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A PRESENTATION FOR THE MONOID OF UNIFORM BLOCK PERMUTATIONS

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The monoid $\mathfrak{F}_n$ of uniform block permutations is the factorisable inverse monoid which arises from the natural action of the symmetric group on the join semilattice of equivalences on an n-set; it has been described in the literature as the factorisable part of the dual symmetric inverse monoid. The present paper gives and proves correct a monoid presentation for $\mathfrak{F}_n$. The methods involved make use of a general criterion for a monoid generated by a group and an idempotent to be inverse, the structure of factorisable inverse monoids, and presentations of the symmetric group and the join semilattice of equivalences on an n-set.

1. INTRODUCTION

A monoid M has an associative multiplication with an identity element 1. A monoid is *inverse* if it possesses a unary operation $x \mapsto x^{-1}$ (called inversion), which is an involutory anti-automorphism satisfying the law $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$; or equivalently, if it is regular ($x \in xMx$ for all $x \in M$) and has commuting idempotents. Inverse monoids model the partial or local symmetries of structures in addition to the total symmetries modelled by groups. The *symmetric inverse monoid* on a set X consists of all bijections between subsets of X, with an appropriate multiplication, and every inverse monoid may be faithfully represented in a symmetric inverse monoid. Further details concerning inverse monoids may be found in [4].

A categorical dual to the symmetric inverse monoid was described in [2] as consisting essentially of bijections between quotient sets of a given set X, or *block permutations* of X. These map the blocks of a 'domain' equivalence (or partition) on X bijectively to

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blocks of a 'range' equivalence, and may also be regarded as special binary relations on X called *biequivalences*. The appropriate multiplication involves the join of equivalences—details are found in [**2**], and an equivalent description in [**4**, pp. 122–124]. Those block permutations which are induced by permutations of X form an inverse submonoid of the whole, in fact the *factorisable part* (see below). In [**2**] this submonoid was denoted by $\mathfrak{F}_X^*$ and its elements were called *uniform*, since they have the characteristic property that corresponding blocks are of equal cardinality. Only finite X is considered here, the set $\{1, 2, \ldots, n\}$ being denoted by $\mathbf{n}$, and the monoid is denoted simply by $\mathfrak{F}_n$ since the asterisk seems redundant in the absence of duality considerations.

As with the symmetric inverse monoid, $\mathfrak{F}_n$ may be regarded as a generalisation of the symmetric group $\mathfrak{S}_n$. Popova [**6**] gave a presentation for the symmetric inverse monoid which extends Moore's presentation [**5**] for $\mathfrak{S}_n$; the present paper does the same for $\mathfrak{F}_n$. An important motivation for the work is that it may guide the construction of presentations for analogous monoids which generalise the braid groups, and thus help elucidate the structure of the space of braids. Moreover, some auxiliary results are of interest in their own right.

2. Factorisable inverse monoids

An inverse monoid M is said to be *factorisable* [**1**] if $M = GE$, where $G = \{g \in M : g^{-1}g = gg^{-1} = 1\}$ is the group of units of M, and $E = \{e \in M : e^2 = e\}$ the semilattice of idempotents. We have the following useful results.

Lemma 1. *Let $M = GE$ be a factorisable inverse monoid, $e, f \in E$ and $g, h \in G$. A homomorphism $\phi : M \to N$ is injective if, and only if, $\phi|_G$ and $\phi|_E$ are injective and $(ge)\phi = e\phi$ implies $ge = e$.*

Proof: Necessity of the conditions is immediate. Suppose, for the converse, that the conditions hold and $(ge)\phi = (hf)\phi$. Then $g\phi e\phi = h\phi f\phi$ in $M\phi$, which is factorisable since the images of units and idempotents under ϕ are themselves units and idempotents respectively. By [**1**, Theorem 2.1 (iv)], $e\phi = f\phi$ and $(g\phi)^{-1}h\phi e\phi = (g^{-1}he)\phi = e\phi$, when $e = f$ and $g^{-1}he = e$ by hypothesis. Then $ge = he = hf$ and ϕ is injective. ∎

Lemma 2. *Let M be a monoid generated by its group of units G and an idempotent $e = e^2 \in M$. Then M is inverse if, and only if, $eg^{-1}eg = g^{-1}ege$ for all $g \in G$, and M is then factorisable with $E = \langle g^{-1}eg : g \in G\rangle$.*

