BULL. AUSTRAL. MATH. SOC. 46L05
VOL. 65 (2002) [57–67]

APPLICATIONS OF THE GAUGE-INVARIANT UNIQUENESS THEOREM FOR GRAPH ALGEBRAS

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We give applications of the gauge-invariant uniqueness theorem, which states that the Cuntz-Krieger algebras of directed graphs are characterised by the existence of a canonical action of $\mathbb{T}$. We classify the C^*-algebras of higher order graphs, identify the C^*-algebras of cartesian product graphs with a certain fixed point algebra and investigate a relation called elementary shift equivalence on graphs and its effect on the associated graph C^*-algebras.

1. INTRODUCTION

In the last few years various authors have considered analogues of the Cuntz-Krieger algebras associated to infinite directed graphs. In [**14, 13**] these *graph* C^*-algebras were studied using a groupoid model and the deep results of Renault on the ideal structure of groupoid C^*-algebras. In [**18, 10**] they were viewed as the Cuntz-Pimsner algebras of appropriate Hilbert bimodules, as introduced in [**16**]. In [**3**] an elementary approach was adopted which enabled the authors to generalise the results of [**4, 5, 13, 14, 18, 10**].

In this paper we consider applications of the *gauge-invariant uniqueness theorem* proved in [**3**](see also [**2**] for a version for arbitrary graphs).

This theorem states that the C^*-algebra of a directed graph is uniquely characterised by the existence of a canonical action of $\mathbb{T}$ called the *gauge action*. The gauge-invariant uniqueness theorem allows us to establish many of the basic properties of graph C^*-algebras without any extra hypotheses on the graph. This is of interest because for many years authors have assumed that their $\{0,1\}$-matrices A satisfied condition (I) of [**5**] merely to ensure that the Cuntz-Krieger algebras $\mathcal{O}_A$ were well-defined. In order to obtain the results proved in this paper we need to restrict our attention to row-finite graphs. However, this is a significant improvement upon the consideration of only finite graphs with $\{0,1\}$-adjacency matrices that satisfy condition (I).

We begin with a brief review of some key facts about graph C^*-algebras in Section 2.

Received 13th December, 2001
The main theorems in Sections 3 and 4 come from the author's Ph.D. thesis which was completed in 1999 at the University of Newcastle under the supervision of Iain Raeburn and David Pask.

In Section 3 we give the main results. We introduce the concept of *higher order directed graphs* (E^p, E^q) and show that, up to isomorphism, their C^*-algebras are classified by the gap $q - p$. This work generalises [**7**, Theorem 3] and [**3**, Corollary 2.4] (see also [**15**, Proposition 4.1]).

In Section 4 we give a new proof of a result from [**11**] which identifies the graph C^*-algebra of a type of cartesian product graph with a certain fixed point algebra. The proof given in [**11**] relies on the theory of groupoid C^*-algebras.

In Section 5 we investigate a relation called strong shift equivalence on directed graphs. This relation was introduced by Ashton in [**1**]. We extend this relation to arbitrary directed graphs and prove that row-finite strong shift equivalent graphs have Morita equivalent graph algebras (a result that was proved for finite graphs in [**1**]).

2. The C^*-algebras of Graphs

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ describing the range and source of the edges. A vertex $v \in E^0$ which emits no edges is called a *sink*, and a vertex $v \in E^0$ which receives no edges is called a *source*. The graph E is *row-finite* if the set $s^{-1}(v) \subseteq E^1$ is finite for every $v \in E^0$.

For $n \geqslant 2$, we define

$$E^n := \left\{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in E^1 \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for } 1 \leqslant i \leqslant n-1\right\}$$

and the finite path space $E^* = \bigcup_{n \geqslant 0} E^n$. For $\alpha \in E^n$, we write $|\alpha| = n$. The maps r, s extend naturally to E^*; for $v \in E^0$, we define $s(v) = r(v) = v$. A path $\mu \in E^*$ is a *loop* if it satisfies $s(\mu) = r(\mu)$. The loop μ is *simple* if the vertices $r(\mu_i)$ for $i \in \left\{1, \ldots, |\mu|\right\}$ are distinct. For $\alpha, \mu \in E^*$ satisfying $s(\mu) = r(\alpha)$, we define $\alpha\mu := (\alpha_1, \ldots, \alpha_{|\alpha|}, \mu_1, \ldots, \mu_{|\mu|})$.

