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RESEARCH ARTICLE

Geometric filling curves on punctured surfaces

Nhat Minh Doan*

Department of Mathematics, University of Luxembourg, Esch-sur-Alzette, Luxembourg and Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam
E-mail: dnminh@math.ac.vn

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Abstract
This paper is about a type of quantitative density of closed geodesics and orthogeodesics on complete finite-area hyperbolic surfaces. The main results are upper bounds on the length of the shortest closed geodesic and the shortest doubly truncated orthogeodesic that are ε-dense on a given compact set on the surface.

1. Introduction

It is well known that if X is a complete finite-area hyperbolic surface, the set of closed geodesics is dense on X and on the unit tangent bundle of X (see e.g. [4] or [7]). A type of quantitative density of closed geodesics on closed hyperbolic surfaces was investigated by Basmajian, Parlier, and Souto in [3]. In particular, for any closed hyperbolic surface X and any positive number ε, they found an upper bound on the length of the shortest closed geodesic that is ε-dense on X, by which it is meant that all points of X are at a distance at most ε from the geodesic. This upper bound is recently used to estimate the complexity of an algorithm of tightening curves and graphs on surfaces [6]. Our goal is to extend their results to the case of complete finite-area hyperbolic surfaces in two directions. The first is that for any complete finite-area hyperbolic surface and any positive number ε less than or equal to 2, we are going to construct a closed geodesic γ_ε so that γ_ε is ε-dense on a given compact set of the surface and its length is bounded above by a quantity that depends on the geometry of X and ε. The second is that we will construct a doubly truncated orthogeodesic that is ε-dense and also of bounded length. These types of orthogeodesics appear for instance in identities [9] related to McShane's identity [8] and Basmajian's identity [1].

Let us begin with a few necessary notations so that one can understand the statement of the main results. Let $\mathcal{M}_{g,n}$ be the moduli space of complete connected orientable finite-area hyperbolic surfaces of genus g and n cusps. For any X in $\mathcal{M}_{g,n}$ and any positive number $\xi \leq 2$, we define X^ξ as a subset of X such that each connected component of the boundary of X^ξ is a horocycle of length ξ. A geodesic arc on X is called a doubly truncated orthogeodesic on X^ξ if it is perpendicular to the horocyclic boundary of X^ξ at its endpoints. Our main results are the following.

Theorem A. *For all $X \in \mathcal{M}_{g,n}$ there exists a constant $C_X > 0$ such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq 2$ there exists a closed geodesic γ_ε that is ε-dense on X^ξ and such that*

$$\ell(\gamma_\varepsilon) \leq C_X \frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\xi}\right).$$

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Theorem B. *Let $X \in \mathcal{M}_{g,n}$, there exists a constant $D_X > 0$ such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq \min\left\{2\log\frac{1}{\xi}, 2\right\}$ there exists a doubly truncated orthogeodesic $\mathcal{O}_\varepsilon$ that is ε-dense on X^ξ and such that*

$$\ell(\mathcal{O}_\varepsilon) \leq D_X \frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\xi}\right).$$

Our main ingredient in the proof of Theorem A and Theorem B is the following result.

Theorem 1. *For any $X \in \mathcal{M}_{g,n}$, there exists a constant K_X such that the following holds. For all $0 < \varepsilon \leq 1$, $0 < \xi \leq 1$ and any finite collection $\{c_i\}_{i=1}^N$ of geodesic arcs of average length $\bar{c}$ in X^ξ, there exist a closed geodesic γ of length at most*

$$N\left(K_X + \bar{c} + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi}\right)$$

containing $\{c_i\}_{i=1}^N$ in its 2ε-neighborhood.

In the following, we will give a brief outline of the arc-replacement idea in [3] for the case of closed surfaces and explain how we adapt it to the case of punctured surfaces in the proof of Theorem 1.

Let X be a complete finite-area hyperbolic surface. When X is a closed surface, the steps in the proof of [3, Theorem 2.4] can be described briefly as follows.

(i) Taking a filling closed geodesic γ_0 on X which decomposes X into polygons.

(ii) For any $\varepsilon > 0$, we take a finite collection $\mathcal{A}_N := \{c_i\}_{i=1}^N$ of geodesic arcs such that the ε-neighborhood of $\mathcal{A}_N$ covers the whole surface X. The number of arcs in this collection is roughly $\frac{1}{\varepsilon}$ up to a constant depending on X.

(iii) Extending these arcs in both directions a certain distance r_ε of roughly $\log\frac{1}{\varepsilon}$ and then keep extending them (at most a distance D the diameter of X) until they connect to γ_0 with good angles.

(iv) Constructing a closed piecewise geodesic forming from the collection of extended arcs and suitable subarcs of the filling closed geodesic γ_0. The resulting closed piecewise geodesic is contained in the ε-neighborhood of the desired closed geodesic γ_ε. The length of γ_ε is bounded above by $C_X\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon}\right)$ where C_X is a constant depending on X.

When X is a punctured surface, we identify the main obstruction and propose the key idea in the proof of Theorem 1 step by step as follows.

- Similar to steps (i) and (ii) above, we take a filling closed geodesic γ_0 on S such that γ_0 cuts S into polygons and once-punctured polygons. For any $\varepsilon > 0$, we take a finite collection $\{c_i\}_{i=1}^N$ of geodesic arcs on the truncated surface S^ξ which contains S^ξ in its ε-neighborhood.
- Major obstruction: it is almost surely possible to extend these arcs in both directions a certain distance r_ε and then keep extending them until they connect to γ_0 with good angles, but it is not always possible to bound the lengths of these extended arcs since they can go into a cusp region for a long time before connecting to γ_0.
- Key idea: we replace the collection $\{c_i\}_{i=1}^N$ by a better one in the sense that the new collection, denoted by $\{\zeta_i\}_{i=1}^N$, still contains $\{c_i\}_{i=1}^N$ in its ε-neighborhood and when we extend them, they will not go too deep into the cusp region before connecting to γ_0 with sufficiently big angles. Denote the collection of these extended arcs by $\{\zeta_i'\}_{i=1}^N$.
- Applying the step (iv) above for the collection $\{\zeta_i'\}_{i=1}^N$.

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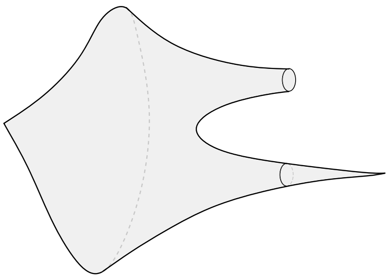
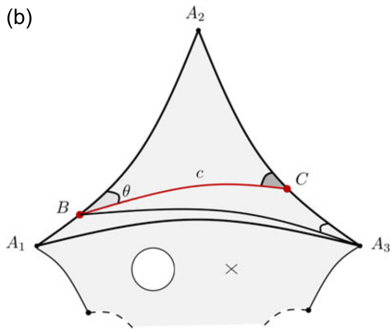


An example of P_1.



The geodesic arc c in the triangle $A_1A_2A_3$.

Figure 1.

2. Geodesics and horocycles in the hyperbolic plane $\mathbb{H}$ and on surfaces

In this section, we introduce some elementary properties of geodesics traveling through subsurfaces which we will use to prove Theorem 1. Let P_n be a hyperbolic subsurface with a single polygonal boundary of n concatenated geodesic edges such that all angles are less than π. Figure 1(a) shows an example of P_1. The following lemma is an extended version of Lemma 1 in [3].