Proof: For all $g \in G$, $g^{-1}eg$ is idempotent and so commutes with e if M is inverse. For the converse, suppose $eg^{-1}eg = g^{-1}ege$ for all $g \in G$. It follows that the set $\{g^{-1}eg : g \in G\}$ consists of commuting idempotents, and so generates a submonoid P consisting of idempotents. Clearly $g^{-1}Pg \subseteq P$ and so $PG \subseteq GP$; it follows that GP is a submonoid. But $G \cup \{e\} \subseteq GP$ and so $M \subseteq GP$. Now let $m \in M$, say $m = gp$ with $p \in P$ and $g \in G$. Since $mg^{-1}m = gp^2 = m$, M is regular. If $m \in E$, that is, $gp = gpgp$, then

$pg^{-1} = pgpg^{-1} \in P$, and so $pg^{-1} = pg^{-1}pg^{-1}$. But then $gp = gpg^{-1}p \in P$, we have $E \subseteq P$, idempotents of M commute, and M is inverse. Finally, $M \subseteq GP \subseteq GE$. ∎

In the main theorem of section 4, it is helpful to use monoid presentations for the group of units and the semilattice of idempotents of $\mathfrak{F}_n$, which are respectively the symmetric group $\mathfrak{S}_n$ and the join semilattice $\mathfrak{Eq}_n$ of equivalences on $\mathbf{n}$.

3. Monoid presentations

This paper considers only generation within the variety of monoids; thus, given a subset $X \subseteq M$, $\langle X \rangle$ denotes the *submonoid* generated by X. Similarly, given a set of generators X and a set of relations R (conventionally written as equations), $M \cong \langle X \mid R \rangle$ means $M \cong X^*/R^\#$, where X^* is the *free monoid* generated by X and $R^\#$ the congruence generated by the relations R (see, for example, [**3**, Section I.6]). Crucial use will be made of the following universal property of $\langle X \mid R \rangle$.

We say that a monoid S *satisfies* R (or that R *holds in* S), *via* a mapping $i_S : X \to S$, if for all $(w_1, w_2) \in R$, $w_1 i_S^* = w_2 i_S^*$ (where $i_S^* : X^* \to S$ is the natural extension of i_S to X^*). Then $\langle X \mid R \rangle$ is the monoid M, unique up to isomorphism, which is universal with the property that it satisfies R (via $i_M : x \mapsto xR^\#$); that is, if a monoid S satisfies R via i_S, there is a *unique* homomorphism $\phi : M \to S$ such that $i_M\phi = i_S$. This ϕ is called the *canonical homomorphism*.

$$\begin{array}{ccc} X & \xrightarrow{i_M} & M \cong \langle X \mid R \rangle \\ & \searrow_{i_S} & \downarrow \phi \\ & & S \end{array}$$

Provided $R^\#$ does not identify any distinct generators, the mapping i_M is injective. If X generates S via i_S, then ϕ is necessarily surjective because i_S^* is.

Moore's monoid presentation for the symmetric group. Let $\mathbf{M} = (m_{ij})$ be the $(n-1) \times (n-1)$ matrix with entries

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } |i-j| = 1, \\ 2 & \text{if } |i-j| \geqslant 2. \end{cases}$$

Theorem 1. (Moore [5].) $\mathfrak{S}_n \cong \langle X \mid R \rangle$, *where:*

Generators X: $s_1, s_2, \dots s_{n-1}$

Relations R: $(s_i s_j)^{m_{ij}} = 1 \quad (i, j = 1, 2, \dots n-1)$.

Notice the presentation is equivalent to one with the same generators but relations

$$\begin{array}{lll} R' : & s_i^2 = 1 & (i = 1, 2, \dots n-1) \\ & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & (i = 1, 2, \dots n-2) \\ & s_i s_j = s_j s_i & (1 \leqslant i < j-1 < n). \end{array}$$

We shall need the following lemma treating multiplication in $\mathfrak{S}_n$. Here and later two notational conventions are helpful:

(i) the empty word is the identity 1;

(ii) if $i \leqslant j$, an expression with ellipsis such as $s_i \dots s_j$ includes *all consecutive subscripts* in increasing order from i to j inclusive;

an analogous decreasing sequence will be denoted $s_i \overset{\searrow}{..} s_j$.