Following [**13**], if E is a row-finite directed graph we say that a *Cuntz-Krieger E-family* in a C^*-algebra B consists of a set $\{p_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{s_e : e \in E^1\}$ of partial isometries satisfying

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e : s(e) = v\}} s_e s_e^* \text{ for } v \in s(E^1).$$

Note that this last equation does not apply if v is a sink, so the projection p_v can be non-zero. The above relations need to be weakened for graphs which are not row-finite (see, for example [**2**]). However, as we shall see, the results we prove in this paper do not hold in this more general setting, and so we shall focus our attention on row-finite graphs.

In [**13**, Theorem 1.2], it is proved that for every row-finite directed graph E there is a universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$ with all projections non-zero. For $\alpha = (\alpha_1, \dots, \alpha_n) \in E^n$, $s_\alpha := s_{\alpha_1} \dots s_{\alpha_n}$ is a partial isometry with initial projection $s_\alpha^* s_\alpha = p_{r(\alpha)}$ and final projection $s_\alpha s_\alpha^* \leqslant p_{s(\alpha)}$. We also set $s_v := p_v$ for $v \in E^0$. Using this notation, we can write $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*\}$.

The *gauge action* α of $\mathbb{T}$ on $C^*(E)$ is a strongly continuous action which satisfies $\alpha_z(s_e) = z s_e$ and $\alpha_z(p_v) = p_v$. The *gauge-invariant uniqueness theorem* [**3**, Theorem 2.1] (see also [**2**, Theorem 2.1]) states that $C^*(E)$ is characterised by the existence of such a canonical action of $\mathbb{T}$.

3. Higher order graphs

Let $E = (E^0, E^1, r, s)$ be a directed graph which contains no sinks. For $q > p$ we define maps $r^p, s^p : E^q \to E^p$ by $r^p(\alpha) := \alpha_{q-p+1} \dots \alpha_q$ and $s^p(\alpha) := \alpha_1 \dots \alpha_p$. For $p, q \in \mathbb{N} \cup \{0\}$ satisfying $p < q$ we can construct a *higher order directed graph* (E^p, E^q, r^p, s^p) from E. We note that if E is a row-finite graph, then (E^p, E^q, r^p, s^p) is also row-finite. We denote the universal C^*-algebra of the directed graph (E^p, E^q, r^p, s^p) by $C^*(E^p, E^q)$, and note that $C^*(E) = C^*(E^0, E^1)$.

The graph $\widehat{E} := (E^1, E^2, r^1, s^1)$ appears quite frequently in the literature. Following [**17**], we shall call this graph the *dual graph*. [**3**, Corollary 2.5] states that if E is a row-finite graph with no sources or sinks, then $C^*(E) \cong C^*(\widehat{E})$. The following theorem is a generalisation of this result.

Theorem 3.1. *Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph which contains no sinks. Let $p, q \in \mathbb{N}$ satisfy $p < q$. Let $\{s_\alpha, p_\delta : \alpha \in E^q, \delta \in E^p\}$ and $\{t_\lambda, q_\beta : \lambda \in E^{q+1}, \beta \in E^{p+1}\}$ be the canonical Cuntz-Krieger (E^p, E^q) and (E^{p+1}, E^{q+1})-families. For $\beta \in E^{p+1}$, define*

$$r_\beta := \sum_{\substack{\alpha \in E^q \\ s^{p+1}(\alpha) = \beta}} s_\alpha s_\alpha^*$$

and for $\lambda \in E^{q+1}$ define

$$u_\lambda := \sum_{\substack{\gamma \in E^q \\ r^{p+1}(\lambda) = s^{p+1}(\gamma)}} s_{s^q(\lambda)} s_\gamma s_\gamma^*$$

Then $\{r_\beta, u_\lambda\}$ is a Cuntz-Krieger (E^{p+1}, E^{q+1})-family and there is an isomorphism $\psi : C^*(E^p, E^q) \to C^*(E^{p+1}, E^{q+1})$ such that $\psi(r_\beta) = q_\beta$ and $\psi(u_\lambda) = t_\lambda$ for $\beta \in E^{p+1}$ and $\lambda \in E^{q+1}$. Moreover, for $\mu \in E^q$, we have $\psi(s_\mu) = \sum_{\substack{\lambda \in E^{q+1} \\ s^q(\lambda) = \mu}} t_\lambda$. The isomorphism ψ restricts to an isomorphism of the fixed point algebras under the canonical gauge actions.

