# ON THE HOMOLOGY THEORY OF ABELIAN GROUPS

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**1. Introduction.** In **(1)** we have introduced the notions of "construction" and "generic acyclicity" in order to determine a homology theory for any class of multiplicative systems defined by identities. Among these classes the most interesting one is the class of associative and commutative systems $\Pi$ with a unit element (containing the class of abelian groups). For this class we have exhibited a "cubical" construction $Q(\Pi)$ and proved its generic acyclicity. We have also indicated the initial stages (in low dimensions) of a second construction $A(\Pi)$. In the present paper, this gap is completed; the construction $A(\Pi)$ is defined (§4) and its generic acyclicity is proved (§5). The complexes $A(\Pi)$ are closely related to the complexes $A(\Pi, n)$ used in the theory of the groups $H(\Pi, n)$ **(2; 3)** which were introduced with a view to topological applications. It is this relation with the groups $H(\Pi, n)$ which was the initial motivation for the introduction of the notion of generic acyclicity.

We have found it convenient to employ extensively the notion of tensor product. In this connection we found a theorem characterizing the natural homomorphisms of tensor products, which, besides its own merits, has the effect of eliminating many explicit calculations that would otherwise be needed in the proofs of §5. This theorem on tensor products is stated in §2 and proved in §3. These two sections form a self-contained unit, which can be read independently of the rest of the paper.

**2. Natural homomorphisms of tensor products.** An *augmented commutative ring* $R$ is a commutative ring together with a ring homomorphism $\alpha$: $R \to J$ into the ring $J$ of integers; an (augmented) ring homomorphism $\beta$: $R \to R'$ of two such rings is an ordinary ring homomorphism satisfying $\alpha'\beta = \alpha$. We recall here the convention that each ring has an identity element and that each ring homomorphism maps the identity into the identity.

Important examples of augmented rings are obtained by considering the integral ring $R = J(\Pi)$ of an associative and commutative system $\Pi$ with a unit element. This ring $J(\Pi)$ is defined as the free abelian group generated by the elements $x \in \Pi$, with multiplication defined by the multiplication in $\Pi$ and with augmentation given by $\alpha x = 1$. In particular, if $\Pi$ is the free associative and commutative system with a set $\{y_\alpha\}$ as base, then $J(\Pi)$ is the free commutative ring with $\{y_\alpha\}$ as base, or equivalently $J(\Pi)$ is the polynomial ring $J[\{y_\alpha\}]$. The augmentation $\alpha$ assigns to each polynomial $P$ the sum of its coefficients; in other words, $\alpha(P) = P(1)$.

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If $R_1$ and $R_2$ are augmented commutative rings, their tensor product $R = R_1 \otimes R_2$ (over the integers) has the multiplication $(x_1 \otimes x_2)(x_1' \otimes x_2') = x_1x_1' \otimes x_2x_2'$ and the augmentation $\alpha(x_1 \otimes x_2) = (\alpha_1 x_1)(\alpha_2 x_2)$. If $\beta_1: R_1 \to R_1'$ and $\beta_2: R_2 \to R_2'$ are (augmented) ring homomorphisms then

$$\beta_1 \otimes \beta_2: R_1 \otimes R_2 \to R_1' \otimes R_2'$$

again is an (augmented) ring homomorphism. With this definition, $R_1 \otimes R_2$ constitutes a covariant functor of two variables.

We shall treat the tensor product as an associative operation and will consider the functors of $t$ variables

$$R_1^{m_1} \otimes \ldots \otimes R_t^{m_t} \tag{2.1}$$

where the $k$th factor is the $m_k$-fold tensor product $R_k \otimes \ldots \otimes R_k$. We adopt the conventions that $R^0 = J$ and that $R \otimes J = R = J \otimes R$; this allows us freely to omit or add terms with exponent 0. In view of the natural isomorphism $R_1 \otimes R_2 \approx R_2 \otimes R_1$, the functors (2.1) are essentially the only ones that can be constructed using the tensor product.

