}) \subseteq M_1$. But $\mathcal{X} = L_1 \vee (L_2 \vee L_3)$ as well. So $F(\mathcal{X}) \subseteq L_1$. But then $F(\mathcal{X}) \subseteq L_1 \cap M_1 = (0)$, *i.e.*, $F = 0$. ■

## 4 Decomposition of Finite Rank Operators in $\mathcal{A}_{\mathcal{L}}$

The following theorem is the main result of this section. It gives a lattice condition for when a finite rank operator is decomposable.

***Theorem 4.1*** *Let $F \in \mathcal{F}_{\mathcal{L}}$. $F$ is decomposable if and only if there is a subspace $M \in \mathcal{L}$ such that $\mathcal{R}(F) \cap M \nsubseteq F(M_-)$.*

**Proof** Suppose $\mathcal{R}(F) \cap M \nsubseteq F(M_-)$. Let $x_1 \in \mathcal{R}(F) \cap M \setminus F(M_-)$. Let $\{x_2, \dots, x_m\}$ be a basis for $F(M_-)$, and use $x_1$ and, if necessary, additional vectors $x_{m+1}, \dots, x_n$ to form a basis $\{x_1, \dots, x_n\}$ for $\mathcal{R}(F)$. (We are assuming $\dim \mathcal{R}(F) = n$.) We claim $x_1$ is not in the closed subspace $F(M_-) \vee \langle x_2, \dots, x_n \rangle = F(M_-) + \langle x_2, \dots, x_n \rangle$. This is clear since the $x_j$'s are linearly independent.

Now, by the Hahn-Banach Theorem, we can choose linear functionals

$$\alpha_1 \in \big(F(M_-) \vee \langle x_2, \dots, x_n \rangle\big)^{\perp},$$

with $\alpha_1(x_1) = 1$, and $\alpha_2, \dots, \alpha_n$ such that $\alpha_i(x_j) = \delta_{ij}$ for all $1 \le i, j \le n$. If $x \in \mathcal{X}$ then $Fx = \sum_{j=1}^{n} \lambda_j x_j$ for some complex numbers $\lambda_j$, and, for each $1 \le k \le n$,

$$\alpha_k(Fx) = \alpha_k\Big(\sum_{j=1}^{n} \lambda_j x_j\Big) = \sum_{j=1}^{n} \lambda_j \alpha_k(x_j) = \lambda_k.$$

So $Fx = \sum_{j=1}^{n} \alpha_j(Fx) x_j$, whence $F = \sum_{j=1}^{n} (\alpha_j \circ F) \otimes x_j$. But $\alpha_1 \circ F \in (M_-)^{\perp}$ since $\alpha_1 \in F(M_-)^{\perp}$. By Longstaff's Lemma (3.1), $\alpha_1 \circ F \otimes x_1 \in \mathcal{A}_{\mathcal{L}}$.

Now suppose $F$ is decomposable, *i.e.*, suppose $F = \alpha_1 \otimes x_1 + R$ where $\alpha_1 \otimes x_1$ and $R$ are in $\mathcal{F}_{\mathcal{L}}$. We may assume the range of $R$ and $\langle x_1 \rangle$ are linearly independent subspaces. By Lemma 3.1, there exists $M \in \mathcal{L}$ such that $x_1 \in M$ and $\alpha_1 \in (M_-)^{\perp}$. Suppose there is a vector $y \in M_-$ with $Fy = x_1$. Then $x_1 = Fy = \alpha_1(y)x_1 + Ry$. But then $\alpha_1(y) = 1$ (and $Ry = 0$), impossible, since $\alpha_1 \in (M_-)^{\perp}$. Thus $\mathcal{R}(F) \cap M \nsubseteq F(M_-)$ ■

***Corollary 4.2*** **(Spanoudakis)** *If $\mathcal{N}$ is a nest and $F \in \mathcal{F}_{\mathcal{N}}$ then $F$ is completely decomposable.*