Proof: We show that [**3**, Theorem 2.1] can be applied to obtain the required isomorphism between $C^*(E^p, E^q)$ and $C^*(E^{p+1}, E^{q+1})$.

First, we claim that $\{u_\lambda, r_\beta : \lambda \in E^{q+1}, \beta \in E^{p+1}\}$ is a Cuntz-Krieger (E^{p+1}, E^{q+1})-family in which each r_β is non-zero. Since there are no sinks in E, there are no sinks in

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(E^p, E^q). Hence there is always at least one path of length q extending each path β of length $p+1$, and $r_\beta \neq 0$ for every $\beta \in E^{p+1}$. Also, for all $\beta \in E^{p+1}$, r_β is a projection in $C^*(E^p, E^q)$ since it is the sum of orthogonal projections $s_\alpha s_\alpha^*$. For $\lambda \in E^{q+1}$, we have

$$u_\lambda^* u_\lambda = \sum_{\substack{\gamma,\delta \in E^q \\ s^{p+1}(\gamma)=s^{p+1}(\delta)=r^{p+1}(\lambda)}} s_\gamma s_\gamma^* s_{s^q(\lambda)}^* s_{s^q(\lambda)} s_\delta s_\delta^* = \sum_{\substack{\gamma \in E^q \\ s^{p+1}(\gamma)=r^{p+1}(\lambda)}} s_\gamma s_\gamma^* = r_{r^{p+1}(\lambda)}$$

which is a projection in $C^*(E^p, E^q)$, and thus u_λ is a partial isometry in $C^*(E^p, E^q)$. For $\beta \in E^{p+1}$, we use the Cuntz-Krieger relations for $C^*(E^p, E^q)$ to obtain

$$r_\beta = \sum_{\substack{\delta \in E^q \\ s^{p+1}(\delta)=\beta}} s_\delta s_\delta^* = \sum_{\substack{\gamma,\delta \in E^q \\ s^{p+1}(\delta)=\beta, r^p(\delta)=s^p(\gamma)}} s_\delta s_\gamma s_\gamma^* s_\delta^* = \sum_{\substack{\gamma \in E^q, \lambda \in E^{q+1} \\ s^{p+1}(\lambda)=\beta \\ r^{p+1}(\lambda)=s^{p+1}(\gamma)}} s_{s^q(\lambda)} s_\gamma s_\gamma^* s_{s^q(\lambda)}^*$$

since (E^p, E^q) contains no sinks and the condition $r^{p+1}(\lambda) = s^{p+1}(\gamma)$ ensures that there is a unique $\lambda \in E^{q+1}$ corresponding to each $\delta \in E^q$ from the preceding sum. Thus $r_\beta = \sum_{\substack{\lambda \in E^{q+1} \\ s^{p+1}(\lambda)=\beta}} u_\lambda u_\lambda^*$, establishing our claim.

Next, we claim that $\{u_\lambda, r_\beta : \lambda \in E^{q+1}, \beta \in E^{p+1}\}$ generates $C^*(E^p, E^q)$. Since (E^p, E^q) contains no sinks, $C^*(E^p, E^q) = C^*(\{s_\alpha : \alpha \in E^q\})$. Let $\alpha \in E^q$. Then

$$s_\alpha = s_\alpha p_{r^p(\alpha)} = \sum_{\substack{\gamma \in E^q \\ s^p(\gamma)=r^p(\alpha)}} s_\alpha s_\gamma s_\gamma^* = \sum_{\substack{\lambda \in E^{q+1}, \gamma \in E^q \\ s^q(\lambda)=\alpha, s^{p+1}(\gamma)=r^{p+1}(\lambda)}} s_{s^q(\lambda)} s_\gamma s_\gamma^* = \sum_{\substack{\lambda \in E^{q+1} \\ s^q(\lambda)=\alpha}} u_\lambda,$$

and our claim follows.

Last we show that the gauge action on $C^*(E^p, E^q)$ satisfies the required properties for the application of [**3**, Theorem 2.1]. Let α^p and α^{p+1} denote the canonical gauge actions on $C^*(E^p, E^q)$ and $C^*(E^{p+1}, E^{q+1})$ respectively (see [**3**, Section 1]). Then for $z \in \mathbb{T}$ and $\lambda \in E^{q+1}$, we have $\alpha_z^p(u_\lambda) = zu_\lambda$, and for $\beta \in E^{p+1}$, $\alpha_z^p(r_\beta) = r_\beta$. Thus we may apply [**3**, Theorem 2.1] to obtain the required isomorphism ψ of $C^*(E^p, E^q)$ and $C^*(E^{p+1}, E^{q+1})$.