Lemma 1. *There exists $\theta_P > 0$ such that any geodesic arc c lying inside P_n with endpoints on edges of the n-gons forms an acute angle of at least θ_P in one of its endpoints. Furthermore, the length of c is at most a constant ℓ_P if one of the angles has value less than or equal θ_P.*

Proof. We first label the vertices of the n-gonal boundary of P_n by $A_1, A_2, ..., A_n$ consecutively. For each $i \in \{1, 2, ..., n\}$, we can connect A_i to A_{i+2} by a shortest geodesic arc lying inside the interior of P_n (in which $A_{n+1} := A_1$ and $A_{n+2} := A_2$) such that there is no cusp or geodesic boundary component in the resulting triangle $A_iA_{i+1}A_{i+2}$. We call each such resulting triangle to be an ear of P_n. In the set of angles of the ears in P_n (three angles for each ear), we denote by θ_P their minimum value. Also, in the set of sides of the ears in P_n, we denote by ℓ_P their maximum value.

Without loss of generality, we can assume that the geodesic arc c leaving from the point B on the interior of a side A_1A_2 of the n-gons forms an angle θ of at most θ_P as in Figure 1(b). Since the triangle BA_2A_3 is contained in the ear $A_1A_2A_3$, Area$(BA_2A_3) \leq$ Area$(A_1A_2A_3)$. As these two triangles are sharing an angle A_2, by Gauss–Bonnet, the sum of the two remaining angles of BA_2A_3 is greater than or equal the sum of the angles A_1 and A_3 of the ear. In other words, we have

$$\measuredangle A_2BA_3 + \measuredangle A_2A_3B \geq \measuredangle A_2A_1A_3 + \measuredangle A_2A_3A_1.$$

Since B is on the interior of the side A_1A_2, we have $\measuredangle A_2A_3B < \measuredangle A_2A_3A_1$ and thus $\measuredangle A_2BA_3 > \measuredangle A_2A_1A_3$. Hence, $\measuredangle A_2BA_3 > \theta_P \geq \theta$ and c lies inside BA_2A_3. This implies that c also lies inside the ear, and the triangle BA_2C is contained in the ear. By the same argument, we can show that the angle at C (i.e., the remaining angle formed by c and an edge of the n-gons) is greater than θ_P. The fact c lies inside the ear also tells us that the length of c has to be less than or equal at least one of three sides of the ear, hence $\ell(c) \leq \ell_P$. Note that θ_P and ℓ_P are optimal as the inner sides of the ears are admissible geodesic arcs. □

In this paper, we only need to focus on the case of once-punctured polygons. We also refer the reader to [5, Chapter 2] and [4, Chapter 7] for all necessary trigonometric fomulae. The following lemma will give us an upper bound on the length of the geodesic arc that traverses inside the polygon with endpoints lying on the boundary of the polygon.

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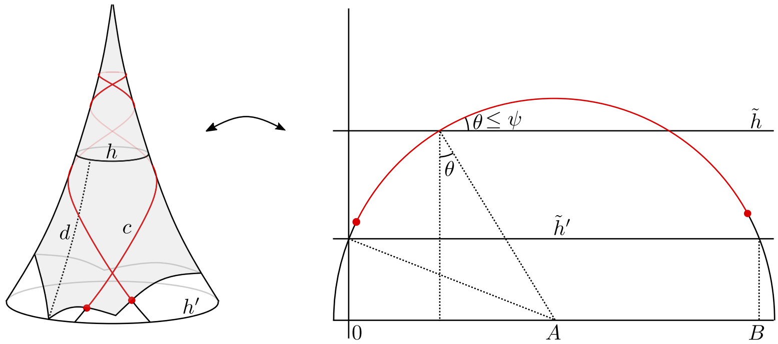

Figure 2.

Lemma 2. *Let P be a once-punctured polygon and $\psi \in \left[0, \frac{\pi}{2}\right)$ a constant. Let h be a closed horocycle lying inside P. Let d be the maximal distance from a point on ∂P to h. Then for any geodesic arc c in P with two end points on ∂P and $\theta := \measuredangle(c, h) \leq \psi$, we have*

$$\ell(c) \leq \operatorname{arccosh}\left(\frac{2e^{2d}}{\cos^2 \psi} - 1\right).$$

Proof. We lift to $\mathbb{H}$ as in Figure 2.

A lift of c is contained in the geodesic segment with endpoints i and $B + i$, where

$$A^2 + 1 = \left(\frac{e^d}{\cos\theta}\right)^2, \text{ and } B = 2A.$$

Hence,

$$\cosh \ell(c) \leq \cosh(d_{\mathbb{H}}(2A + i, i)) = 1 + \frac{|2A|^2}{2} = \frac{2e^{2d}}{\cos^2\theta} - 1 \leq \frac{2e^{2d}}{\cos^2\psi} - 1.$$

□

The next lemma describes some properties in a certain type of quadrilateral. Let h_1 and h_2 be two disjoint horocycles in $\mathbb{H}$. Let A_1A_2 be the common orthogonal between h_1 and h_2. For $i = 1, 2$, let B_i, C_i be points on h_i so that $B_1B_2C_2C_1$ becomes a quadrilateral with two horocyclic edges $\{B_1C_1, B_2C_2\}$ and two geodesic edges $\{B_1B_2, C_1C_2\}$.

Lemma 3. *Suppose the inner acute angles of the quadrilateral $B_1B_2C_2C_1$ are of the same value ψ. Then*

$$\ell(B_1C_1) = \ell(B_2C_2) = 2\sqrt{\tan^2\psi + e^{-\ell(A_1A_2)} + 1} - 2\tan\psi.$$

Furthermore, every geodesic segment, which only meets h_1 and h_2 at its endpoints, lies totally inside the quadrilateral if and only if each of two acute angles at the endpoints are of value at least ψ.

Proof. Denote by u the length of A_1A_2. Let h_1 be the horizontal line $y = ie^u$, h_2 be the horocycle centered at 0 and going through i as in Figure 3. We can also suppose that $A_1 = ie^u$, $A_2 = i$, hence $C_1 = \ell e^u + ie^u$ where ℓ is defined by the length of the horocyclic segment A_1C_1. By symmetry of the quadrilateral, we can find an involution f which is a nonorientation-preserving isometry sending A_1 to A_2, B_1 to B_2, and C_1 to C_2. By a standard computation, $f(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$ where $a = d = 0, b = e^{\frac{u}{2}}$, and $c = e^{\frac{-u}{2}}$. As a consequence,

$$C_2 = f(C_1) = f(\ell e^u + ie^u) = \frac{\ell}{\ell^2 + 1} + \frac{i}{\ell^2 + 1}.$$

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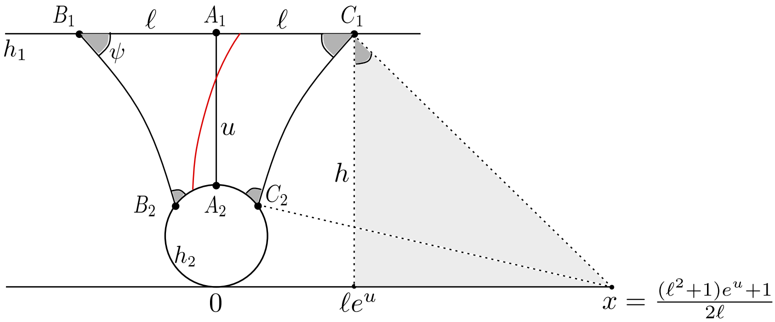

Figure 3.