**Proof** It is enough to show every $F \in \mathcal{F}_{\mathcal{N}}$ is decomposable, since then we can write $F = F_1 + F_2$ where $F_1$ is rank one, $\dim \mathcal{R}(F_2) = \dim \mathcal{R}(F) - 1$, and $F_2$ is itself decomposable. Let $M = \bigcap\{N \in \mathcal{N} : \mathcal{R}(F) \cap N \neq (0)\}$. Since $\mathcal{R}(F)$ is finite dimensional and $\mathcal{N}$ is totally ordered, $M \neq (0)$. If $N \subset M$ then $F(N) \subseteq N \cap \mathcal{R}(F) = (0)$. Thus $F(M_-) = (0)$, and we may apply Theorem 4.1. ■

Suppose $\mathcal{L}$ is atomic, and $F \in \mathcal{F}_{\mathcal{L}}$ is non-zero. If $F(A) = (0)$ for all atoms $A \in \mathcal{L}$, then $F = 0$. So there must exist an atom $A$ with $F(A) \neq (0)$. If in addition $A \nsubseteq A_-$, then $F(A_-) \cap A \subseteq A_- \cap A = (0)$. In this case we can apply Theorem 4.1. In particular, if every atom in $\mathcal{L}$ satisfies $A \nsubseteq A_-$, then every $F \in \mathcal{F}_{\mathcal{L}}$ is decomposable, hence completely decomposable. These observations prove the following two results.

***Corollary 4.3*** **(Lambrou)** *If $\mathcal{L}$ is an* ABSL *and $F \in \mathcal{F}_{\mathcal{L}}$ then $F$ is completely decomposable.*

***Corollary 4.4*** *If $\mathcal{L}$ is a pentagon lattice (see 5.10 below) and $F \in \mathcal{F}_{\mathcal{L}}$ then $F$ is completely decomposable.*

The next result is a partial decomposition result for finite rank operators in $\mathcal{A}_{\mathcal{L}}$ where $\mathcal{L}$ is $\vee$-distributive.

***Proposition 4.5*** *Suppose $\mathcal{L}$ is $\vee$-distributive and $F \in \mathcal{F}_{\mathcal{L}}$. Then $F$ can be written as $F = F_1 + F_2$ where $F_1$ and $F_2$ are in $\mathcal{F}_{\mathcal{L}}$, $F_1$ is completely decomposable, and $F_2$ is nilpotent.*

**Proof** It is enough to show every non-nilpotent finite rank operator in $\mathcal{A}_{\mathcal{L}}$ is decomposable. So assume $F$ is not nilpotent. Let $\mathcal{E}$ be the linear span of all eigenvectors corresponding to non-zero eigenvalues of $F$. Choose $M' \in \mathcal{L}$ such that $\dim(M' \cap \mathcal{E})$ is non-zero and minimal. Let $M = \bigcap\{L \in \mathcal{L} : \mathcal{E} \cap M' \subseteq L\}$. Then if $L \in \mathcal{L}$ and $L \subset M$, $L$ does not contain an eigenvector for $F$ with nonzero eigenvalue.

We claim $M \nsubseteq M_-$. To see this, assume the contrary. We will reach a contradiction. If $M \subseteq M_-$ then there are subspaces $\{K_\alpha\}$ in $\mathcal{L}$ with $M \nsubseteq K_\alpha$ for each $\alpha$, and

$$M \subseteq M \cap M_- = M \cap \Big(\bigvee K_\alpha\Big) = \bigvee (M \cap K_\alpha).$$

Thus $M = \bigvee\{L \in \mathcal{L} : L \subset M\}$. Let $x \in M \cap \mathcal{E}$, *i.e.*, $x = x_1 + \cdots + x_n$ where each $x_j$ is an eigenvector for $F$ with nonzero eigenvalue. Then $F^m x \neq 0$ for all $m > 0$. This is true since if $Fx_j = \lambda_j x_j$ for some complex numbers $\lambda_j \neq 0$ (the eigenvalues corresponding to the $x_j's$) then

$$F^m x = \sum_{j=1}^{n} F^m x_j = \sum_{j=1}^{n} \lambda_j^m x_j.$$

But

$$F(M) = F\Big(\overline{\sum\{L \subset M\}}\Big) \subset \overline{F\Big(\sum\{L \subset M\}\Big)} = F\Big(\sum\{L \subset M\}\Big),$$

since $F$ is finite rank. Thus $Fx \in \sum\{L \in \mathcal{L} : L \subset M\}$. So we may write $Fx = y_1 + \cdots + y_m$ where each $y_j \in L_j \subset M$. Now, the restriction of $F$ to each $L_j$ is nilpotent, by the definition of $M$ (since $L_j \subset M$ can not contain an eigenvector corresponding to a non-zero eigenvalue of $F$). Hence there exists $r > 0$ such that $F^r(y_j) = 0$ for all $j$. But then we have

$$0 \neq F^{r+1}x = F^r(y_1 + \cdots + y_m) = 0,$$

a contradiction. This proves the claim.