LEMMA 3. *Suppose $i \leqslant j$. Then*

$$(s_i \dots s_j)s_k = \begin{cases} s_k(s_i \dots s_j) & \text{if } j+1 < k \text{ or } k < i-1, \\ s_i \dots s_{j+1} & \text{if } k = j+1, \\ s_i \dots s_{j-1} & \text{if } k = j, \\ s_{k+1}(s_i \dots s_j) & \text{if } i \leqslant k \leqslant j-1, \\ s_k s_{k+1}(s_{i-1} \dots s_j) & \text{if } k = i-1. \end{cases}$$

PROOF: By the relations. ∎

COROLLARY 1. *Every word $v \in \{s_1, \dots s_{n-1}\}^*$ is R-reducible to one of the form $w = (s_{i_1} \dots s_{j_1}) \dots (s_{i_k} \dots s_{j_k})$ for some $k \geqslant 0$ and some $i_1 \leqslant j_1, \dots i_k \leqslant j_k$ such that $n-1 \geqslant i_1 > i_2 > \dots > i_k \geqslant 1$. (For $k = 0$ we have the empty word 1.)*

PROOF: Call a subword of v a *run* if it is maximal with the property of having successive subscripts, and say that there is a *breach* between two successive runs $(s_i \dots s_j)$ and $(s_k \dots s_l)$ if $k \geqslant i$. Then the claim is that v is equivalent to a word without breaches. But repeated application of Lemma 3 to a breach produces an equivalent word with fewer breaches, so the claim is valid. ∎

A MONOID PRESENTATION FOR THE JOIN SEMILATTICE OF EQUIVALENCES

THEOREM 2. *With join written as multiplication, $\mathfrak{Eq}_n \cong \langle X \mid R \rangle$ where, for all $i, j, k, l \in \mathbf{n}$ satisfying the stated constraints,*

Generators X: $\quad t_{ij} \quad (i < j)$

Relations R:

$(E1) \dots \quad t_{ij}^2 = t_{ij} \quad (i < j)$

$(E2) \dots \quad t_{ij}t_{kl} = t_{kl}t_{ij} \quad (i < j \text{ and } k < l)$

$(E3) \dots \quad t_{ij}t_{ik} = t_{ij}t_{jk} = t_{ik}t_{jk} \quad (i < j < k).$

PROOF: Via the map $t_{ij} \mapsto (i, j \mid \dots, id, \dots)$ (the equivalence generated by the pair (i, j)), the relations R hold in $\mathfrak{Eq}_n$. This map is onto generators and so the canonical homomorphism $\phi : \langle X \mid R \rangle \to \mathfrak{Eq}_n$ is surjective.

Consider the collection $\mathcal{N}$ of digraphs on the vertex set $\mathbf{n}$ which are unions of trails (including singletons) such that each edge satisfies $i < j$. For any non-singleton trail X, say

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_b,$$

with $i_1 < i_2 < \cdots < i_b$, define

$$e_X = t_{i_1 i_2} t_{i_2 i_3} \ldots t_{i_{b-1} i_b} \in X^*.$$

If $\Gamma \in \mathcal{N}$ has non-singleton connected components (hence trails) $X_1, X_2, \ldots X_q$, define

$$e_\Gamma = e_{X_1} e_{X_2} \ldots e_{X_q}.$$

Then $e_{\_} : \Gamma \mapsto [e_\Gamma] = e_\Gamma R^{\#}$ is, by (E2), a well-defined mapping of $\mathcal{N}$ to $\langle X \mid R \rangle$.

Let $w \in X^*$, say $w = t_{i_1 j_1} t_{i_2 j_2} \ldots t_{i_d j_d}$. The digraph Γ' with vertex set $\mathbf{n}$ and edge set $(i_1, j_1), (i_2, j_2), \ldots (i_d, j_d)$ is not necessarily in $\mathcal{N}$, because its components need not be trails; however by a finite sequence of steps, replacing any subgraphs

$$\begin{tikzcd} i \circ \arrow[r] & \circ\, k \\ j \circ \arrow[ur] & \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} i \circ \arrow[r] \arrow[dr] & \circ\, k \\ & \circ\, j \end{tikzcd}$$

in which $i < j < k$ by the subgraph $\overset{i}{\circ} \longrightarrow \overset{j}{\circ} \longrightarrow \overset{k}{\circ}$, Γ' may be transformed to $\Gamma \in \mathcal{N}$. The corresponding sequence of applications of (E3) transforms w to e_Γ, and so $(w, e_\Gamma) \in R^{\#}$. Thus the map $e_{\_}$ is surjective, and so $|\mathcal{N}| \geqslant \langle X \mid R \rangle$.