Let x be a point on the real line of $\mathbb{H}$ such that the Euclidean distances from C_1 and C_2 to x are the same. By computation, $x = \frac{(\ell^2+1)e^u+1}{2\ell}$. Now applying the Euclidean trigonometric formula for the shaded Euclidean right triangle in Figure 3, noting that value of the angle at C_1 of this triangle is exactly ψ, we get

$$\tan\psi = \frac{\frac{(\ell^2+1)e^u+1}{2\ell} - \ell e^u}{e^u} = \frac{-\ell^2+1+e^{-u}}{2\ell}.$$

From this, we obtain the value of ℓ in terms of ψ and u.

For the second part, we fix an angle ϕ of value between ψ and $\frac{\pi}{2}$ at one endpoint of c and observe what happens to the acute angle at the other endpoint of c while moving c along the horocycles and keeping the value of the angle ϕ. The behavior of the values of the remaining acute angle is exactly that of a concave function. By symmetry of the quadrilateral, c lies inside the quadrilateral if and only if both acute angles at the endpoints are of value at least ψ. □

Next we recall two useful facts.

Lemma 4. [3, Lemmas 2.2] *Let* $\frac{\pi}{2} \geq \theta_0 > 0$, *and set* $m(\theta_0) := 2\log\left(\frac{1}{\sin\theta_0}\right) + 2\log(1+\cos\theta_0)$. *If c is an oriented geodesic segment in $\mathbb{H}$ of length at least $m(\theta_0)$ between two (complete) geodesics γ_1, γ_2 such that the starting (resp. end) point of c lies on γ_1 (resp. γ_2) and $\measuredangle(c, \gamma_i) \geq \theta_0$ for $i = 1, 2$, then γ_1 and γ_2 are disjoint.*

Lemma 5. [3, Lemmas 2.3] *Let* $\frac{\pi}{2} \geq \theta_0 > 0$ *be a fixed constant. Let c be a geodesic segment in $\mathbb{H}$ and γ_c the complete geodesic containing c. Fix $\varepsilon \in (0, 2]$ and let γ_1 and γ_2 be geodesics that intersect γ_c such that intersection points p_1, p_2 lie on different sides of c. There exists an r_ε (depending only on ε and θ_0), so that if $\measuredangle(\gamma_i, \gamma_c) \geq \theta_0$ and $d(c, p_i) \geq r_\varepsilon$, for $i = 1, 2$, then γ_1 and γ_2 are disjoint.*

Furthermore, for any geodesic γ intersecting both γ_1 and γ_2, we have the following properties:

(P1) $c \subset B_\varepsilon(\gamma)$.

(P2) The image of the orthogonal projection of c on γ is contained in the middle part of γ (i.e., it lies between γ_1 and γ_2).

Proof. The properties: "γ_1 and γ_2 are disjoint" and (P1) are proved in [3, Lemma 2.3], here we will fix a minor mistake in their proof to obtain a correct value of r_ε under the requirement that $0 < \varepsilon \leq 2$.

(P1) Keeping all notations introduced in [3, Lemma 2.3], from the proof we already had:

$$\cosh\frac{\ell(\mu)}{2} = \cosh\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sin(\theta_0) \ \ ; \ \ \sinh d' = \frac{1}{\sinh\frac{\ell(\mu)}{2}} \ \ ; \ \ \sinh h' = \sinh d' \cosh\frac{\ell(c)}{2}. \quad (2.1)$$

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In [3], the authors deduced from Equalities (2.1) the following incorrect relation:

$$\sin(\theta_0)\cosh\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sinh h' = \cosh\frac{\ell(c)}{2}$$

In the end of their proof, they deduced the following relation

$$r_\varepsilon \geq \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{4}{\sin\theta_0}\right),$$

which holds for any $\varepsilon > 0$. Then they chose $r_\varepsilon := \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{4}{\sin\theta_0}\right)$. Fortunately, this incorrect value of r_ε does not affect the overall conclusion of the main theorems since later on one will see that the condition $0 < \varepsilon \leq 2$ is necessary and then the correct value of r_ε can be chosen as $\log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2e}{\sin\theta_0}\right)$. From Equalities (2.1), we deduce that

$$\sinh^2(h') = \sinh^2(d')\cosh^2\left(\frac{\ell(c)}{2}\right) = \frac{\cosh^2\left(\frac{\ell(c)}{2}\right)}{\cosh^2\left(\frac{\ell(\mu)}{2}\right) - 1} = \frac{\cosh^2\left(\frac{\ell(c)}{2}\right)}{\cosh^2\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sin^2(\theta_0) - 1}.$$

We want to show that $h' \leq \frac{\varepsilon}{2}$ thus that

$$\frac{\cosh^2\left(\frac{\ell(c)}{2}\right)}{\cosh^2\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sin^2(\theta_0) - 1} \leq \sinh^2\left(\frac{\varepsilon}{2}\right) \tag{2.2}$$

and Inequality (2.2) is equivalent to the following:

$$\cosh^2\left(r_\varepsilon + \frac{\ell(c)}{2}\right) \geq \frac{\cosh^2\left(\frac{\ell(c)}{2}\right) + \sinh^2\left(\frac{\varepsilon}{2}\right)}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)}.$$

By using the identities $\cosh(2x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1$, the last inequality can be expressed differently as follows:

$$2r_\varepsilon \geq \operatorname{arccosh}\left(\frac{\cosh\ell(c) + \cosh\varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)} - 1\right) - \ell(c). \tag{2.3}$$

The right hand of Inequality (2.3) can be considered as a function:

$$f(x) = \operatorname{arccosh}(ax + b) - \operatorname{arccosh} x$$

on the domain $[1, \infty)$, in which

$$a = \frac{1}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)} > 0,$$

$$b = \frac{\cosh\varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)} - 1 \geq \frac{\cosh\varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)} - 1 = 1 + \frac{1}{\sinh^2\left(\frac{\varepsilon}{2}\right)} > 1.$$

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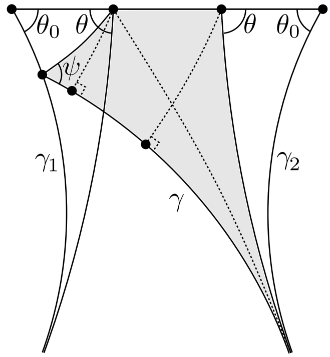

Figure 4. *The worst-case scenario.*

This function reaches its maximum $x = 1$. Hence, Inequality (2.3) will hold if

$$2r_\varepsilon \geq \operatorname{arccosh}\left(\frac{1+\cosh\varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)} - 1\right). \tag{2.4}$$

For simplicity, we set $A := \dfrac{1+\cosh\varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)}$. Note that

$$\operatorname{arccosh}(A-1) < \operatorname{arccosh} A = \log\left(A + \sqrt{A^2-1}\right) < \log(2A)$$

and

$$\log(2A) = \log\left(\frac{2+2\cosh\varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)}\right) = 2\log\left(\frac{2}{\sin\theta_0}\right) + 2\log\left(\frac{e^\varepsilon+1}{e^\varepsilon-1}\right)$$

in which $\log\left(\dfrac{e^\varepsilon+1}{e^\varepsilon-1}\right) < \log\left(\dfrac{1}{\varepsilon}\right) + 1$ for all $\varepsilon \in (0, 2]$.