To prove the final statement we note that since α^p, α^{p+1} are strongly continuous, ψ is norm continuous, and for all $z \in \mathbb{T}$, $\alpha_z^{p+1} \circ \psi$ and $\psi \circ \alpha_z^p$ agree on the canonical Cuntz-Krieger (E^p, E^q)-family, we must have $\psi \circ \alpha_z^p = \alpha_z^{p+1} \circ \psi$. It follows that $\psi \circ \phi_p = \phi_{p+1} \circ \psi$ where $\phi_p : b \mapsto \int \alpha_z^p(b) dm(z)$ and $\phi_{p+1} : b \mapsto \int \alpha_z^{p+1}(b) dm(z)$ are the canonical expectations of $C^*(E^p, E^q)$ and $C^*(E^{p+1}, E^{q+1})$ onto their respective fixed point algebras (see [**3**, Section 1]). Thus the isomorphism ψ restricts to an isomorphism of the fixed point algebras, establishing our result. $\square$

REMARK 3.2. It is not clear how one would define a higher order directed graph (E^p, E^q) for a graph E which contains sinks. Indeed, there may be no paths of length p or q in E.

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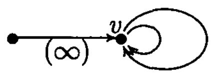
We may apply Theorem 3.1 inductively to obtain the following Corollary.

COROLLARY 3.3. *Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph which contains no sinks. Let $p, q, l, m \in \mathbb{N} \cup \{0\}$ satisfy the conditions that $p < q$, $l < m$ and $q - p = m - l$. Then $C^*(E^p, E^q) \cong C^*(E^l, E^m)$.*

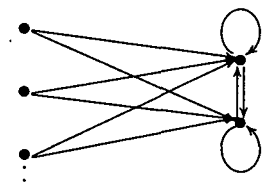
The following example shows that we cannot weaken our hypotheses to include graphs which are not row-finite.

EXAMPLE 3.4. Consider the directed graph

$E :=$ 

Since E satisfies condition (K) (every vertex lies either on 0 or 2 loops), the results of [**2**, Section 3] tell us that $C^*(E)$ is not simple: there is a non-trivial ideal generated by the projection associated to the vertex v.

However, E has dual graph

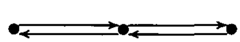
$\widehat{E} :=$ 

which is row-finite and $C^*(\widehat{E})$ is simple by [**3**, Proposition 5.1]. Thus $C^*(E) \not\cong C^*(\widehat{E})$.

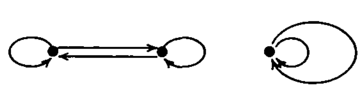
Note that one can also obtain examples of non-row-finite graphs which contain no sources and for which $C^*(E) \not\cong C^*(\widehat{E})$.

REMARK 3.5. For unequal gaps $q - p$ and $m - l$, we may not always have $C^*(E^p, E^q) \cong C^*(E^l, E^m)$ for a given directed graph E. For example, if E is a simple loop of length n, then $C^*(E) \cong M_n(C(\mathbb{T}))$ by, for example, [**9**, Lemma 2.4], but $C^*(E^0, E^n) \cong C(\mathbb{T})^n \not\cong C^*(E)$.

Even for graphs with simple C^*-algebras we may not have $C^*(E) \cong C^*(E^p, E^q)$ for $q - p \neq 1$. Consider the directed graph

$F :=$ 

which has higher order graph

$(F^0, F^2) =$ 

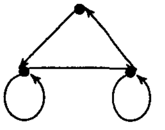
Here $C^*(F)$ is simple by [**14**, Corollary 6.8], but $C^*(F^0, F^2)$ can be written as the direct sum of the C^*-algebras of the two connected components of (F^0, F^2), and hence is not simple.

It is natural to ask whether graphs E and F which satisfy $C^*(E) \cong C^*(F)$ also satisfy $C^*(E^p, E^q) \cong C^*(F^p, F^q)$ for all $p, q \in \mathbb{N} \cup \{0\}$ such that $p < q$. The following example shows that this is not the case.