Since $\mathcal{N}$ is in one-to-one correspondence with the set of partitions of $\mathbf{n}$, $\langle X \mid R \rangle \leqslant |\mathfrak{Eq}_n| < \infty$. It follows that the canonical homomorphism ϕ is actually an isomorphism. ∎

4. Presentation for $\mathfrak{F}_n$

Theorem 3. *$\mathfrak{F}_n \cong \langle X \mid R \rangle$, where:*

Generators X: $s_1, s_2, \ldots s_{n-1}, t$

Relations R:

(S)	$(s_i s_j)^{m_{ij}} = 1$	$(i, j = 1, 2, \ldots n-1)$
$(F2)$	$t^2 = t$	
$(F3)$	$ts_1 = t = s_1 t$	
$(F4)$	$s_i t = ts_i$	$(i = 3, \ldots n-1)$
$(F5)$	$s_2 t s_2 t = t s_2 t s_2$	
$(F6)$	$s_2 s_1 s_3 s_2 t s_2 s_3 s_1 s_2 t = t s_2 s_1 s_3 s_2 t s_2 s_3 s_1 s_2.$	

Proof: We first prove that $\langle X \mid R \rangle$ satisfies the conditions of Lemma 2. Clearly $\langle X \mid R \rangle = \langle G, t \rangle$ where $G = \langle s_1, s_2, \ldots s_{n-1} \rangle$; we know $G \cong \mathfrak{S}_n$ because relation (S) defines $\mathfrak{S}_n$ (Theorem 1) and any sequence of R-transitions between words of G remains in G. So take any $g \in G$, and let g be written in normal form according to Corollary 1 as

$$g = h(s_2 \ldots s_j)(s_1 \ldots s_k) = h s_2^j s_3^j (s_4 \ldots s_j) s_1^k s_2^k (s_3 \ldots s_k),$$

where $h \in \langle s_3, s_4, \dots s_{n-1}\rangle$, empty products evaluate to 1, and

$$s^i_\alpha = \begin{cases} s_\alpha & \text{if } i \geqslant \alpha, \\ 1 & \text{if } i < \alpha, \end{cases}$$

with $j \geqslant 1$ and $k \geqslant 0$. (The reader may think of s^i_α as abbreviating $s_\alpha^{i \geqslant \alpha}$, where $i \geqslant \alpha$ is a truth value equal to 1 if $i \geqslant \alpha$ holds and 0 otherwise.) Then

$$\begin{aligned} g^{-1}tg &= (s_k \mathrel{\cdot\!\searrow\!\cdot} s_3)(s_j \mathrel{\cdot\!\searrow\!\cdot} s_4) s^k_2 s^k_1 s^j_3 s^j_2 (h^{-1}th) s^j_2 s^j_3 s^k_1 s^k_2 (s_4 \dots s_j)(s_3 \dots s_k) \\ &= u^{-1}wu, \end{aligned}$$

where $u \in \langle s_3, \dots s_{n-1}\rangle$ and $w = s^k_2 s^k_1 s^j_3 s^j_2 t s^j_2 s^j_3 s^k_1 s^k_2$. Thus $g^{-1}tg$ commutes with t if and only w does. Table 1 shows that $tw = wt$ is a consequence of the relations by listing all cases.

	$k=0$	$k=1$	$k \geqslant 2$
$j=0,1$	t	$s_1ts_1 = t$ $(F3)$	s_2ts_2 $(F5)$
$j=2$	s_2ts_2 $(F5)$	$s_1s_2ts_2s_1$ $(F3,F5)$	$s_1s_2ts_2s_1$ $(F3,F5)$
$j \geqslant 3$	$s_3s_2ts_2s_3$ $(F4,F5)$	$s_1s_3s_2ts_2s_3s_1$ $(F3,F4,F5)$	$s_2s_1s_3s_2ts_2s_3s_1s_2$ $(F6)$

Table 1. Evaluation of $w = s^k_2 s^k_1 s^j_3 s^j_2 t s^j_2 s^j_3 s^k_1 s^k_2$, for $j = 0, 1, 2, \geqslant 3$ and $k = 0, 1, \geqslant 2$, together with the non-group relations which ensure that $tw = wt$. (See text for explanation.)

It follows from Lemma 2 that $\langle X \mid R\rangle$ is a factorisable inverse monoid with $E = \langle g^{-1}tg : g \in G\rangle$.