We shall be concerned with the natural transformations

$$\rho: R_1^{m_1} \otimes \ldots \otimes R_t^{m_t} \to R_1^{n_1} \otimes \ldots \otimes R_t^{n_t} \tag{2.2}$$

of one of the functors (2.1) into another. Such a $\rho$ is a family of maps, one for each $t$-tuple $(R_1, \ldots, R_t)$, satisfying the following naturality condition: if $\beta_k: R_k \to Q_k (k = 1, \ldots, t)$ are augmented ring homomorphisms then the diagram

$$\begin{array}{ccc} R_1^{m_1} \otimes \ldots \otimes R_t^{m_t} & \xrightarrow{\rho} & R_1^{n_1} \otimes \ldots \otimes R_t^{n_t} \\ \beta_1^{m_1} \otimes \ldots \otimes \beta_t^{m_t} \downarrow & & \downarrow \beta_1^{n_1} \otimes \ldots \otimes \beta_t^{n_t} \\ Q_1^{m_1} \otimes \ldots \otimes Q_t^{m_t} & \xrightarrow[\rho]{} & Q_1^{n_1} \otimes \ldots \otimes Q_t^{n_t} \end{array}$$

is commutative. In considering the transformations $\rho$ three points of view are possible:

(1) Each map (2.2) is a group homomorphism; then $\rho$ is called a *natural group homomorphism.* With fixed $(m_1, \ldots, m_t)$ and $(n_1, \ldots, n_t)$ the natural group homomorphisms form an abelian group.

(2) Each map (2.2) is a ring homomorphism; then $\rho$ is called a *natural ring homomorphism.*

(3) Each map (2.2) is an augmented ring homomorphism; then $\rho$ is called a *natural augmented ring homomorphism.* Important examples of natural augmented ring homomorphisms (for one variable $R$) are the following:

$$\rho: R^m \to R^{m+1}, \quad \rho(x_1 \otimes \ldots \otimes x_m) = x_1 \otimes \ldots \otimes x_m \otimes 1, \tag{2.3}$$

$$\rho: R^{m+1} \to R^m, \quad \rho(x_1 \otimes \ldots \otimes x_{m+1}) = (\alpha x_1)\, x_2 \otimes \ldots \otimes x_{m+1}, \tag{2.4}$$

$$\rho: R^{m+1} \to R^m, \quad \rho(x_1 \otimes \ldots \otimes x_{m+1}) = (x_1x_2) \otimes x_3 \otimes \ldots \otimes x_{m+1}, \tag{2.5}$$

$$\rho: R^m \to R^m, \quad \rho(x_1 \otimes \ldots \otimes x_m) = x_{\pi 1} \otimes \ldots \otimes x_{\pi m}, \tag{2.6}$$

the latter for any permutation $\pi$ of the digits $1, \ldots, m$.

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THEOREM I. *For given* $(m_1, \ldots, m_t)$ *and* $(n_1, \ldots, n_t)$, *the natural group homomorphisms* (2.2) *form a free abelian group with the set of natural ring homomorphisms as base. A natural ring homomorphism* $\rho$ *may written uniquely as*

$$\rho = \rho_1 \otimes \ldots \otimes \rho_t$$

*where* $\rho_k\colon R_k^{m_k} \to R_k^{n_k}$ *is a natural ring homomorphism* $(k = 1, \ldots, t)$. *Each such* $\rho_k$ *can be obtained by composition of the natural augmented ring homomorphisms of the type* (2.3)–(2.6).

COROLLARY. *All natural ring homomorphisms are augmented natural ring homomorphisms.*

COROLLARY. *The only natural ring homomorphism*

$$\rho\colon R_1^{m_1} \otimes \ldots \otimes R_t^{m_t} \to J$$

*is the augmentation map.*

Theorem I states in effect that natural group homomorphisms (2.2) are exactly those given by the obviously possible formulas. The proof will be given in the next section.

In all our exposition we have considered the ring $J$ of integers as the "ground-ring"; this could be replaced by an arbitrary commutative ring $K$. Then $R$ will be a $K$-algebra, $\alpha\colon R \to K$ will be a $K$-algebra homomorphism and all tensor products will be tensor products over $K$. Our proof of Theorem I necessitates the assumption that $K$ is an integral domain of characteristic zero.

**3. Proof of Theorem I.** The proof employs two elementary lemmas on polynomial identities.

LEMMA 1. *If* $x_1, \ldots, x_n$ *are independent indeterminates over an integral domain* $D$ *of characteristic* 0, *and if the polynomial* $f \in D[x_1, \ldots, x_n]$ *satisfies the identity*

$$f(x_1 + y_1 - z_1, \ldots, x_n + y_n - z_n) \\ = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n) - f(z_1, \ldots, z_n) \tag{3.1}$$

*then* $f$ *has degree at most* 1.