Thus Inequality (2.4) certainly holds provided

$$r_\varepsilon \geq \log\left(\frac{2}{\sin\theta_0}\right) + \log\left(\frac{1}{\varepsilon}\right) + 1.$$

Hence, we set

$$r_\varepsilon := \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2e}{\sin\theta_0}\right).$$

(P2) We consider the worst-case scenario: c is a complementary θ_0-transversal of γ_1 and γ_2 (see Figure 4). Consider the limit case which is when γ and γ_2 are ultra-parallel. Now we orient c from γ_1 to γ_2. Let ψ denote the angle between γ and the geodesic segment connecting the endpoint of γ on γ_1 and the starting point of the oriented geodesic segment c. Let θ denote the angle between the extended part of c toward γ_2 and the geodesic ray starting at the endpoint of c and ending at the endpoint of γ at infinity. Notice that the image of the orthogonal projection of c on γ lies between γ_1 and γ_2 if and only if the angle ψ is acute. Since the sum of four inner angles in a quadrilateral is always less than 2π, $\psi \leq \dfrac{\pi}{2}$ holds if $\theta < \dfrac{\pi}{4}$. By using the same formula as in the proof of Lemma 4, we have

$$\cos\theta = \frac{\tanh r_\varepsilon - \cos\theta_0}{1 - \tanh r_\varepsilon \cos\theta_0}.$$

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Hence, $\theta < \frac{\pi}{4}$ holds provided

$$\frac{\tanh r_\varepsilon - \cos\theta_0}{1 - \tanh r_\varepsilon \cos\theta_0} > \frac{1}{\sqrt{2}}. \tag{2.5}$$

By a small manipulation, Inequality (2.5) is equivalent to the following:

$$r_\varepsilon > \frac{1}{2}\log\left(\frac{1}{\sin\theta_0}\right) + \log(1+\sqrt{2}) + \log(1+\cos\theta_0).$$

And this last inequality holds by definition of r_ε. □

3. Main tools

Moduli space $\mathcal{M}_{g,n}$ we think of as the space of complete hyperbolic structures up to isometry on a punctured orientable topological surface $\Sigma_{g,n}$ of genus g with n punctures (with $2g+n \geq 3$). A cusp region of area ξ is a portion of the surface isometric to $\{z : \mathrm{Im} z \geq 1\}/z \mapsto z+\xi$. For any X in $\mathcal{M}_{g,n}$ and any positive number $\xi \leq 2$, we can define

$$X^\xi := \mathrm{cl}(X \setminus \{\text{all cusp regions of area } \xi\}).$$

In other word, X^ξ is a surface of genus g with n boundary components and each connected component of its boundary is a horocycle of length ξ. The following theorem is the main technical part in this paper:

Theorem 1. *For any $X \in \mathcal{M}_{g,n}$, there exists a constant K_X such that the following holds. For all $0 < \varepsilon \leq 1$, $0 < \xi \leq 1$ and any finite collection $\{c_i\}_{i=1}^N$ of geodesic arcs of average length $\bar{c}$ in X^ξ, there exists a closed geodesic γ of length at most*

$$N\left(K_X + \bar{c} + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi}\right)$$

which contains $\{c_i\}_{i=1}^N$ in its 2ε-neighborhood.

Proof. The proof is structured in three parts. The first part will introduce some necessary geometric quantities and a classification of geodesics traveling through (punctured) polygons on the surface X forming big or small angles with the sides of the polygons. The second part is about the details of the arc-replacement technique and some upper bounds of length of extended arcs. The final part is a recap of parts 1 and 2 followed by the construction of the closed geodesic γ.

Part 1: Setup.

We take a closed geodesic γ_0 on X of minimal length such that $X \setminus \gamma_0$ consists of a finite collection of ordinary polygons $\{P_i\}_{i\in I}$ and once-punctured polygons $\{P_i\}_{i\in J}$ (I and J are two disjoint finite index sets). Recall that, for each polygon $P_i (i \in I \cup J)$ we have the constants θ_{P_i} and ℓ_{P_i} as mentioned in Lemma 1. Also in each ordinary polygon P_i, we denote by D_{P_i} the value of its intrinsic diameter. Note that there is no intrinsic diameter in once-punctured polygons. We define:

$$\theta_0 := \min_{i\in I\cup J}\{\theta_{P_i}\} \text{ and } D := \max_{i\in I, j\in J}\{D_{P_i}, \ell_{P_j}\}.$$

In this part, we aim to define a classification for geodesics traveling through polygons in the following way.

We begin by defining a closed horocycle that lies inside a once-punctured polygon (hence γ_0 and this horocycle have no intersection) such that the distance between this horocycle and the horocycle of length ξ (namely h_ξ) is at least

$$r_\varepsilon := \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2e}{\sin\theta_0}\right).$$

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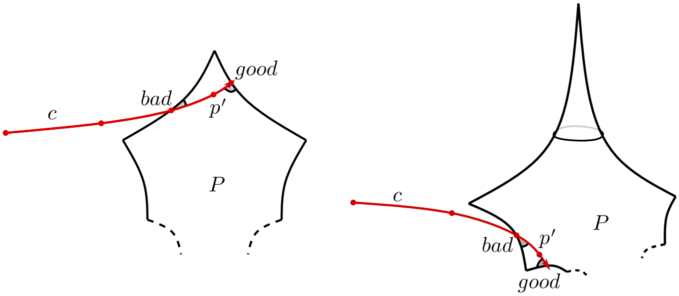

Figure 5. *Case* 1.

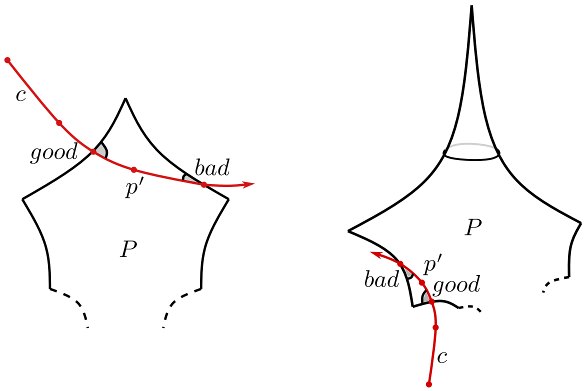

Figure 6. *Case* 2.

Since γ_0 wraps around each cusp at most once, it will not cross transversely the horocycle of length 1 in each cusp region. We also note that ξ is less than 1. Hence, one option that satisfies the above condition is the horocycle which is at distance r_ε from h_ξ. We denote this horocycle by h. Since the decay of length of horocycle in a cusp is e,

$$\ell(h) = \frac{\xi}{e^{r_\varepsilon}} = \frac{\varepsilon\xi}{2e}\sin\theta_0.$$

Now, let c be an arbitrary geodesic arc on X, we extend c by r_ε in one direction to get a new arc c' and the new endpoint p'. Then we continue to extend c' from p'. In the process of extending, the geodesic can intersect γ_0 several times and form angles. An intersection is called a good intersection if the acute angle at it is at least θ_0, and otherwise, it will be called a bad intersection. The extension will stop at the first good intersection from p'. By Lemma 1, the extensions can be divided into 5 cases as follows:

1. From p', the previous intersection is bad and the next intersection is good (see Figure 5).
2. From p', the previous intersection is good, and the next intersection is bad (see Figure 6).
3. p' lies inside an ordinary polygon, the previous intersection and the next intersection are both good (see Figure 7).
4. p' lies inside a once-punctured polygon P, the previous intersection, and the next intersection are both good (see Figure 7) and so that the geodesic arc, namely c'', between these two intersections is not too long, more precisely, this arc either intersects the horocycle h at an angle less than a given angle ψ or does not intersect h.
5. p' lies inside a once-punctured polygon and if we continue to extend c' from p', it will intersect the horocycle h at an angle at least ψ (see Figure 7).