Via the map $f : s_i \mapsto (i \quad i+1) \in \mathfrak{S}_n$, $\quad t \mapsto (1, 2 \mid \cdots id \cdots) \in \mathfrak{Eq}_n$, the relations R are satisfied in $\mathfrak{F}_n$, as the reader may compute. Moreover, f is onto generators, so the canonical homomorphism $\phi : \langle X \mid R\rangle \to \mathfrak{F}_n$ is surjective. It was remarked previously that $\phi|_G$ is an isomorphism; it is also clear now that $\phi|_E : E \to \mathfrak{Eq}_n$ is surjective. We consider $\phi|_E$ further, using special idempotents defined next. Henceforth it is convenient to use the conjugation notation $v^g = g^{-1}vg$ for $v \in M, g \in G$.

Definition 1. *For $1 \leqslant i < j \leqslant n$,*

$$t_{i,j} = t^{(s_2 \dots s_{j-1})(s_1 \dots s_{i-1})}.$$

For brevity write t_{ij} for $t_{i,j}$. The usual conventions apply, so that $t_{12} = t$ and $t_{13} = t^{s_2}$. By calculations using Lemma 3, one verifies that for $i, j, k \in \mathbf{n}$,

$$t^{s_k}_{i,j} = \begin{cases} t_{i-1,j} & \text{if } k = i-1 \\ t_{i+1,j} & \text{if } k = i < j-1 \\ t_{i,j-1} & \text{if } i < k = j-1 \\ t_{i,j+1} & \text{if } k = j \\ t_{i,j} & \text{if otherwise.} \end{cases} \tag{1}$$

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Now $\phi|_G : G \to \mathfrak{S}_n$ defines the usual n-transitive (right) action of G on $\mathbf{n}$, $(i,g) \mapsto ig = i(g\phi)$, under which

$$is_k = \begin{cases} i-1 & \text{if} \quad k = i-1 \\ i+1 & \text{if} \quad k = i \\ i & \text{if} \quad \text{otherwise.} \end{cases}$$

Using a symmetrised version of the $t_{i,j}$ obtained by defining, for the case $i > j$, $t_{i,j} = t_{j,i}$, we see from equation (1) that $t_{i,j}^{s_k} = t_{is_k, js_k}$ for all $i, j \in \mathbf{n}$ and hence the action of G by conjugation on the idempotents $t_{i,j}$ satisfies

$$t_{i,j}^{g} = t_{ig,jg} \quad \text{for all} \quad i, j \in \mathbf{n} \quad \text{and} \quad g \in G.$$

In particular, $t^g = t_{12}^g = t_{1g,2g}$ and so the t_{ij} generate E. From (F3) and (F5) we have

$$tt^{s_2 s_1} = ts_1 s_2 t s_2 s_1 = ts_2 t s_2 s_1 = s_2 t s_2 t s_1 = t^{s_2} t,$$

that is, $t_{12}t_{23} = t_{13}t_{12}$. By the transitivity of G acting on $\mathbf{n}$, there exists $g \in G$ such that $(i, j, k) = (1g, 2g, 3g)$. Then $t_{12}^g t_{23}^g = t_{13}^g t_{12}^g$, that is, $t_{ij}t_{jk} = t_{ik}t_{ij}$. Similarly, $t_{ij}t_{jk} = t_{ik}t_{jk}$. Thus $\{t_{ij} : 1 \leqslant i < j \leqslant n\}$ satisfy the relations (E1–E3) of Theorem 2. It follows that E is a homomorphic image of $\mathfrak{Eq}_n$; it was remarked previously that $\mathfrak{Eq}_n$ is a homomorphic image of E under ϕ, and since $\mathfrak{Eq}_n$ is finite it follows that $\phi|_E$ is an isomorphism. We may proceed to apply Lemma 1 to ϕ.