*Proof.* Set $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - f(0, \ldots, 0)$. Then $g$ satisfies

$$g(x_1 + y_1, \ldots, x_n + y_n) = g(x_1, \ldots, x_n) + g(y_1, \ldots, y_n).$$

This is the usual identity characterizing a homogeneous linear polynomial.

LEMMA 2. *If a polynomial* $f \in D[x_1, \ldots, x_n]$ *with coefficients in an integral domain* $D$ *satisfies the identity*

$$f(x_1y_1, \ldots, x_ny_n) = f(x_1, \ldots, x_n)\, f(y_1, \ldots, y_n), \tag{3.2}$$

*then* $f$ *is a monomial or zero.*

Here and in the sequel the word "monomial" will be understood as "monomial with coefficient 1."

*Proof.* The identity yields

$$f(x_1, \ldots, x_n) = f(x_1, 1, \ldots, 1) f(1, x_2, \ldots, x_n).$$

Since $f(x_1, 1, \ldots, 1)$ and $f(1, x_2, \ldots, x_n)$ each satisfy the identity (3.2) for 1 and $n - 1$ variables respectively, the proposition is reduced to the case $n = 1$. For this case, if $f \neq 0$, set $f(x) = ax^m + g(x)$, with $a \neq 0$ and $g(x)$ of degree $k$ less than $m$. Then $f(xy) = f(x)f(y)$ becomes

$$ax^m y^m + g(xy) = a^2x^m y^m + ax^m g(y) + ay^m g(x) + g(x)\, g(y).$$

Comparison of coefficients yields $a = a^2$ and hence $a = 1$, since $D$ is an integral domain. Consideration of terms of degree $m + k$ then shows that $g(x) = 0$. Thus $f(x) = x^m$, as desired.

We first prove Theorem I for $t = 1$. As a "model" ring for this proof we use the polynomial ring

$$P = J[y_1, \ldots, y_m], \qquad \alpha(y_i) = 1,$$

in the independent indeterminates $y_1, \ldots, y_m$ with integer coefficients. Let

$$\kappa_j\colon P \to P^n, \qquad j = 1, \ldots, n,$$

be the natural injection of $P$ into the $j$th factor of $P^n$. The ring $P^n$ will be identified with the polynomial ring $J[Y]$ in the set

$$Y = \{\kappa_1 y_1, \ldots, \kappa_n y_1; \ldots ; \kappa_1 y_m, \ldots, \kappa_n y_m\}$$

of $mn$ independent indeterminates $\kappa_j y_i$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, each with augmentation 1. A polynomial in the ring $J[Y]$ will be called *generic* if it is of degree at most one in each of the $m$ strings of variables $\kappa_1 y_i, \ldots, \kappa_n y_i$.

With each natural group homomorphism $\rho\colon R^m \to R^n$ we associate a polynomial $f_\rho \in J[Y]$ as follows:

$$f_\rho = \rho(y_1 \otimes \ldots \otimes y_m), \quad \rho\colon P^m \to P^n.$$

THEOREM I′. *The correspondence $\rho \to f_\rho$ establishes an isomorphism between the group of natural group homomorphisms and the subgroup of generic polynomials in $J[Y]$. Furthermore $\rho$ is a natural ring homomorphism if and only if $f_\rho$ is a monomial.*

Theorem I′ includes the first part of Theorem I, for $t = 1$. The remaining two parts of Theorem I will also be established in the course of the proof. We begin with

LEMMA 3. *If $x_i \in R$ $(i = 1, \ldots, m)$ are elements with $\alpha(x_i) = 1$, then*

$$\rho(x_1 \otimes \ldots \otimes x_m) = f_\rho(\kappa_j x_i).$$

The last symbol is to be interpreted as the value of the polynomial $f_\rho$ when each of the indeterminates $\kappa_j y_i$ is replaced by the element $\kappa_j x_i$ of $R^n$.