Now for $x \in M \cap \mathcal{E}$, if there is a vector $y \in M_-$ with $Fy = x$, then $x \in M \cap M_- \subset M$, contradicting the definition of $M$. So $F(M_-) \nsupseteq \mathcal{R}(F) \cap M$. Apply Theorem 4.1. ■

## 5 Applications

Throughout this section we assume each lattice $\mathcal{L}$ is reflexive, *i.e.*, $\mathcal{L} = \mathrm{Lat}\big(\mathrm{Alg}(\mathcal{L})\big)$. Recall, if $\mathcal{L} = \mathrm{Lat}(\mathcal{A})$, then $\mathcal{L}$ is reflexive. Let $(N, M)$ be a gap in $\mathcal{L}$. We can define (as in [3]) a representation $\pi_{M/N} \colon \mathcal{A}_{\mathcal{L}} \to B(M/N)$, called a *quotient representation*, by restriction of an operator $T$ in $\mathcal{A}_{\mathcal{L}}$ to the subspace $M$ and then reduction to the quotient space $M/N$. Thus for $T \in \mathcal{A}_{\mathcal{L}}$,

$$\pi_{M/N}(T)(x + N) = Tx + N \qquad (x \in M).$$

These representations (one for each gap in $\mathcal{L}$) are continuous in the norm topology on $\mathcal{A}_{\mathcal{L}}$. Note that $T \in \ker(\pi_{M/N})$ precisely when $TM \subseteq N$. It is not difficult to check $\ker(\pi_{M/N})$ is a weakly closed, two-sided ideal of $\mathcal{A}_{\mathcal{L}}$. Moreover, by virtue of the reflexivity of $\mathcal{L}$, there are no closed $\mathcal{A}_{\mathcal{L}}$-invariant subspaces in $\mathcal{L}$ contained properly between $N$ and $M$. Thus $\pi_{M/N}$ is a topologically irreducible representation, and so $\ker(\pi_{M/N})$ is a prime ideal. (Recall that an ideal $\mathcal{P}$ is prime if whenever $\mathcal{I}$ and $\mathcal{J}$ are ideals with $\mathcal{I}\mathcal{J} \subseteq P$ then one of $\mathcal{I}$ or $\mathcal{J}$ must be contained in $\mathcal{P}$.) To see that $\ker(\pi_{M/N})$ is prime suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{A}_{\mathcal{L}}$ and that their product $\mathcal{I}\mathcal{J}$ is contained in $\ker(\pi_{M/N})$, but $\mathcal{J} \nsubseteq \ker(\pi_{M/N})$. Then $\overline{\pi_{M/N}(\mathcal{J})(M/N)} = M/N$, since $\pi_{M/N}$ is topologically irreducible. So

$$\begin{aligned}\pi_{M/N}(\mathcal{I})(M/N) &= \pi_{M/N}(\mathcal{I})\Big(\overline{\pi_{M/N}(\mathcal{J})(M/N)}\Big)\\ &\subseteq \overline{\pi_{M/N}(\mathcal{I}\mathcal{J})(M/N)} = (0).\end{aligned}$$

Thus $\mathcal{I}$ is contained in $\ker(\pi_{M/N})$.

Our first result allows us to characterize the spectrum of a Riesz operator in terms of its invariant subspaces. It is a generalization of theorems of Ringrose [17], and Clauss [3]. Note that $\mathcal{L}$ can be taken to be $\mathrm{Lat}(T)$.