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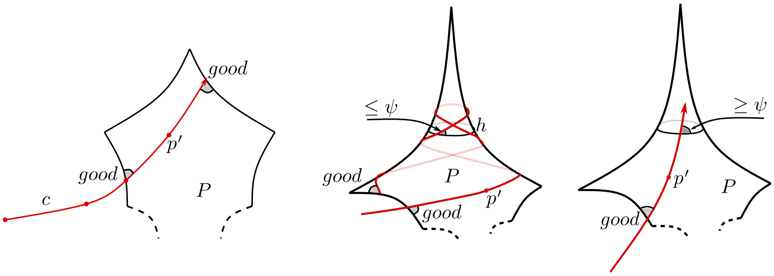

Figure 7. *Cases* 3, 4, *and* 5, *respectively.*

Now in order to stop the extension, by Lemma 1 and the definition of D above, the distance we need to extend from p' is at most D (in cases 1 and 3) and $2D$ (in case 2). In case 4, let $s_0 \in \partial P$ such that

$$d_X(s_0, h) = \max_{s \in \partial P} \{d_X(s, h)\}.$$

Let d_P be the distance from s_0 to the closed horocycle of length 1 of the same cusp. Note that the distance between this horocycle and h is $\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right)$. Thus,

$$d_X(s_0, h) = d_P + \log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right).$$

Then applying Lemma 2 and the inequality $\operatorname{arccosh}(x-1) < \log(2x)$ we have

$$\ell(c'') \leq \operatorname{arccosh}\left(\frac{2e^{2d_P + 2\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right)}}{\cos^2\psi} - 1\right) < 2d_P + 2\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right) + 2\log\left(\frac{2}{\cos\psi}\right). \tag{3.1}$$

Note that, in part 2, we will define ψ as the angle formed by $\tilde{h}$ and $\tilde{\eta}_1$ (see Figure 8). By simple computations, we obtain

$$\psi = \arccos\left(\frac{\frac{\varepsilon\xi}{e}\sin\theta_0}{1 + \frac{\varepsilon^2\xi^2}{4e^2}\sin^2(\theta_0)}\right).$$

From there we have

$$2\log\left(\frac{2}{\cos\psi}\right) = 2\log\left(1 + \frac{\varepsilon^2\xi^2}{4e^2}\sin^2(\theta_0)\right) + 2\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right) < 2 + 2\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right). \tag{3.2}$$

Combining (3.1) and (3.2), one has

$$\ell(c'') < 2d_P + 4\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right) + 2.$$

Moreover, if one denotes by d_{γ_0} the maximal value of $\{d_{P_i}\}_{i \in J}$, then

$$m_A := 2D + 2d_{\gamma_0} + 2 + 4\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right)$$

is an upper bound on the length of the extension in class A (i.e., cases 1, 2, 3, and 4). Note that, in class B (i.e., case 5), the length of the extension is unbounded when $\measuredangle(c', h)$ goes to $\frac{\pi}{2}$.

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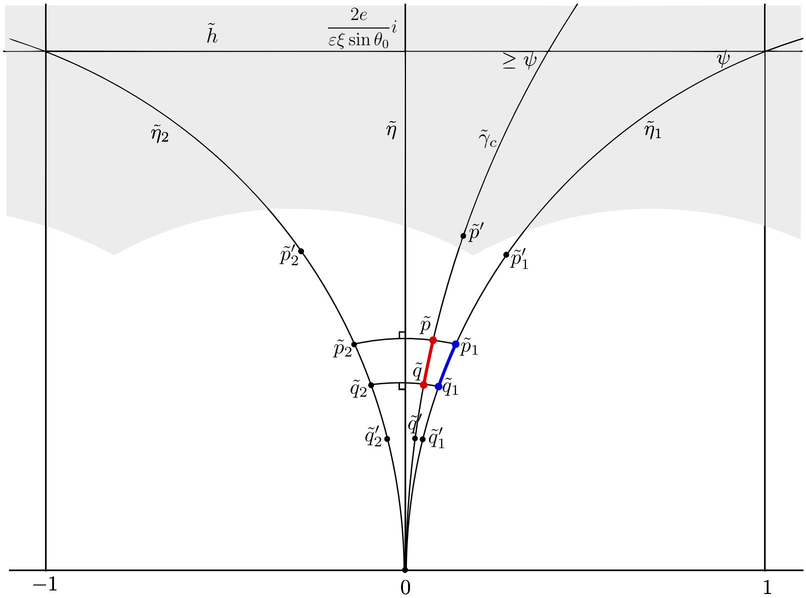

Figure 8. *Lifting to $\mathbb{H}$ in Case B. The shaded part is a lift of polygon P.*

Part 2: Replacements and estimates.

Now, let c be an arbitrary geodesic arc in the collection $\{c_i\}_{i=1}^N$. Denote the two endpoints of c by p and q. We extend c by r_ε in both directions to get a new geodesic arc c'. We now look at different cases.

Case A: The extensions in both directions are in class A.

In order to get good intersections in both directions, we need to extend c' by at most m_A for each of its directions. Hence, an upper bound on the length of c after being extended is

$$\ell(c) + 2r_\varepsilon + 2m_A$$

or more precisely,

$$\ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + 4D + 4d_{\gamma_0} + 4 + 10\log\left(\frac{2e}{\sin\theta_0}\right). \tag{3.3}$$

Case B: There is a direction where the extension is in class B.

Let p' be the endpoint of c' in this direction, we can suppose p' lies inside a once-punctured polygon, namely P.

What we aim to do is to replace c by another geodesic arc, which is very close to c and controlled in both directions (i.e., the extension in each direction is in class A). Denote the complete geodesic containing c by γ_c. We assume that the horizontal line $y = \frac{2e}{\varepsilon\xi\sin\theta_0}i$, namely $\tilde{h}$, is a lift of the closed horocycle h of length $\frac{\varepsilon\xi}{2e}\sin\theta_0$ in P. From there we can suppose the complete geodesic $\tilde{\gamma}_c$ with an endpoint at 0, forming an angle at least ψ with $\tilde{h}$, is a lift of γ_c. Hence, $\tilde{\gamma}_c$ lies between $\tilde{\eta}_1$ and $\tilde{\eta}_2$, in which $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are the complete geodesics with a common endpoint at 0, containing $\frac{2e}{\varepsilon\xi\sin\theta_0}i + 1$ and $\frac{2e}{\varepsilon\xi\sin\theta_0}i - 1$, respectively.

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Now we construct lifts of other points from there. Let $\tilde{p}$ and $\tilde{q}$ be lifts of p and q, respectively (see Figure 8). Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be geodesics going though $\tilde{p}$ and $\tilde{q}$, respectively, and orthogonal to the axis $x = 0$. Let $\tilde{p}_i := \tilde{\gamma}_1 \cap \tilde{\eta}_i$ and $\tilde{q}_i := \tilde{\gamma}_2 \cap \tilde{\eta}_i$, for $i = 1, 2$.

For $i = 1, 2$, we denote by $p_i q_i$ the projection of $\tilde{p}_i \tilde{q}_i$ to the surface X. Extend the geodesic arc $p_i q_i$ by r_ε in both directions to get a new geodesic arc $p'_i q'_i$. If $p'_i \notin P$, we only need to extend by an extra at most $\frac{\varepsilon\xi}{e} \sin\theta_0 + 2\ell(\gamma_0)$ to get into P. This can be proved by using the inequality in the triangle formed by $\tilde{\eta}_i$, the geodesic segment $\tilde{p}_i{}'\tilde{p}'$ and a lift of γ_0 (one of the boundary components of the shaded part in Figure 8).