*Proof.* Let $\beta$: $P \rightarrow R$ be the (unique) augmented ring homomorphism satisfying $\beta(y_i) = x_i\,(i = 1, \ldots, m)$. Since $\beta^m$ is a ring homomorphism we obtain

$$\rho(x_1 \otimes \ldots \otimes x_m) = \rho\,\beta^m(y_1 \otimes \ldots \otimes y_m) = \beta^n f_\rho(\kappa_j y_i)$$
$$= f_\rho(\beta\kappa_j y_i) = f_\rho(\kappa_j \beta y_i) = f_\rho(\kappa_j x_i).$$

We are now ready to tackle the proof of Theorem I′. It is clear that the correspondence $\rho \rightarrow f_\rho$ is a homomorphism. To show that it is a monomorphism assume $f_\rho = 0$. Then, by Lemma 3, $\rho(x_1 \otimes \ldots \otimes x_m) = 0$ whenever the elements $x_i$ have augmentation 1. Now for any augmented ring $R$ and any element $x \in R$ we have

$$x = (\alpha x - 1)\,\theta + (x + (1 - \alpha x)\,\theta),$$

where $\theta$ denotes the identity element of $R$. Since $\theta$ and $x + (1 - \alpha x)\,\theta$ both have augmentation 1, it follows from the multilinearity of the tensor product that $\rho = 0$.

Next we show that the polynomial $f_\rho$ is generic. To simplify the notation, we shall show only that $f_\rho$ is of degree at most 1 in the string of variables $\kappa_1 y_1, \ldots, \kappa_n y_1$. Let $Z$ be the set of the remaining indeterminates of $Y$ and let $D = J[Z]$. Then $D$ is an integral domain of characteristic zero and we may identify $J[Y]$ with $D[\kappa_1 y_1, \ldots, \kappa_n y_1]$. Consider the ring

$$Q = P[u, v] = J[y_1, \ldots, y_m, u, v],$$

where the additional indeterminates $u$ and $v$ also have augmentation 1. Let $w = y_1 + u - v$; then $\alpha w = 1$. We apply Lemma 3 to the ring $Q$. Regarding $f_\rho$ as a polynomial with coefficients in $D$, we obtain

$$\rho(w \otimes y_2 \otimes \ldots \otimes y_m) = f_\rho(\kappa_1 w, \ldots, \kappa_n w).$$

If on the left-hand side we replace $w$ by $y_1 + u - v$, expand by linearity, and again apply Lemma 3, we obtain

$$f_\rho(\kappa_1 y_1, \ldots, \kappa_n y_1) + f_\rho(\kappa_1 u, \ldots, \kappa_n u) - f_\rho(\kappa_1 v, \ldots, \kappa_n v).$$

Thus $f_\rho \in D[\kappa_1 y_1, \ldots, \kappa_n y_1]$ satisfies the identity (3.1) of Lemma 1, and therefore $f_\rho$ is of degree at most 1 in the string of variables $\kappa_1 y_1, \ldots, \kappa_n y_1$.

Next, suppose that $\rho$ is a natural ring homomorphism. We apply $\rho$ to the ring $R = J[u_1, \ldots, u_m, v_1, \ldots, v_m]$, where $\alpha u_i = \alpha v_i = 1$. The definition of the product in $R^m$ yields

$$(u_1 \otimes \ldots \otimes u_m)(v_1 \otimes \ldots \otimes v_m) = u_1 v_1 \otimes \ldots \otimes u_m v_m.$$

Then applying Lemma 3 and the fact that $\rho$ is a ring homomorphism we obtain

$$f_\rho((\kappa_j u_i)(\kappa_j v_j)) = f_\rho(\kappa_j u_i)\, f_\rho(\kappa_j v_i).$$

Since $\kappa_j u_i$ and $\kappa_j v_i$ are arbitrary indeterminates, Lemma 2 implies that $f_\rho$ is a monomial.

To complete the proof of Theorem I′ we must show that each generic monomial $f \in J[Y]$ is of the form $f_\rho$ for some $\rho$. Observe that those natural ring homomorphisms $\rho\colon R^m \to R^n$ which are composites of homomorphisms of the type (2.3)–(2.6) may be described as follows. Let $A_0 \cup \ldots \cup A_n$ be a decomposition of the set $\{1, \ldots, m\}$ into disjoint sets $A_j$. Then

$$\rho(x_1 \otimes \ldots \otimes x_m) = \alpha(x_{A_0})\, x_{A_1} \otimes \ldots \otimes x_{A_n},$$

where $x_{A_j}$ is the product of the $x_i$'s with $i \in A$; and $x_{A_j} = 1$ if $A_j = 0$. The monomial $f_\rho$ of this homomorphism contains the indeterminate $\kappa_j y_i$ if and only if $x_i \in A_j$. Clearly any generic monomial may be obtained in this way from a suitable decomposition $A_0 \cup \ldots \cup A_n$.