EXAMPLE 3.6. Consider the graphs

$G :=$ 

and the directed graph F discussed in Remark 3.5. Both $C^*(F)$ and $C^*(G)$ are simple by [**14**, Corollary 6.8] and have trivial K-theory by [**14**, Corollary 6.12]. It follows by [**19**, Theorem 6.5] that $C^*(F) \cong C^*(G) \cong \mathcal{O}_2$. However, $C^*(F^0, F^2)$ is isomorphic to a direct sum of two simple C^*-algebras, but the graph C^*-algebra of

$(G^0, G^2) =$ 

is simple by [**3**, Proposition 5.1]. Thus $C^*(F^0, F^2)$ and $C^*(G^0, G^2)$ cannot be isomorphic.

REMARK 3.7. The above example also illustrates the fact that the K-theory gives us little insight into whether $C^*(E) \cong C^*(E^p, E^q)$ for $p, q \in \mathbb{N} \cup \{0\}$ satisfying $p < q$. In both cases the K-theory of the graph C^*-algebra and the K-theory of the higher order graph C^*-algebra are trivial by [**14**, Corollary 6.12], and hence isomorphic. However, $C^*(F)$ and $C^*(F^0, F^2)$ are not isomorphic.

4. Cartesian Products of Directed Graphs

Following [**12**], if E and F are two row-finite directed graphs, we define the *Cartesian product graph*

$$E \times F := (E^0 \times F^0, E^1 \times F^1, r, s)$$

where for $(e, f) \in E^1 \times F^1$, $r(e, f) := \big(r(e), r(f)\big)$ and $s(e, f) := \big(s(e), s(f)\big)$. We warn that this is not the standard graph-theoretic definition for the Cartesian product of directed graphs (see, for example, [**8**, Definition 1.3.3]). However, the graph $E \times F$ is of interest as its C^*-algebra appears quite frequently as a crossed product. See, for example, [**12**].

As in [**11**], given C^*-algebras A and B and a compact Abelian group G with actions $\mu : G \to \operatorname{Aut} A$ and $\nu : G \to \operatorname{Aut} B$, we define $A \otimes_G B$ to be the fixed point algebra $(A \otimes B)^\lambda$ under the action $\lambda : G \to \operatorname{Aut}(A \otimes B)$ defined by $\lambda_g(a \otimes b) = \mu_g(a) \otimes \nu_{g^{-1}}(b)$.

PROPOSITION 4.1. ([**11**, Fact 2.7]) *Let E and F be row-finite directed graphs with no sinks. Then there is an isomorphism*

$$\phi : C^*(E \times F) \to C^*(E) \otimes_{\mathbb{T}} C^*(F)$$

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which is equivariant for the gauge action $\alpha \otimes id$ on $C^*(E \times F)$. In particular, for integers $m, n \geqslant 1$, one has $\mathcal{O}_{mn} \cong \mathcal{O}_m \otimes_{\mathbb{T}} \mathcal{O}_n$ and, for non-negative integer matrices A, B with no zero row or column, one has $\mathcal{O}_{A\otimes B} \cong \mathcal{O}_A \otimes_{\mathbb{T}} \mathcal{O}_B$.

PROOF: Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family and let $\{t_f, q_w\}$ be the canonical Cuntz-Krieger F-family. We construct a Cuntz-Krieger $E \times F$-family in $C^*(E) \otimes C^*(F)$ in the following manner. For $(e, f) \in E^1 \times F^1$, we define

$$u_{(e,f)} := s_e \otimes t_f$$

and for $(v, w) \in E^0 \times F^0$, we define

$$r_{(v,w)} := p_v \otimes q_w.$$

Routine calculations show that $\{u_{(e,f)}, r_{(v,w)}\}$ is a Cuntz-Krieger $E \times F$-family. Its elements are non-zero partial isometries and projections by construction.

We claim that $C^*\big(\{r_{(v,w)}, u_{(e,f)}\}\big) = C^*(E) \otimes_{\mathbb{T}} C^*(F)$. Let λ denote the action on $C^*(E) \otimes C^*(F)$ induced by the canonical gauge actions on $C^*(E)$ and $C^*(F)$. Then for all $z \in \mathbb{T}$ we have $\lambda_z(u_{(e,f)}) = u_{(e,f)}$ for all $(e, f) \in E^1 \times F^1$ and $\lambda_z(r_{(v,w)}) = r_{(v,w)}$ for all $(v, w) \in E^0 \times F^0$. It follows by the linearity, multiplicativity and strong continuity of λ that $C^*\big(\{r_{(v,w)}, u_{(e,f)}\}\big) \subseteq \big(C^*(E) \otimes C^*(F)\big)^{\lambda}$. We prove the reverse inclusion.