There are two different subcases of case B:

Subcase BA: The extension in the direction of either q_1 or q_2 is in class A.

Without loss of generality, we assume that the extension in the direction of q_1 is in class A. Note that the geodesic segments $\tilde{p}\tilde{p}_1$ and $\tilde{q}\tilde{q}_1$ are of length at most $\frac{\varepsilon\xi}{e} \sin\theta_0$. Since $\xi \leq 1$ and $\sin\theta_0 \leq 1$, $\frac{\varepsilon\xi}{e} \sin\theta_0 < \frac{\varepsilon}{2}$. Thus, any geodesic containing $p_1 q_1$ in its ε-neighborhood contains c in its $\left(\frac{\varepsilon}{2} + \varepsilon\right)$-neighborhood. Hence in this case, we will replace c by $p_1 q_1$.

Recall that we extended $p_1 q_1$ by r_ε in both directions to obtain the geodesic arc $p'_1 q'_1$. In order to get good intersections in both directions, we continue to extend $p'_1 q'_1$ by at most

$$2m_A + \frac{\varepsilon\xi}{e} \sin\theta_0 + 2\ell(\gamma_0).$$

Hence, an upper bound on the length of $p_1 q_1$ after being extended is

$$\ell(c) + 2r_\varepsilon + 2m_A + \frac{\varepsilon\xi}{e} \sin\theta_0 + 2\ell(\gamma_0)$$

which is less than or equal to

$$\ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + 4D + 4d_{\gamma_0} + 4 + 10\log\left(\frac{2e}{\sin\theta_0}\right) + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0). \tag{3.4}$$

Subcase BB: The extensions in the directions of q_1 and q_2 are both in class B.

Since η, η_1, and η_2 asymptotic in the direction of q, q_1, and q_2, length of the geodesic arc orthogonal to η and connecting q'_1 and q'_2 is very small, roughly less than $\frac{\varepsilon\xi}{2e} \sin\theta_0$. Thus, we can suppose that q'_1 and q'_2 lie in the same once-punctured polygon, denoted by P'. Let h' be the closed horocycle of length $\frac{\varepsilon\xi}{2e} \sin\theta_0$ in P'. From there we construct a lift of h', denoted by $\tilde{h}'$. We keep all the notations $A_1, A_2, B_1, B_2, C_1, C_2$, and u as introduced in Lemma 3 (see Figure 9).

Now we would like to apply Lemma 3 to the two horocycles $\tilde{h}$ and $\tilde{h}'$ with the angle ψ. Recall that $u = \ell(A_1 A_2)$ is the distance from $\tilde{h}$ to $\tilde{h}'$. One can estimate a lower bound and an upper bound on u as follows:

$$2\log\frac{4e}{\varepsilon\xi\sin\theta_0} \leq u \leq \ell(c) + 2r_\varepsilon + \frac{\varepsilon\xi}{e} \sin\theta_0 + 2\ell(\gamma_0) + 2\left(d_{\gamma_0} + \log\frac{2e}{\varepsilon\xi\sin\theta_0}\right).$$

This implies that:

$$0 < u \leq \ell(c) + 4\log\frac{1}{\varepsilon} + 2\log\frac{1}{\xi} + k_X$$

in which $k_X := 2d_{\gamma_0} + 4\log\frac{2e}{\sin\theta_0} + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0)$.

Recall that $\ell := \frac{\ell(B_1 C_1)}{2}$. Then by Lemma 3, we have

$$\ell = \sqrt{\tan^2\psi + e^{-u} + 1} - \tan\psi = \frac{e^{-u} + 1}{\sqrt{\tan^2\psi + e^{-u} + 1} + \tan\psi} < \frac{2}{\sqrt{\tan^2\psi + 1} + \tan\psi}$$

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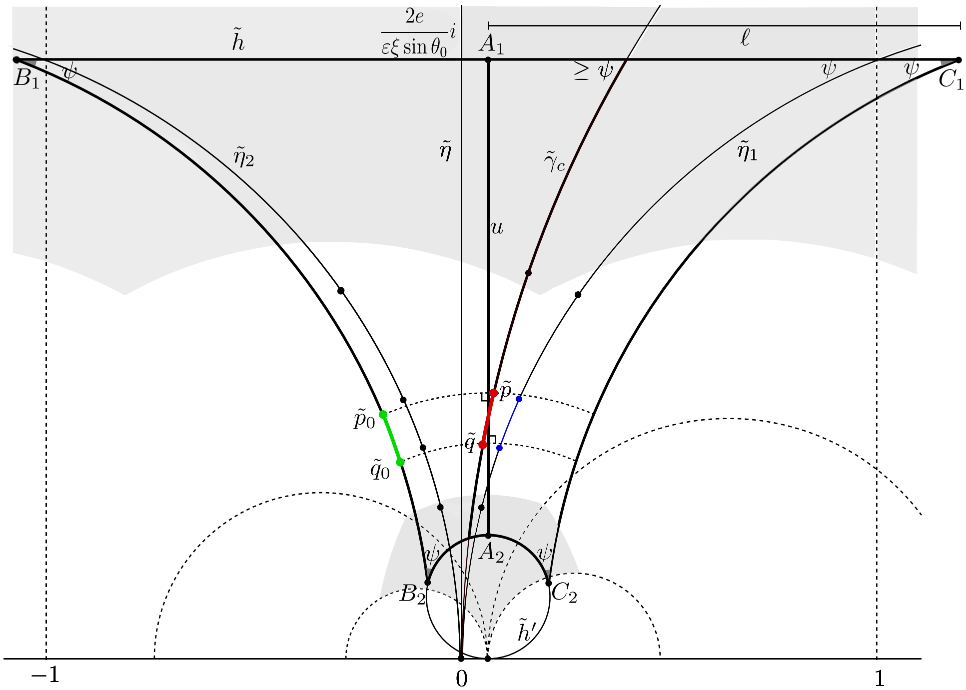

Figure 9. *Lifting to $\mathbb{H}$ in sub-case BB.*

Thus, since $\psi = \arccos\left(\dfrac{\frac{\varepsilon\xi}{e}\sin\theta_0}{1+\frac{\varepsilon^2\xi^2}{4e^2}\sin^2(\theta_0)}\right)$, $0<\varepsilon\leq 1$, $0<\xi\leq 1$ and $\dfrac{2e}{\sin\theta_0}>2e$ we have

$$\ell < \frac{2}{\sqrt{\tan^2\psi+1}+\tan\psi} = \frac{2\varepsilon\xi}{\frac{2e}{\sin\theta_0}} < 2e^{-1},$$

hence

$$\ell(B_1B_2)=\ell(C_1C_2)\leq 2\ell+u<4e^{-1}+\ell(c)+4\log\frac{1}{\varepsilon}+2\log\frac{1}{\xi}+k_X.$$

Now we draw a geodesic going through $\tilde{p}$, resp. $\tilde{q}$, orthogonal to A_1A_2, meeting B_1B_2 at a point, denoted by $\tilde{p}_0$, resp. $\tilde{q}_0$ (see the Figure 9). We also denote by $\tilde{\zeta}$ the geodesic segment $\tilde{p}_0\tilde{q}_0$. By projecting the geodesic segment $\tilde{\zeta}$ to X, we get a geodesic arc on X, denoted by ζ. Note that the geodesic segments $\tilde{p}\tilde{p}_0$ and $\tilde{q}\tilde{q}_0$ are of length at most $2\ell<\dfrac{4\varepsilon\xi\sin\theta_0}{2e}<\varepsilon$. Thus, any geodesic containing ζ in its ε-neighborhood contains c in its 2ε-neighborhood. In this case, we will replace c by ζ.