The proof of Theorem I for general $t$ is like that above for $t = 1$, with the use of $t$ "model" rings

$$P_k = J[y_{k,1}, \ldots, y_{k,m_k}], \qquad \alpha(y_{k,i}) = 1;\ k = 1, \ldots, t.$$

For any $\rho$ as in (2.2) we again have $f_\rho$ in $J[Y]$, where $Y$ is now a set of $\sum m_k n_k$ independent indeterminates $\kappa_j y_{k,i}$; $k = 1, \ldots, t, j = 1, \ldots, n_k, i = 1, \ldots, m_k$, and

$$f_\rho = \rho(y_{1,1} \otimes \ldots \otimes y_{1,m_1} \otimes \ldots \otimes y_{t,1} \otimes \ldots \otimes y_{t,m_t}).$$

In concluding the proof, we observe that each generic monomial $f$ in the indeterminates $Y$ admits a unique factorization $f = f_1 \ldots f_t$, where each $f_k$ is a generic monomial in the indeterminates $\kappa_j y_{k,i}$ with $k$ fixed. Each $f_k$ determines a natural ring homomorphism $\rho_k\colon R_k^{m_k} \to R_k^{n_k}$ and $\rho = \rho_1 \otimes \ldots \otimes \rho_t$ satisfies $f_\rho = f_{\rho_1} \ldots f_{\rho_t} = f_1 \ldots f_t = f$.

**4. The complexes $A^n(R)$.** Let $R$ be an augmented commutative ring. We convert $R$ into a graded ring by assigning the degree zero to all the elements of $R$. Further by introducing a differentiation (i.e., boundary operator) which is identically zero we convert $R$ into an augmented graded $\partial$-ring. We may now apply the bar construction **(2**, §7**)** and define

$$A(R, 1) = B(R).$$

From now on we continue by applying the normalized bar construction **(2**, §12**)** and define

$$A(R, n) = B_N(A(R, n - 1)), \qquad n > 1.$$

The elements of $A(R, n)$ may be described as follows. For each sequence of integers $k_1, \ldots, k_{r-1}$, $1 \leqslant k_i \leqslant n$, consider the $r$-fold tensor product $T_r(R) = R \otimes \ldots \otimes R$. A typical element of this tensor product will be written as

$$[x_1|_{k_1} x_2|_{k_2} \ldots |_{k_{r-1}} x_r] \tag{4.1}$$

instead of the usual $x_1 \otimes x_2 \otimes \ldots \otimes x_r$. The degree (i.e., dimension) of the element (4.1) is $n + k_1 + \ldots + k_{r-1}$. As a graded group $A(R, n)$ is the direct sum of all these tensor products for all sequences $k_1, \ldots, k_{r-1}$ as above and of the group $J$ whose elements have degree zero. Thus $A(R, n)$ has no elements of degrees $0 < d < n$ while, in degree $n$, $A(R, n)$ consists of $R$ itself. In addition, $A(R, n)$ is equipped with a boundary operator $\partial$ and a product $*_n$ which are natural and which convert $A(R, n)$ into a graded $\partial$-ring. The complexes $A(R, n)$ for various $n$ are compared by means of the suspension map

$$S: A(R, n) \to A(R, n + 1)$$

which is a monomorphism of the additive group structure, raises the dimension by 1, and anticommutes with the boundary operator.

The details of the definition of $A(R, n)$ were motivated by geometrical applications. Indeed, if $\Pi$ is an abelian group, then the homology groups of $A(J(\Pi), n)$ are those of a space $X$ with vanishing homotopy groups $\pi_i(X)$ except for $\pi_n(X) \approx \Pi$. For algebraic purposes it is convenient to introduce the complexes $A^n(R)$ obtained from $A(R, n)$ by the following three modifications: (i) The group $J$ in degree zero is removed. (ii) All degrees are lowered by $n - 1$; the element (4.1) has thus degree $1 + k_1 + \ldots + k_{r-1}$. (iii) The boundary operator $\partial$ is replaced by $\partial' = (-1)^n\partial$.