Consider the conditional expectation $P : C^*(E) \otimes C^*(F) \to C^*(E) \otimes_{\mathbb{T}} C^*(F)$ defined by $P(t) = \int \lambda_z(t)dm(z)$. Let $t \in C^*(E) \otimes C^*(F)$. Then t may be approximated by a finite linear combination of elementary tensors of the form $s_\alpha s_\beta^* \otimes t_\gamma t_\delta^*$. Now

$$P(s_\alpha s_\beta^* \otimes t_\gamma t_\delta^*) = \int_{\mathbb{T}} z^{|\alpha|-|\beta|-|\gamma|+|\delta|} s_\alpha s_\beta^* \otimes t_\gamma t_\delta^* \; dm(z)$$

which is non-zero if and only if $|\alpha| - |\beta| = |\gamma| - |\delta|$, in which case, $P(s_\alpha s_\beta^* \otimes t_\gamma t_\delta^*) = s_\alpha s_\beta^* \otimes t_\gamma t_\delta^*$. So, if $a = P(a) \in C^*(E) \otimes_{\mathbb{T}} C^*(F)$, the continuity of P implies that a can be approximated by linear combinations of $\big\{s_\alpha s_\beta^* \otimes t_\gamma t_\delta^* : \alpha, \beta \in E^*, \gamma, \delta \in F^*, |\alpha| - |\beta| = |\gamma| - |\delta|\big\}$. To show that $\big\{s_\alpha s_\beta^* \otimes t_\gamma t_\delta^* : \alpha, \beta \in E^*, \gamma, \delta \in F^*, |\alpha| - |\beta| = |\gamma| - |\delta|\big\} \subseteq C^*\big(\{u_{(e,f)}, r_{(v,w)}\}\big)$, we note that since the graphs E and F contain no sinks, we may repeatedly apply the Cuntz-Krieger relations to extend paths $\alpha, \beta, \gamma, \delta$ as necessary, and thereby write each term $s_\alpha s_\beta^* \otimes t_\gamma t_\delta^*$ as a finite linear combination of terms $s_{\alpha'} s_{\beta'}^* \otimes t_{\gamma'} t_{\delta'}^*$ where $\alpha', \beta' \in E^*$ and $\gamma', \delta' \in F^*$ satisfy $|\alpha'| = |\gamma'|$ and $|\beta'| = |\delta'|$. It follows that $C^*(E) \otimes_{\mathbb{T}} C^*(F) \subseteq C^*\big(\{u_{(e,f)}, r_{(v,w)}\}\big)$, establishing our claim.

The gauge action $(\alpha \otimes \mathrm{id})$ on $C^*(E) \otimes_{\mathbb{T}} C^*(F)$ satisfies $(\alpha_z \otimes \mathrm{id})(u_{(e,f)}) = z u_{(e,f)}$ and $(\alpha_z \otimes \mathrm{id})(r_{(v,w)}) = r_{(v,w)}$. Our result follows by [**3**, Theorem 2.1]. ∎

REMARK 4.2. The map $\phi : C^*(E \times F) \to C^*(E) \otimes_{\mathbb{T}} C^*(F)$ will not, in general, be surjective for graphs containing sinks or infinite valence vertices. We cannot extend paths ending at sinks or sum projections at infinite valence vertices to prove that $C^*(E) \otimes_{\mathbb{T}} C^*(F) \subseteq C^*\big(\{u_{(e,f)}, r_{(v,w)}\}\big)$.

5. Elementary Strong Shift Equivalence

Generalising [**1**] (see also [**6**] for some related work using the adjacency matrices of the graphs) we give the following definition of elementary strong shift equivalence for directed graphs.

Definition 5.1: Let $E_i = (E_i^0, E_i^1, r^i, s^i)$ for $i = 1, 2$ be directed graphs. Suppose there is a directed graph $E_3 = (E_3^0, E_3^1, r_3, s_3)$ which has the following properties:

(a) $E_3^0 = E_1^0 \cup E_2^0$, and $E_1^0 \cap E_2^0 = \emptyset$.

(b) $E_3^1 = E_{12}^1 \cup E_{21}^1$ where $E_{ij}^1 := \{e \in E_3^1 : s_3(e) \in E_i^0, r_3(e) \in E_j^0\}$.

(c) For $i \in \{1,2\}$, there exist source and range-preserving bijections $\theta_i : E_i^1 \to E_3^2(E_i^0, E_i^0)$ where for $i \in \{1,2\}$, $E_3^2(E_i^0, E_i^0) := \{\alpha \in E_3^2 : s_3(\alpha) \in E_i^0, r_3(\alpha) \in E_i^0\}$.