Suppose that $(ge)\phi = e\phi$ for $g \in G, e \in E$. If $e = 1$ then $g = 1$ and $ge = e$. So take $e \neq 1$ and write, as in the proof of Theorem 2, $e = e_\Gamma = e_{X_1} e_{X_2} \dots e_{X_q}$ where the X_α $(1 \leqslant \alpha \leqslant q)$ are the non-singleton blocks of the partition $e\phi$ of $\mathbf{n}$. Write $n_\alpha = |X_\alpha|$ and define

$$\begin{aligned} & & j_0 &= 0, \\ i_1 &= 1, & j_1 &= n_1, \\ i_\alpha &= j_{\alpha-1} + 1, & j_\alpha &= j_{\alpha-1} + n_\alpha, \end{aligned}$$

for $1 \leqslant \alpha \leqslant q$. Let $\sigma = v\phi \in \mathfrak{S}_n$ be the permutation which, for each α, maps the elements of X_α, in their trail order inherited from $\mathbf{n}$, to $(i_\alpha, \dots j_\alpha)$. Then

$$e^v = e_{X_1\sigma} e_{X_2\sigma} \dots e_{X_q\sigma} = e_{[i_1,j_1]} e_{[i_2,j_2]} \cdots e_{[i_q,j_q]},$$

where $[i_\alpha, j_\alpha]$ denotes the integer interval $\{i_\alpha, i_\alpha + 1, \dots j_\alpha\}$. Moreover, $(g^v e^v)\phi = e^v\phi$, that is,

$$g^v \phi e^v \phi = (i_1 \dots j_1 | \cdots | i_q \dots j_q | \cdots id \cdots).$$

Thus for each $i \in \mathbf{n}$, $i(g^v\phi)$ and i lie in the same block of $e^v\phi$, so that $g^v\phi$ stabilises each block $X_\alpha\sigma = [i_\alpha, j_\alpha]$ and fixes each singleton block. It follows that

$$g^v \phi \in \mathrm{Sym}[i_1, j_1] \times \mathrm{Sym}[i_2, j_2] \times \cdots \times \mathrm{Sym}[i_q, j_q]$$

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and hence that $g^v = u_1 u_2 \dots u_q$, with $u_\alpha \in \langle s_{i_\alpha}, \dots s_{j_\alpha - 1} \rangle$ for $1 \leqslant \alpha \leqslant q$. Therefore

$$g^v e^v = u_1 u_2 \dots u_q e_{[i_1, j_1]} e_{[i_2, j_2]} \dots e_{[i_q, j_q]}.$$

This expression simplifies with the aid of

Lemma 4. *For $1 \leqslant \alpha \neq \beta \leqslant q$,*

(i) $u_\beta e_{[i_\alpha, j_\alpha]} = e_{[i_\alpha, j_\alpha]} u_\beta$, *and*

(ii) $u_\alpha e_{[i_\alpha, j_\alpha]} = e_{[i_\alpha, j_\alpha]}$.

Proof: If $\alpha \neq \beta$ and $k \in [i_\beta, j_\beta - 1]$ then $k \notin [i_\alpha - 1, j_\alpha]$ and we may use the last row of equation (1) to deduce $s_k e_{[i_\alpha, j_\alpha]} = e_{[i_\alpha, j_\alpha]} s_k$, and thus (i) holds. If $k \in [i_\alpha, j_\alpha - 1]$, let $\tau \in G$ be chosen so that $i\tau = i + k - 1 (\operatorname{mod} n)$. Set $Y_\alpha = \{n - (k - i_\alpha) + 1, \dots n, 1, \dots j_\alpha - k + 1\}$, so that $Y_\alpha \tau = [i_\alpha, j_\alpha]$.

Now $s_1 t_{12} = t_{12}$, $s_1 t_{n1} t_{12} = t_{n2} t_{12} = t_{n1} t_{12}$, and s_1 commutes with $t_{34}, \dots t_{n-1,n}$. Thus $s_1 e_{Y_\alpha} = e_{Y_\alpha}$, and so

$$s_k e_{[i_\alpha, j_\alpha]} = s_1^\tau e_{Y_\alpha \tau} = (s_1 e_{Y_\alpha})^\tau = e_{Y_\alpha}^\tau = e_{[i_\alpha, j_\alpha]}. \qquad \square$$

Returning to complete the main proof,

$$g^v e^v = u_1 e_{[i_1, j_1]} u_2 e_{[i_2, j_2]} \dots u_q e_{[i_q, j_q]} = e_{[i_1, j_1]} e_{[i_2, j_2]} \dots e_{[i_q, j_q]} = e^v,$$

by the two parts of Lemma 4, and so $ge = e$. By Lemma 1, ϕ is an isomorphism and the main theorem is proved. $\square$

Added in Proof: There are related results in M. Kosuda, *Ryukyu Math. J.* **13** (2000), 7–22 (noted by Professor T. Halvorsen).

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