With these changes the suspension operator $S: A^n(R) \to A^{n+1}(R)$ preserves degree and boundary and may be regarded as an inclusion. We thus obtain the nested sequence of complexes

$$A^1(R) \subset A^2(R) \subset \ldots \subset A^n(R) \subset \ldots,$$

whose union will be denoted by $A(R)$. In this complex we have no product; each of the subcomplexes $A^n(R)$ has a product $*_n$ (inherited from $A(R, n)$) which, however, does not have a unit element.

The complexes $A^n(R)$ and $A(R)$ commence in degree 1 with $R$ while in degree 2 we have the group $R \otimes R$ with boundary given by

$$\partial[x_1|_1x_2] = (\alpha x_2)[x_1] - [x_1x_2] + (\alpha x_1)[x_2]. \tag{4.2}$$

The 1-dimensional homology group thus is the quotient of $R$ by the subgroup generated by the elements (4.2). We may denote this quotient by $h(R)$ and adjoin it to the complex $A^n(R)$ (or $A(R)$) in dimension zero, with the natural map $R \to h(R)$ as boundary operator. The resulting complex will be called the augmented complex $A^n(R)$ (or $A(R)$); its homology groups are trivial in dimensions $<2$.

If $\Pi$ is a commutative associative system with a unit element and $R = J(\Pi)$ then it will be customary to write $A(\Pi, n)$, $A^n(\Pi)$, $A(\Pi)$, and $h(\Pi)$ instead of $A(J(\Pi), n)$, etc.

In particular, let $\Pi$ be a free system with base $\{y_\alpha\}$. Then $R = J(\Pi)$ is the ring of polynomials $J[y_\alpha]$. Let $\sigma = [x_1|_{k_1} \ldots |_{k_{r-1}}x_r]$ be an element of $A(\Pi, n)$ (or of $A^n(\Pi)$ or of $A(\Pi)$) with $x_1, \ldots, x_r \in \Pi$. We shall say that $\sigma$ is *generic* if the product $x_1 \ldots x_r$ is a monomial of degree $\leqslant 1$ in each variable $y_\alpha$; i.e., if no generator $y_\alpha$ is repeated in $x_1 \ldots x_r$. The unit element of $A(\Pi, n)$ also is regarded as generic. The generic elements span a subgroup which will be denoted by $A(y_\alpha; n)$ (or $A^n(y_\alpha)$ or $A(y_\alpha)$). The complexes considered here are direct sums of tensor products and the boundary operator is a natural group homomorphism, so Theorem I may be applied. It asserts that the boundary operator can be obtained from linear combinations of tensor products of composites of the special homomorphisms (2.3)–(2.6). The formulas displayed for these special homomorphisms clearly show that they carry generic elements into generic elements, hence the same applies to the boundary operator. In other words, the subgroup $A^n(y_\alpha, n)$ (or $A^n(y_\alpha)$ or $A(y_\alpha)$) is stable under the boundary operator, and thus forms a subcomplex called the *generic* subcomplex of $A(\Pi, n)$ (or of $A^n(\Pi)$ or $A(\Pi)$, respectively). The generic complexes $A^n(y_\alpha)$ and $A(y_\alpha)$ may be augmented by adjoining in dimension zero the generic subgroup $h(y_\alpha)$ of $h(\Pi)$ which is defined as the image of the generic subgroup of $J(\Pi)$ under the natural homomorphism $J(\Pi) \to h(\Pi)$.

The determination of the homology groups of the generic subcomplexes is our main objective here. The results are contained in the three theorems below. In stating these results it is convenient to assume that the base $\{y_\alpha\}$ of the free system $\Pi$ is indexed by a simply ordered set $\{\alpha\}$.

THEOREM II. *The homology groups of the generic complex $A(y_\alpha; n)$ are as follows: $H_0$ is the infinite cyclic group generated by the unit element of $A(\Pi, n)$: $H_{nr}(r = 1, 2, \ldots)$ is the free abelian group generated by the homology classes of the cycles*

$$[y_{\alpha_1}]*_n \ldots *_n[y_{\alpha_r}], \qquad \alpha_1 < \ldots < \alpha_r.$$

*In other dimensions the homology groups are zero.*

THEOREM IIa. *The homology groups of the generic complex $A^n(y_\alpha)$ are as follows: $H_{n(r-1)+1}$ $(r = 1, 2, \ldots)$ is the free abelian group generated by the homology classes of the cycles*

$$[y_{\alpha_1}]*_n \ldots *_n[y_{\alpha_r}], \qquad \alpha_1 < \ldots < \alpha_r.$$