Then we say that E_1 and E_2 are *elementary strong shift equivalent* ($E_1 \sim_{ES} E_2$) *via* E_3.

We define *strong shift equivalence* (denoted by $\sim_S$) to be the equivalence relation on row-finite directed graphs generated by elementary strong shift equivalence.

A version of the following theorem was proved in [**1**] for finite graphs which satisfy condition (L) (every loop has an exit). Our result holds for general row-finite graphs. Note that if $\{s_e, p_v\}$ is the canonical Cuntz-Krieger E-family for some row-finite graph E, then it follows easily from [**3**, Lemma 1.1], that for any subset $X \subseteq E^0$, the infinite sum $\sum_{v \in X} p_v$ converges strictly to a projection $P \in \mathcal{M}(C^*(E))$.

Theorem 5.2. *Let E_1, E_2 and E_3 be row-finite directed graphs which contain no sinks and which satisfy the conditions that $E_1 \sim_{ES} E_2$ via E_3. Let $\theta_1 : E_1^1 \to E_3^2(E_1^0, E_1^0)$ denote the canonical bijection. Suppose $\{s_e, p_v\}$ is the canonical Cuntz-Krieger E_3-family and that $\{t_f, q_w\}$ is the canonical Cuntz-Krieger E_1-family. Then there is an isomorphism*

$$\phi : PC^*(E_3)P \to C^*(E_1)$$

where $P = \sum_{v \in E_1^0} p_v \in \mathcal{M}(C^*(E_3))$. This isomorphism satisfies the conditions that $\phi(s_{\theta_1(e)}) = t_e$ for all $e \in E_1^1$ and $\phi(p_v) = q_v$ for all $v \in E_1^0$. Moreover, $PC^*(E_3)P$ is a full corner in $C^*(E_3)$ and $C^*(E_1) \sim_{SME} C^*(E_2)$. If E_1 and E_2 are row-finite graphs which satisfy $E_1 \sim_S E_2$, then $C^*(E_1) \sim_{SME} C^*(E_2)$.

Proof: We begin by constructing a Cuntz-Krieger E_1-family in $C^*(E_3)$. Let $e \in E_1^1$, write $\theta_1(e) = e_1 e_2$ and define $u_e := s_{\theta_1(e)} = s_{e_1} s_{e_2}$. For $v \in E_0^1$, we define the projection $r_v := p_v$. Then $\{u_e, r_v\}$ is a family of partial isometries and projections in $C^*(E_3)$ with $r_v \neq 0$ for all $v \in E_1^0$ and the $u_e u_e^*$ pair-wise orthogonal. We claim that $\{u_e, r_v\}$ is a

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Cuntz-Krieger E_1-family. Let $v \in s(E_1^0)$. Then $v \in s(E_3^0)$ and

$$\begin{aligned} r_v = p_v &= \sum_{\{e \in E_3^1 : s_3(e) = v\}} s_e s_e^* \\ &= \sum_{\{ef \in E_3^2(E_1^0, E_1^0) : s_3(ef) = v\}} s_{ef} s_{ef}^* \\ &= \sum_{\{\theta_1^{-1}(ef) \in E_1^1 : s_1(\theta_1^{-1}(ef)) = v\}} u_{\theta_1^{-1}(ef)} u_{\theta_1^{-1}(ef)}^* \\ &= \sum_{\{k \in E_1^1 : s_1(k) = v\}} u_k u_k^* \end{aligned}$$

where we have repeatedly used condition c) Definition 5.1. Also, for $e \in E_1^1$, we have

$$u_e^* u_e = s_{\theta_1(e)}^* s_{\theta_1(e)} = s_{e_1 e_2}^* s_{e_1 e_2} = p_{r_3(e_1 e_2)} = p_{r_3(\theta_1(e))} = p_{r_1(e)} = r_{r_1(e)}$$

since θ_1 is a range-preserving bijection, establishing our claim.

Next we claim that $\{u_e, r_v\}$ generates $PC^*(E_3)P$. Note that since $Pu_eP = u_e$ for all $e \in E_1^1$ and $Pr_vP = r_v$ for all $v \in E_1^0$ we have $C^*(\{u_e, r_v\}) \subseteq PC^*(E_3)P$. We prove the reverse inclusion.