Since B_1B_2 contains $\tilde{\zeta}$, we will extend B_1B_2 instead of $\tilde{\zeta}$. By Lemma 2, in order to get good intersections in both directions, we need to extend B_1B_2 in each direction by a distance v, where v satisfies:

$$\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right)+\log\frac{1+\sin\psi}{1-\sin\psi}\leq v\leq\frac{1}{2}\operatorname{arccosh}\left(\frac{2e^{2d}}{\cos^2\psi}-1\right)+\frac{1}{2}\log\frac{1+\sin\psi}{1-\sin\psi},$$

where $d := d_{\gamma_0}+\log\left(\dfrac{2e}{\varepsilon\xi\sin\theta_0}\right)$. Similarly to (3.1) and (3.2), one can show that:

$$\log\frac{1}{\varepsilon}+\log\left(\frac{2e}{\xi\sin\theta_0}\right)+\log\frac{1+\sin\psi}{1-\sin\psi}\leq v<d_{\gamma_0}+1+3\log\frac{1}{\xi}+3\log\frac{1}{\varepsilon}+3\log\left(\frac{2e}{\sin\theta_0}\right).$$

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Since the lower bound of v is greater than r_ε, we do not need to extend the segment B_1B_2 in two steps as in the previous cases.

In this way, we obtain an upper bound on the length of B_1B_2 after the extension:

$$4e^{-1} + \ell(c) + 4\log\frac{1}{\varepsilon} + 2\log\frac{1}{\xi} + k_X + 2d_{\gamma_0} + 2 + 6\log\frac{1}{\xi} + 6\log\frac{1}{\varepsilon} + 6\log\left(\frac{2e}{\sin\theta_0}\right)$$

or

$$\ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + 4d_{\gamma_0} + 10\log\left(\frac{2e}{\sin\theta_0}\right) + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0) + 4e^{-1} + 2. \tag{3.5}$$

Finally, after comparing upper bounds (3.3), (3.4) and (3.5) in cases A, BA, and BB, respectively, we set

$$M(c, \varepsilon, \xi, X) := \ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + k'_X$$

the upper bound of all cases, in which $k'_X := 4D + 4d_{\gamma_0} + 4 + 10\log\left(\frac{2e}{\sin\theta_0}\right) + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0)$ is a quantity that depends only on X.

Part 3: Construction of the geodesic γ.

Since X is orientable, γ_0 has two opposite sides denoted by γ_0^+ and γ_0^-. Let $\mu_\pm$ be an oriented geodesic arc from γ_0 to itself, orthogonal γ_0 in both end points, and which leaves and returns to $\gamma_0^\pm$. Note that these two geodesic arcs μ_+ and μ_- are constructed using a finite cover of X which lifts γ_0 to a simple closed geodesic. The lengths of these two arcs are constants depending on X.

In the previous parts, we replaced the collection $\{c_i\}_{i=1}^N$ by a new collection, denoted by $\{\zeta_i\}_{i=1}^N$. We also defined a collection of the extended geodesic arcs of $\{\zeta_i\}_{i=1}^N$, denoted by $\{\zeta'_i\}_{i=1}^N$. In this collection, each element ζ'_i, is of length at most $M(c, \varepsilon, \xi, X)$, has endpoints lying on γ_0 and forms good angles ($\geq \theta_0$) with γ_0, and is an extension of ζ_i by at least r_ε in each direction. In short, each ζ_i is an example of the geodesic segment c in Lemma 5. Furthermore, we showed that any geodesic containing ζ_i in its ε-neighborhood contains c_i in its 2ε-neighborhood.

With the new collection $\{\zeta_i\}_{i=1}^N$ and its extension $\{\zeta'_i\}_{i=1}^N$ in hand, following exactly the same algorithm in the proof of [3, Theorem 2.4], one can construct a closed piecewise geodesic forming from these arcs with suitable choices of subsegments of the filling closed geodesic γ_0 and $\mu_\pm$ as following steps.

- Cyclically ordering and orienting each ζ'_i arbitrarily.
- If ζ'_{i+1} starts on the opposite side of γ_0 that ζ'_i ends on, we join the endpoint of ζ'_i to the starting point of ζ'_{i+1} by the shortest subarc of γ_0 which does this.
- If ζ'_{i+1} starts and ζ'_i ends on the same side, say γ_0^+, of γ_0, we join the endpoint of ζ'_i to the starting point of μ_- by the shortest subarc of γ_0 which does this. Then we join the endpoint of μ_- to the starting point of ζ'_{i+1} by the shortest subarc of γ_0 which does this.

Note that each connecting shortest subarc introduced in each step is of length at most $\frac{\gamma_0}{2}$.The resulting closed piecewise geodesic, denoted by γ', is contained in the ε-neighborhood of γ, where γ is the unique closed geodesic in the free homotopy class of γ'. Due to the above construction, γ is a nontrivial loop. Denote $\bar{c}$ the average length of the collection $\{c_i\}_{i=1}^N$, the length of γ is bounded above by

$$N\left(K_X + \bar{c} + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi}\right),$$

where K_X is a constant depending on X. □

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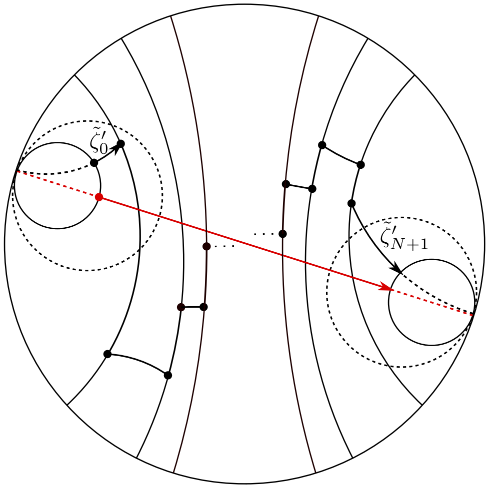

Figure 10. *Lifting to* $\mathbb{H}$.

A geodesic arc on X is called a doubly truncated orthogeodesic on X^{ξ} if it is perpendicular to the horocyclic boundary of X^{ξ} at its endpoints. As a consequence of Theorem 1, we can also construct a doubly truncated orthogeodesic $\mathcal{O}$ with the same properties:

Theorem 2. *For any $X \in \mathcal{M}_{g,n}$, there exists a constant K_X such that the following holds. For all $0 < \xi \leq 1$, $0 < \varepsilon \leq \min\left\{\log\frac{1}{\xi}, 1\right\}$, and any finite collection $\{c_i\}_{i=1}^{N}$ of geodesic arcs of average length $\bar{c}$ in X^{ξ}, there exist a doubly truncated orthogedesic $\mathcal{O}$ of length at most*

$$(N+1)\left(K_X + \bar{c} + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi}\right)$$

containing $\{c_i\}_{i=1}^{N}$ in its 2ε-neighborhood.