***Theorem 5.1*** *Let $T \in \mathcal{A}_{\mathcal{L}}$ be a Riesz operator. Then*

$$\sigma(T) \cup \{0\} = \bigcup \{\sigma\big(\pi_{M/N}(T)\big) : (N, M) \textit{ is a gap in } \mathcal{L}\} \cup \{0\}.$$

**Proof** If $T$ is invertible in $\mathcal{A}_{\mathcal{L}}$, and $(N, M)$ is a gap in $\mathcal{L}$, then we have $\pi_{M/N}(T^{-1}) = \pi_{M/N}(T)^{-1}$. So $\pi_{M/N}(T)$ is also invertible. Thus $\sigma\big(\pi_{M/N}(T)\big) \subseteq \sigma_{\mathcal{A}_{\mathcal{L}}}(T)$. Moreover, since $\mathcal{A}_{\mathcal{L}}$ is a closed subalgebra of $B(\mathcal{X})$, and since $\sigma(T)$ is equal to its own boundary, standard Banach algebra theory tells us $\sigma_{\mathcal{A}_{\mathcal{L}}}(T) = \sigma(T)$. So $\sigma\big(\pi_{M/N}(T)\big) \subseteq \sigma(T)$.

Conversely, suppose $0 \neq \lambda \in \sigma(T)$. Let $\mathcal{N}$ be any maximal nest in $\mathcal{L}$. By Theorem 3.13 of [17], there is a finite rank idempotent in $\mathcal{A}_{\mathcal{N}}$ with range $K = \ker(\lambda - T)^n$, where $n < \infty$ is the smallest positive integer such that $\ker(\lambda - T)^n = \ker(\lambda - T)^{n+1}$ (the ascent of $\lambda - T$). By the proof Proposition 3.3, there exists a gap in $\mathcal{N}$ at

$$M = \bigcap \{L \in \mathcal{N} : L \cap K \neq (0)\}.$$

Let $N = \mathrm{pred}_{\mathcal{N}}(M)$. Choose $z \in M \cap \ker(\lambda - T)^n$. Then $z \notin N$. Let $d < n$ be the smallest integer such that $w = (\lambda - T)^d z \neq 0$ but $(\lambda - T)^{d+1} z = 0$. Since

$$w \in \ker(\lambda - T) \subseteq \ker(\lambda - T)^n,$$

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and since $\ker(\lambda - T)^n \cap N = (0)$, $w \notin N$. But $w \in M$. Thus $\big(\pi_{M/N}(\lambda - T)\big)(w + N) = N$ whence $\lambda \in \sigma\big(\pi_{M/N}(T)\big)$. ■

Suppose $\mathcal{N}$ is a nest and $\mathcal{A} = \mathcal{A}_{\mathcal{N}}$. By Corollary 4.2, if $T \in \mathrm{J}(\mathcal{A}) \cap \mathcal{F}(\mathcal{A})$ then we may write $T = \sum_{j=1}^{n} \alpha_j \otimes x_j$, where each $\alpha_j \otimes x_j$ is in $\mathrm{J}(\mathcal{A})$. By Lemma 3.2, each $\alpha_j \otimes x_j \in \ker(\pi_{M/M_-})$ whenever there is a gap at $M$. Hence $T$ is also in $\ker(\pi_{M/M_-})$. A much more general statement is true, however, and forms the basis for the rest of this section. It was suggested to me by Bruce A. Barnes.

***Lemma 5.2*** *Let $\mathcal{A}$ be a unital subalgebra of $B(\mathcal{X})$ and let $\mathcal{P}$ be a prime ideal in $\mathcal{A}$. Then* $\mathrm{J}(\mathcal{A}) \cap \mathcal{F}(\mathcal{A}) \subseteq \mathcal{P}$.

**Proof** Let $T$ be in $\mathrm{J}(\mathcal{A}) \cap \mathcal{F}(\mathcal{A})$, and let $n = \dim\big(\mathcal{R}(T)\big)$. Then the ideal $\mathcal{A}T\mathcal{A}$ generated by $T$ is contained in $\mathrm{J}(\mathcal{A}) \cap \mathcal{F}(\mathcal{A})$, and for each $S \in \mathcal{A}T\mathcal{A}$, $\dim\big(\mathcal{R}(S)\big) \leq n$. We claim $S^{n+1} = 0$ for all $S \in \mathcal{A}T\mathcal{A}$. Indeed, if we choose a basis for $\mathcal{R}(S)$ with respect to which the restriction of $S$ is an upper triangular matrix, then the diagonal elements are each zero, since $S$ has zero spectrum. By the Nagata-Higman Theorem (Proposition 4.4.10 of [15]), $\mathcal{A}T\mathcal{A}$ is a nilpotent ideal. Thus there exists $m$ such that $(\mathcal{A}T\mathcal{A})^m = (0) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is prime, $\mathcal{A}T\mathcal{A} \subseteq \mathcal{P}$, and since $I \in \mathcal{A}$, we must have $T \in \mathcal{P}$. ■