Since $\{s_\mu s_\nu^* : \mu, \nu \in E_3^*\}$ spans a dense subalgebra of $C^*(E_3)$, it is enough to show that $Ps_\mu s_\nu^* P \in C^*(\{u_e, e_v\})$ for all paths $\mu, \nu \in E_3^*$. Now

$$Ps_\mu s_\nu^* P = \begin{cases} s_\mu s_\nu^* & \text{if } s_3(\mu), r_3(\nu) \in E_1^0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $r_3(\mu) = r_3(\nu)$ for $s_\mu s_\nu^* \neq 0$, then $\mu, \nu \in E_3^*$ either both end and start in E_1^0, or start in E_1^0 and end in E_2^0. So either both $|\mu|$ and $|\nu|$ are even or both are odd by construction of E_3. If both are $|\mu|$ and $|\nu|$ are odd, then since E_3 contains no sinks we can write $s_\mu s_\nu^* = \sum_{e \in E_{21}^1 : s(e) = r(\mu)} s_{\mu e} s_{\nu e}^*$, where each of the paths μe and νe has even length. Thus in both cases we can write $s_\mu s_\nu^*$ as a finite sum of products of $u_{\theta_1}(f)$ and r_v for $f \in E_1^1$ and $v \in E_1^0$ and hence $s_\mu s_\nu^* \in C^*(\{u_e, r_v\})$. It follows that that $\{u_e, r_v\}$ generates $PC^*(E_3)P$.

Define a strongly continuous $\mathbb{T}$-action α on $C^*(E_3)$ by

$$\alpha_z s_e = \begin{cases} z s_e & \text{if } e \in E_{12}^1 \\ s_e & \text{if } e \in E_{21}^1 \end{cases}$$

and

$$\alpha_z p_v = p_v$$

for $z \in \mathbb{T}$. Then for $z \in \mathbb{T}$ we have $\alpha_z(u_e) = z u_e$ and $\alpha_z(r_v) = r_v$. The gauge action γ fixes each partial sum of P and so P remains fixed under the extension of γ to $\mathcal{M}(C^*(E_3))$.

It follows that $P(C^*(E_3))P$ carries a gauge action which satisfies the properties required for an application of the gauge-invariant uniqueness theorem. By [**3**, Theorem 2.1] there is an isomorphism $\phi : PC^*(E_3)P \to C^*(E_1)$ which satisfies the properties required for our result.

To see that the corner $PC^*(E_3)P$ is full in $C^*(E_3)$ we note that E_3^0 is the smallest saturated hereditary set containing E_1^0. It follows from [**3**, Theorem 4.4] that any ideal containing the corner $PC^*(E_3)P$ must also contain $C^*(E_3)$. Thus $PC^*(E_3)P$ is full in $C^*(E_3)$.

A symmetric argument shows that $C^*(E_2)$ is isomorphic to a full corner in $C^*(E_3)$ and the strong Morita equivalence of $C^*(E_1)$ and $C^*(E_2)$ follows. The final statement follows by induction. $\square$

REMARK 5.3. We can also talk about elementary strong shift equivalence for graphs with sinks, but no isolated vertices. However we need to modify our definition of E_3 slightly to ensure that both graphs E_1 and E_2 involved have the same number of sinks. If we modify the third condition in Definition 5.1 to

(c) There are range and source preserving bijections:

$$\theta_1 : E_1^1 \cup \{\text{sinks}\} \to E_3^2(E_1^0, E_1^0) \cup \{e \in E_{12}^1 : r(e) \text{ a sink }\}$$

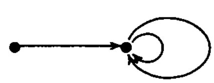
and

$$\theta_2 : E_2^1 \cup \{\text{sinks}\} \to E_3^2(E_2^0, E_2^0) \cup \{\text{sinks}\}$$

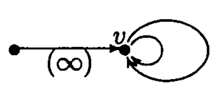
then the proof above will carry over with only slight modifications.

The following example shows that we cannot expand the hypotheses for Theorem 5.2 to include arbitrary graphs which are not row-finite.

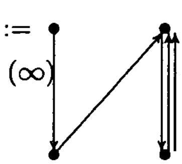
EXAMPLE 5.4. Consider the graphs

$E :=$ 

and

$F :=$ 

We have $F \sim_{ES} E$ via E_3 where

$E_3 :=$ 

However $C^*(E)$ is simple and $C^*(F)$ has a non-trivial ideal generated by the projection associated to the vertex v. Thus $C^*(E) \not\sim_{SME} C^*(F)$.

https://doi.org/10.1017/S0004972700020670 Published online by Cambridge University Press

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