Proof. Let P_0 and P_1 be two arbitrary once-punctured polygons of the partition by γ_0 on X. First, we will construct a doubly truncated orthogeodesic $\mathcal{O}_1$ with endpoints on the horocycles of length 1 associated to the two polygons so that $\mathcal{O}_1$ contains $\{\zeta_i\}_{i=1}^{N}$ in its ε-neighborhood. We take a shortest one-sided orthogeodesic arc, denoted by ζ'_0, oriented with the starting point on the horocycle of length 1 of P_0 and the endpoint on γ_0. We take another shortest one-sided orthogeodesic arc, denoted by ζ'_{N+1}, oriented with the starting point on γ_0 and the endpoint on the horocycle of length 1 of P_1. For each $i \in \{1, 2, ..., N\}$, we orient ζ'_i arbitrarily. The new sequence $\{\zeta'_i\}_{i=0}^{N+1}$ is ordered linearly by its index. We apply the connecting algorithm to this new sequence. Noting that ζ'_0 is in the first step and ζ'_{N+1} is in the last step of the algorithm, one will obtain a doubly truncated orthogeodesic $\mathcal{O}_1$ as desired (see Figure 10).

Since $\mathcal{O}_1$ is an arc, it may not contain totally either c_1 or c_N in its 2ε-neighborhood. In this case, by applying Lemma 5 (P2), we only need to extend $\mathcal{O}_1$ by a small extra segment of length at most ε in both directions. Note that, $\varepsilon \leq \log\frac{1}{\xi}$, and the distance between the horocycle of length 1 and the horocyle of length ξ is $\log\frac{1}{\xi}$, by extending $\mathcal{O}_1$ in both directions until it hits the boundary of X^{ξ}, we obtain $\mathcal{O}$ as desired. □

4. Quantitative density on surface

We now apply Theorem 1 to prove results about quasi-dense geodesics.

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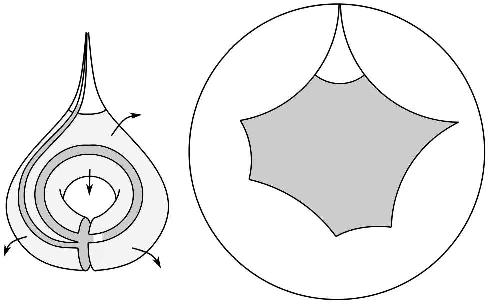

Figure 11. *An example when $g = 1$, $n = 1$.*

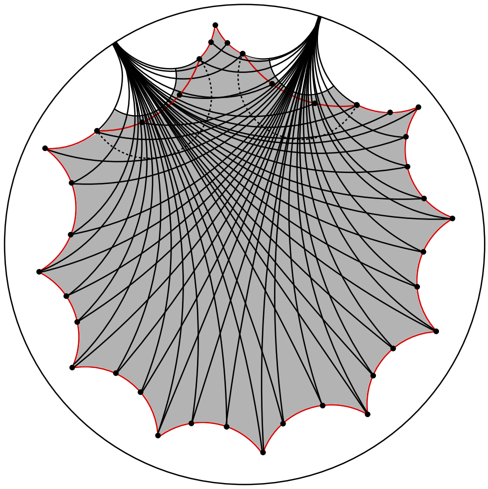

Figure 12. *An example when $g = 2$, $n = 2$. Note that $CH(F^2)$ is the polygon with red edges.*

Theorem 3. *For all $X \in \mathcal{M}_{g,n}$ there exists a constant $C_X > 0$ such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq 2$ there exists a closed geodesic γ_ε that is ε-dense on X^ξ and such that*

$$\ell(\gamma_\varepsilon) \leq C_X \frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\xi}\right).$$

Proof. On $\mathbb{H}$, there is a fundamental polygon F whose boundary consists $4g + 2n$ paired geodesic segments (or rays) which, when glued in pairs, turn the polygon into X. This polygon has n ideal vertices and $4g + n$ ordinary vertices. See Figure 11 for an example.

Since $X^2 \subset X^\xi \subset X$, there is a fundamental polygon of X^ξ in F, say F^ξ, and a fundamental polygon of X^2 in F^ξ, say F^2. We note that the boundary of F^2 consists of n horocyclic segments of length 2 and $4g + 2n$ geodesic segments. By replacing each horocyclic segment by a geodesic segment of length $2\operatorname{arcsinh}1$ with the same endpoints, we obtain the convex hull of F^2, denoted by $CH(F^2)$. We denote by P_X the perimeter of $CH(F^2)$ and note that this value depends only on X. On an edge of $CH(F^2)$, we choose the first point at a vertex, then choose the next points such that the segment on the boundary connecting two consecutive points is of length ε. If the length of the segment connecting the last point and the remaining vertex of the same edge is less than ε, that vertex will be chosen as the first point of the next edge, we then continue the choosing process. Eventually, we have chosen at most

$$\frac{P_X}{\varepsilon} + 4g + 2n$$

points on the boundary of $CH(F^2)$. See Figure 12.

Now we connect each ideal vertex to the points on the boundary of $CH(F^2)$. Since $CH(F^2)$ is convex, the parts of those geodesic rays in F^ξ are exactly one-sided orthogeodesic segments. By gluing back paired geodesic segments of F in pairs, these segments become one-sided orthogeodesic arcs on X and we have at most

$$n\left(\frac{P_X}{\varepsilon}+4g+2n\right)$$

one-sided orthogeodesic arcs on X. By construction, each segment is of length at most

$$\frac{P_X}{2}+\log\left(\frac{2}{\xi}\right).$$

Moreover, the collection of the one-sided orthogeodesic arcs is $\frac{\varepsilon}{2}$-dense on X^ξ. Thus by applying Theorem 1 to this collection, we obtain the closed geodesic γ_ε containing every arcs in its $\frac{\varepsilon}{2}$ neighborhood where length satisfies

$$\ell(\gamma_\varepsilon)\leq n\left(\frac{P_X}{\varepsilon}+4g+2n\right)\left(K_X+\frac{P_X}{2}+\log\frac{2}{\xi}+10\log\frac{2}{\varepsilon}+8\log\frac{1}{\xi}\right). \tag{4.1}$$

Then by manipulating the right hand of inequality (4.1), we obtain a constant C_X depending only on X so that:

$$\ell(\gamma_\varepsilon)\leq C_X\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon}+\log\frac{1}{\xi}\right).$$

□

By using the same collection of geodesic segments as in Theorem 3, and by the connecting algorithm in the proof of Theorem 2, we also obtain the following result:

Theorem 4. *Let $X\in\mathcal{M}_{g,n}$, there exists a constant $D_X>0$ such that for all $0<\xi\leq 1$ and all $0<\varepsilon\leq \min\left\{2\log\frac{1}{\xi},2\right\}$ there exists a doubly truncated orthogeodesic $\mathcal{O}_\varepsilon$ that is ε-dense on X^ξ and such that*

$$\ell(\mathcal{O}_\varepsilon)\leq D_X\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon}+\log\frac{1}{\xi}\right).$$

We end this section with a corollary of Theorem 3 where we apply Theorem 1.2 [2] to obtain an upper bound on the number of self-intersections of the quasi ε-dense closed geodesic.

Corollary 1. *Let $X\in\mathcal{M}_{g,n}$, there exists a constant $C_X>0$ such that for all $0<\xi\leq 1$ and all $0<\varepsilon\leq 2$ there exists a closed geodesic γ_ε that is ε-dense on X^ξ and such that*

$$2i(\gamma_\varepsilon,\gamma_\varepsilon)\leq C_X^{\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon}+\log\frac{1}{\xi}\right)}.$$

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