An immediate consequence of this lemma, and our main use for it, is the following: any prime ideal which does not contain all the finite rank operators of $\mathcal{A}$ must contain $\mathrm{J}(\mathcal{A})$. The kernels of representations on gaps are the necessary prime ideals. We first apply this lemma to algebras of Riesz operators. If $\mathcal{A}$ is not unital, let $\mathcal{A}_u$ be the unitization of $\mathcal{A}$. That is, $\mathcal{A}_u = \mathbf{C}I \oplus \mathcal{A}$. The following is stated for convenience. Its proof is elementary.

***Proposition 5.3*** *Let $\mathcal{A}$ be an algebra of operators. Then $\mathcal{F}(\mathcal{A}) = \mathcal{F}(\mathcal{A}_u)$, $\mathrm{J}(\mathcal{A}) = \mathrm{J}(\mathcal{A}_u)$, and* $\mathrm{Lat}(\mathcal{A}) = \mathrm{Lat}(\mathcal{A}_u)$.

The following proposition is of independent interest. Its proof contains a key idea in the proof of our next main result.

***Proposition 5.4*** *Suppose $\mathcal{A}$ is a closed algebra of Riesz operators, and $(N, M)$ is a gap in* $\mathrm{Lat}(\mathcal{A})$. *Then $\mathcal{A}/\ker \pi_{M/N}$ is semisimple.*

**Proof** Write $\pi = \pi_{M/N}$. Let $T \in \mathcal{F}(\mathcal{A}_u)$ with $\pi(T) \in \mathrm{J}\big(\pi(\mathcal{A}_u)\big)$. Since $\pi(T)$ is a finite-rank operator, $\pi(T^n) = \pi(T)^n = 0$, where $n = \dim\big(\mathcal{R}(T)\big)$. Thus $T^n \in \ker \pi$. But every operator $RTS \in \mathcal{A}_u T \mathcal{A}_u$ is rank-$n$ or less and in $\mathrm{J}(\mathcal{A}_u)$, so $(RTS)^n \in \ker \pi$ for all $RTS$. By the Nagata-Higman Theorem, $\pi(\mathcal{A}_u T \mathcal{A}_u)$ is a nilpotent ideal in $\pi(\mathcal{A}_u)$. Since $\ker \pi$ is a prime ideal, $\mathcal{A}_u T \mathcal{A}_u$ is contained in $\ker \pi$. Thus $T \in \ker \pi$. We have shown

$$\mathcal{F}(\mathcal{A}_u) \cap \Big(\mathrm{J}\big(\pi(\mathcal{A}_u)\big)\Big)^{-1} \subseteq \ker \pi.$$

Now suppose that $\mathcal{F}(\mathcal{A}_u) \subseteq \ker \pi$. Then $\pi(\mathcal{A}_u) = \mathrm{J}\big(\pi(\mathcal{A}_u)\big)$. (For if $\pi(T) \notin \mathrm{J}\big(\pi(\mathcal{A}_u)\big)$ then there exists $S$ such that $\pi(ST)$ is not quasinilpotent, whence there exists a finite rank spectral idempotent $E \in \mathcal{A}_u$ with $\pi(E) \neq 0$. But then $\mathcal{F}(\mathcal{A}_u) \not\subseteq \ker \pi$.) So, by a theorem

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of Barnes and Katavolos [2, Theorem 3.1], $\pi(\mathcal{A}_u)$ is simultaneously triangularizable. This implies $\dim(M/N) = 1$, which forces $\pi(\mathcal{A}_u)$ to be (0). So in this case $\pi(\mathcal{A}_u) = \mathcal{A}_u/\ker\pi$ is certainly semisimple. On the other hand, if $\mathcal{F}(\mathcal{A}_u) \nsubseteq \ker\pi$ then, since $\ker\pi$ is prime, $\Big(\mathrm{J}\big(\pi(\mathcal{A}_u)\big)\Big)^{-1} \subseteq \ker\pi$. That is, $\mathrm{J}\big(\pi(\mathcal{A}_u)\big) = (0)$. ■

Our first application of Lemma 5.2 is to characterize the Jacobson radical of closed algebras of Riesz operators in $B(\mathcal{X})$.