*In other dimensions the homology groups are zero. In the augmented generic complex the group $H_1$ (corresponding to $r = 1$ above) is zero while the remaining groups are unchanged.*

THEOREM IIb. *The homology groups of the generic complex $A(y_\alpha)$ are zero, except in dimension 1, where the group $H_1$ is the free abelian group generated by the homology classes of the cycles $[y_\alpha]$. In the augmented generic complex all the homology groups are zero.*

This last result implies that $A(\Pi)$ is a generically acyclic (augmented) construction in the sense of **(1)**. Hence the homology of $A(\Pi)$ is naturally isomorphic to that of the generically acyclic cubical construction on $\Pi$, as described in **(1)**.

In the complex $A(R)$, $[x_1|_{n+1}x_2]$ is a chain of the lowest dimension not contained in $A^n(R)$. Since this chain has dimension $n + 2$, it follows that

$$H_q(A(R)) = H_q(A^n(R)) = H_{q-1+n}(A(R, n)), \qquad q < n + 1.$$

This implies that for an abelian group $\Pi$ (or more generally, for an associative and commutative system $\Pi$ with a unit element) the homology groups $H_q(\Pi) = H_q(A(\Pi))$, furnished by the theory of generic acyclicity, coincide with the groups $H_{q-1+n}(\Pi, n) = H_{q-1+n}(A(R, n))$ $(q < n + 1)$. The latter are the *stable* groups (under suspension) in the theory of the groups $H(\Pi, n)$ corresponding to the "jump" $q - 1$.

**5. Proofs of results of §4.** We first observe that in Theorems IIa and IIb the second halves, dealing with the augmented complexes, are immediate consequences of the first halves. It is also clear that Theorem II implies Theorem IIa.

Next we derive Theorem IIb from Theorem IIa. Since $A(y_\alpha)$ is the union of the increasing sequence of complexes $A^n(y_\alpha)$ the homology group $H_q(A(y_\alpha))$ is the limit of the direct sequence of groups

$$H_q(A^1(y_\alpha)) \to H_q(A^2(y_\alpha)) \to \ldots \to H_q(A^n(y_\alpha)) \to \ldots$$

where the maps are induced by inclusion (i.e., suspension). For $q = 1$, it follows from Theorem II that the maps are isomorphisms and thus give the desired description of $H_1(A(y_\alpha))$. If $q > 1$ then for $n$ sufficiently large, $q$ is not of the form $n(r-1)+1$. Thus by Theorem IIa, $H_q(A^n(y_\alpha)) = 0$ for all $n$ sufficiently large, and therefore $H_q(A(y_\alpha)) = 0$.

Thus Theorem II is the only one that still requires a proof. Since all the operations involved commute with direct limits, it is clear that it suffices to prove Theorem II in the case the (simply ordered) set $\{y_\alpha\}$ is a finite sequence $\{y_1, \ldots, y_t\}$.

Let $E(y_1, \ldots, y_t; n)$ be the free abelian group generated by the symbols

$$y_{i_1}y_{i_2}\ldots y_{i_r}, \quad 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant t, \quad 0 \leqslant r. \tag{5.1}$$

The symbol corresponding to $r = 0$ will be denoted by 1. We introduce a grading in $E(y_1, \ldots, y_t; n)$ by assigning to the element (5.1) the degree $rn$. We also introduce a boundary operator which is identically zero. We shall prove that

(5.2) *The correspondence*

$$\phi(y_{i_1}\ldots y_{i_r}) = [y_{i_1}]*_n \ldots *_n[y_{i_r}]$$

*is a chain equivalence*

$$\phi = E(y_1, \ldots, y_t;\ n) \to A(y_1, \ldots, y_t;\ n).$$

This will clearly imply that the homology groups of $A(y_1, \ldots, y_t; n)$ are as described in Theorem II.