***Theorem 5.5*** *Let $\mathcal{A}$ be a closed algebra of Riesz operators. Then*

$$\mathrm{J}(\mathcal{A}) = \bigcap\{\ker\pi_{M/N} : (N,M) \textit{ is a gap in } \mathrm{Lat}(\mathcal{A})\}.$$

**Proof** By Theorem 5.1, and since $\mathrm{Lat}(\mathcal{A}) \subseteq \mathrm{Lat}(T)$ for all $T \in \mathcal{A}$,

$$\bigcap\{\ker\pi_{M/N} : (N,M) \text{ is a gap in } \mathrm{Lat}(\mathcal{A})\}$$

is a quasinilpotent ideal, hence contained in $\mathrm{J}(\mathcal{A})$. Conversely, since $\ker\pi_{M/N}$ is a prime ideal, we have $\mathcal{F}(\mathcal{A}_u) \cap \mathrm{J}(\mathcal{A}_u) \subseteq \ker\pi_{M/N}$. Now if $\mathcal{F}(\mathcal{A}_u) = \mathcal{F}(\mathcal{A}) \subseteq \ker\pi_{M/N}$ then, as in the proof of Proposition 5.4, $\pi_{M/N}(\mathcal{A}) = (0)$, whence $\mathrm{J}(\mathcal{A}) \subseteq \ker\pi_{M/N}$. On the other hand, if $\mathcal{F}(\mathcal{A}_u) \nsubseteq \ker\pi_{M/N}$ then $\mathrm{J}(\mathcal{A}_u) = \mathrm{J}(\mathcal{A}) \subseteq \ker\pi_{M/N}$. ■

Next we apply Lemma 5.2 to $\mathcal{A}_\mathcal{L}$ for various reflexive lattices $\mathcal{L}$.

***Proposition 5.6*** *If $\mathcal{L}$ is strongly reflexive then $\mathrm{J}(\mathcal{A}_\mathcal{L}) \subseteq \ker(\pi_{M/N})$ for every gap $(N,M)$ in $\mathcal{L}$.*

**Proof** Suppose $\mathcal{F}_\mathcal{L} \subseteq \ker\pi_{M/N}$ where $(N,M)$ is a gap in $\mathcal{L}$. Since $\mathcal{L}$ is strongly reflexive, we may apply a theorem of Lambrou [11] which states, given $x \in M \setminus N$ and $\varepsilon > 0$, there exists $F \in \mathcal{F}_\mathcal{L}$ such that $\|Fx - x\| < \varepsilon$. But $F \in \ker\pi_{M/N}$ by assumption, so $Fx \in N$. This implies, since $N$ is closed, that $x$ is in $N$, a contradiction. Thus $\mathcal{F}_\mathcal{L} \nsubseteq \ker\pi_{M/N}$. By Lemma 5.2, $\mathrm{J}(\mathcal{A}_\mathcal{L}) \subseteq \ker(\pi_{M/N})$. ■

***Corollary 5.7*** *Let $\mathcal{L}$ be a finite distributive lattice. Then $\mathrm{J}(\mathcal{A}_\mathcal{L}) = \bigcap\{\ker\pi_{M/N} : (N,M)$ is a gap in $\mathcal{L}\}$.*

**Proof** Let $\mathcal{J} = \bigcap\{\ker\pi_{M/N} : (N,M) \text{ is a gap in } \mathcal{L}\}$. Since $\mathcal{L}$ is finite, $\mathcal{J}$ is a nilpotent ideal, hence contained in $\mathrm{J}(\mathcal{A}_\mathcal{L})$. Since a finite distributive lattice is strongly reflexive, the converse follows from Proposition 5.6. ■

The next corollary strengthens a result of Halmos [8], who proves it for the case where $\mathcal{L}$ is an ABSL.

***Corollary 5.8*** *If $\mathcal{L}$ is strongly reflexive and atomic then $\mathcal{A}_\mathcal{L}$ is semisimple. (The class of such lattices includes* ABSL*s.)*

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**Proof** Note there is a gap below each atom $A \in \mathcal{L}$. Since $\mathcal{L}$ is strongly reflexive, if $T$ is in $\mathrm{J}(\mathcal{A}_{\mathcal{L}})$ then $TA = (0)$ for all atoms, whence

$$T(\mathcal{X}) = T\Big(\bigvee\{A : A \text{ is an atom }\}\Big) = (0),$$

*i.e.*, $T = 0$. ■

We remark that while distributive lattices are reflexive, the converse may not hold. The ensuing two propositions involve the so-called double triangle and pentagon lattices. These play a special role in lattice theory because of their relationships to the conditions of distributivity and modularity of a lattice.

***Proposition 5.9*** **(Longstaff, [12])** *A double-triangle lattice is a five element lattice $\mathcal{L} = \{(0), K, L, M, \mathcal{X}\}$ such that $K \cap L = K \cap M = L \cap M = (0)$ and $K \vee L = K \vee M = L \vee M = \mathcal{X}$. Suppose further that the vector sum $K+L$ is closed, i.e., $K+L = K \vee L$. Then $\mathcal{A}_{\mathcal{L}}$ is semisimple.*

**Proof** Let $Q$ be a projection onto $K$ along $L$. (We do not assert that $Q \in \mathcal{A}_{\mathcal{L}}$.) Choose $x \in M$ such that $Qx \neq 0$. Choose $\alpha \in M^{\perp}$ such that $\alpha(Qx) \neq 0$. Let

$$R = \big(\alpha \circ (I - Q)\big) \otimes x - \alpha \otimes Qx = \alpha \otimes (I - Q)x - (\alpha \circ Q) \otimes x.$$

We check that $R \in \mathcal{A}_{\mathcal{L}}$: If $y \in K$ then

$$Ry = \alpha(y - Qy)x - \alpha(y)Qx = -\alpha(y)Qx \in K$$

(since $(I - Q)y = 0$.) If $y \in L$ then

$$Ry = \alpha(y)(x - Qx) - \alpha(Qy)x = \alpha(y)(I - Q)x \in L$$

(since $Qy = 0$.) Finally, if $y \in M$ then

$$Ry = \alpha(y - Qy)x - \alpha(y)Qx = \alpha(y - Qy)x \in M$$

(since $\alpha(y) = 0$.) So $R \in \mathcal{A}_{\mathcal{L}}$, and indeed in $\mathcal{F}_{\mathcal{L}}$. Now, $RQx = -\alpha(Qx)Qx \neq 0$, and

$$R(I - Q)x = \alpha\big((I - Q)x\big)(x - Qx) = -\alpha(Qx)(x - Qx) \neq 0.$$

So $\mathcal{F}_{\mathcal{L}} \nsubseteq \ker \pi_{K/(0)}$ and $\mathcal{F}_{\mathcal{L}} \nsubseteq \ker \pi_{L/(0)}$. Therefore $\mathrm{J}(\mathcal{A}_{\mathcal{L}}) \subseteq \ker \pi_{K/(0)} \cap \ker \pi_{L/(0)} = (0)$. ■

***Proposition 5.10*** **(Longstaff, [12])** *A pentagon lattice is a five element lattice $\mathcal{L} = \{(0), K, L, M, \mathcal{X}\}$ where $L \subset K$, $K \cap M = L \cap M = (0)$ and $K \vee M = L \vee M = (0)$. If $\mathcal{L}$ is a pentagon then $\mathcal{A}_{\mathcal{L}}$ is semisimple.*

**Proof** Since $L_{-} = M$ and $M_{-} = K$, there are strong gaps below $M$ and $L$. Thus, by the proof of Proposition 3.4, there are finite rank operators not contained in $\ker \pi_{M/(0)}$, and finite rank operators not contained in $\ker \pi_{L/(0)}$. By Lemma 5.2, $\mathrm{J}(\mathcal{A}_{\mathcal{L}}) \subseteq \ker \pi_{M/(0)} \cap \ker \pi_{L/(0)} = (0)$. ■

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