We shall now apply the "tensor product theorem" of **(3)** to reduce the proof of (5.2) to the case $t = 1$. The "tensor product theorem" establishes a chain equivalence

$$A(R_1, n) \otimes A(R_2, n) \underset{f}{\overset{g}{\rightleftarrows}} A(R_1 \otimes R_2, n),$$

with natural $g$ and $f$ and natural homotopies $\Phi$: $gf \simeq$ ident., $\Psi$: $fg \simeq$ ident. Further $g$ has the form $g(\sigma \otimes \tau) = \sigma *_n \tau$, provided that we agree to regard $A(R_1, n)$ and $A(R_2, n)$ as subcomplexes of $A(R_1 \otimes R_2, n)$ in the obvious way. Since $f$, $g$, $\Phi$, and $\Psi$ are natural, they must have the form prescribed in Theorem I. Take $R_i = J(\Pi_i)$, where $\Pi_1$ is the free system with base $y_1, \ldots, y_{t-1}$ and $\Pi_2$ is that with base $y_t$. These maps will then carry generic elements to generic elements, and therefore we have an induced chain equivalence

$$A(y_1, \ldots, y_{t-1};\, n) \otimes A(y_t;\, n) \underset{f}{\overset{g}{\rightleftarrows}} A(y_1, \ldots, y_t;\, n).$$

for $t > 1$. Now consider the commutative diagram

$$\begin{array}{ccc}
E(y_1, \ldots, y_{t-1};\, n) \otimes E(y_t;\, n) & \xrightarrow{h} & E(y_1, \ldots, y_t;\, n) \\
\downarrow \phi_1 \otimes \phi_2 & & \downarrow \phi \\
A(y_1, \ldots, y_{t-1};\, n) \otimes A(y_t;\, n) & \xrightarrow[g]{} & A(y_1, \ldots, y_t;\, n),
\end{array}$$

where

$$h(y_{i_1} \ldots y_{i_r} \otimes 1) = y_{i_1} \ldots y_{i_r}$$

and

$$h(y_{i_1} \ldots y_{i_r} \otimes y_t) = y_{i_1} \ldots y_{i_r} y_t,$$

and where $\phi_1$ and $\phi_2$ are the maps of (5.2). Clearly $h$ is an isomorphism. If therefore $\phi_1$ and $\phi_2$ are chain equivalences then so is $g(\phi_1 \otimes \phi_2) = \phi h$. Thus $\phi$ also is a chain equivalence.

The above argument reduces the proof to the case $t = 1$. To establish Theorem II in this case we employ the "Normalization theorem" of **(2)**. An element

$$[x_1|_{k_1} \ldots |_{k_{r-1}} x_r]$$

of $A(R, n)$ is called a *norm* if $r > 1$ and $x_i = 1$ for at least one $i = 1, \ldots, r$. The norms generate a subcomplex; factoring $A(R, n)$ by this subcomplex we obtain the *normalized* complex $A_N(R, n)$. In **(2**, Theorems 4.1 and 12.1**)** we have established a chain equivalence

$$A(R, n) \underset{g}{\overset{f}{\rightleftarrows}} A_N(R, n)$$

where $fg =$ identity, $f$ is the natural factorization map and both $g$ and the homotopy $\Phi$: $gf \simeq$ ident. are natural. Furthermore $gf \equiv$ ident., modulo norms.

Passing to the generic complexes $A(y_\alpha; n)$, we introduce the normalized complex $A_N(y_\alpha; n)$ just as above and denote by $f'$: $A(y_\alpha; n) \to A_N(y_\alpha; n)$ the natural factorization homomorphism. The map $gf$: $A(R, n) \to A(R, n)$ and the homotopy $\Phi$: $gf \simeq$ ident. are both natural, and hence by Theorem I induce a map $k$: $A(y_\alpha; n) \to A(y_\alpha; n)$ and a homotopy $\Phi'$: $k \simeq$ ident. Since $\ker f' \subset \ker k$, it follows that $k$ admits a factorization $k = g'f'$ with $g'$: $A_N(y_\alpha; n) \to A(y_\alpha; n)$.

Since $gf$ and hence $k$ are congruent to the identity, modulo norms, it follows that $f'g' =$ ident. We therefore obtain a chain equivalence

$$A(y_\alpha;\ n) \underset{g'}{\overset{f'}{\rightleftarrows}} A_N(y_\alpha;\ n).$$

Now consider the generic complex $A(y;\ n)$ for a single generator $y$. In this complex all the chains of dimension $>n$ are norms. Since $\partial[y] = 0$ it follows that the complexes $A_N(y;\ n)$ and $E(y;\ n)$ may be identified. Under this identification we have $f'\phi =$ ident. Therefore

$$g' = g'f'\phi \simeq \phi,$$

so that $\phi$ also is a chain equivalence. This completes the proof.

## References


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