PAPER 9

WAVES IN A HOT IONIZED GAS IN A MAGNETIC FIELD

ERNST ÅSTRÖM
Royal Institute of Technology, Stockholm, Sweden

The electromagnetic state of vacuum is characterized by two vector quantities, namely **E** and **B**. They are related to current and charge density by the equations

$$\text{curl } \mathbf{B}/\mu_0 - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{i}, \tag{1}$$

$$\text{div } \mathbf{E} = \rho/\epsilon_0, \tag{2}$$

$$\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{3}$$

$$\text{div } \mathbf{B} = 0. \tag{4}$$

Sometimes it is suitable and possible to introduce two new quantities, **H** and **D**, so defined that the equations (1) and (2) appear in the new form (Stratton [1])

$$\text{curl } \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0, \tag{5}$$

$$\text{div } \mathbf{D} = \rho \tag{6}$$

and with a linear relation between **E** and **D** and also between **B** and **H**. This formalism is common when dealing with fluid and solid media, and has also been introduced to ionized media of zero temperature (Nichols and Schelleng [2], Alfvén [3], p. 85, Åström [4]). Here we shall say a few words about this matter for ionized media of non-zero temperature. In this connexion we also get an opportunity to discuss the meaning of the conception of diamagnetism.

Let us assume a medium which initially is homogeneous in a homogeneous magnetic field. Let us assume that we can neglect collisions. In this case it seems suitable to introduce fictitious particles situated at the guiding centres. Their motion shall be equal to the drift velocity. The magnetic moment due to the spiralling motion of the actual particle becomes an intrinsic property of the fictitious particle. By introducing these particles we have established that the motion of these particles is a single valued function of the space co-ordinates. The random motion is taken into

https://doi.org/10.1017/S0074180900237650 Published online by Cambridge University Press

account by introducing a temperature, and the corresponding electric current by introducing the magnetic moment (compare the alternative view discussed by Cowling [5] and Spitzer [6], p. 25). It may be necessary to keep the frequencies well below the gyro-frequency for the treatment to be valid, but we get simple relations for this case.

In the first approximation the motion of charged particles is the drift in crossed electric and magnetic fields. The drift is independent of the sign of the charge and hence, in a neutral plasma, the corresponding electric current vanishes. Therefore we ought to use the second approximation (Alfvén [3], p. 18),

$$v_{\text{drift}} \approx \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \omega^{-2}(e/m)\,\frac{\partial \mathbf{E}}{\partial t}, \tag{7}$$

where $\mathbf{E}$ is the perturbing electric field and ω is the gyro-angular frequency. If n is the density then the current is

$$\mathbf{i}_1 = \Sigma ne\mathbf{v} \approx \frac{\mathbf{E}\times\mathbf{B}}{B^2}\,\Sigma ne + \frac{\Sigma nm}{B^2}\,\frac{\partial \mathbf{E}}{\partial t}, \tag{8}$$

where the sum is to be extended over all types of particle present. Since we treat a macroscopically neutral medium the first term on the right-hand side vanishes.

In a region where the magnetic field is inhomogeneous we also get a drift of charged particles. In contrast to the drift in crossed fields the direction does depend on the sign of the charge of the particles, and hence we do not get any cancellation of the corresponding current in a neutral plasma. This current is

$$\mathbf{i}_2 \approx \frac{\Sigma nev_\perp^2}{4B^4}\,\mathbf{B}\times\nabla B^2. \tag{9}$$

The current due to the spiralling motion is a multivalued function of the space co-ordinates and therefore ought to be introduced in another way. Let us compute the field at a point P in a homogeneous plasma. Assume a cylinder generated by the magnetic-field lines through a circle with its centre at P and the radius equal to the Larmor radius R. The contribution from particles with their centres of circular motion outside is easily seen to be zero and the contribution from those inside is

$$(1/2)\,\mu_0 nev_\perp R.$$

This contribution has opposite direction to the magnetic field created by other sources. If therefore $\mathbf{B}$ is the field we actually have and $\mathbf{B}_0$ is the field from all sources but with the present one excluded we get

$$\mathbf{B} = \mathbf{B}_0 - a\mathbf{B}; \tag{10}$$

$$a = \mu_0 B^{-2}\Sigma\tfrac{1}{2}nmv_\perp^2. \tag{11}$$

https://doi.org/10.1017/S0074180900237650 Published online by Cambridge University Press

From what is said we find that $\mathbf{B}_0$ and not $\mathbf{B}$ shall appear in Eq. (1). Then Eqs. (1), (7)–(11) give

$$\text{curl } \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + (\mathbf{grad}\ a) \times \mathbf{B}, \tag{12}$$

$$\mu^{-1} = 1 + 2a, \tag{13}$$

$$\epsilon = 1 + \mu_0 B^{-2} \Sigma nmc^2, \tag{14}$$

$$\mathbf{H} = \mu^{-1} \mu_0{}^{-1} \mathbf{B} \quad \text{and} \quad \mathbf{D} = \epsilon\epsilon_0 \mathbf{E}. \tag{15}$$

These equations are of well-known form when the last term in (12) vanishes, i.e. when the ratio of the thermal to the magnetic-field energy is independent of the co-ordinates perpendicular to the magnetic-field lines. Let us for a moment keep to the case when this condition is fulfilled. We have thus defined the permeability of the medium. Since $0 < \mu < 1$ the medium is diamagnetic.

That the magnetic moment $\frac{1}{2}mv_\perp^2/B$ is a constant for motion along a flux tube and for time variations in the magnetic field is well known for slow variations, but it seems to be true also for fast variations. If we accept the Minkowski notation this quantity is also constant for relativistic velocities (cp. Leverett Davis, Jr. [7]).

If we discuss only small disturbances from homogeneity along the magnetic-field lines n/B, also, where n is the particle density, is constant. From what is said follows that the permeability is also a constant, even for large disturbances. If we had kept the whole discussion relativistically correct we should have found that the permeability is not constant when account is taken of the relativistic effects.

The dielectric constant is known before but we may add that it cannot be treated as a constant for large disturbances.

By our procedure we have eliminated the mechanical quantities but we find that the magnetic-field energy of the medium, (1/2) **BH**, now is the sum of $\frac{1}{2}B^2/\mu_0$ and 2/3 of the thermal-energy density if the velocity distribution is isotropic and (1/2) **ED** includes both $\frac{1}{2}\epsilon_0 E^2$ and the kinetic-energy density due to the drift in crossed fields.

Let us discuss waves travelling perpendicular to the magnetic-field lines. The phase velocity is $c(\mu\epsilon)^{-\frac{1}{2}}$. After introducing the values of μ and ϵ we find three cases of special interest.

1. $\frac{1}{2}B^2/\mu_0 \gg nmc^2$. The magnetic field is so strong that the motion of the charged particles is greatly hindered. In the limit we get electromagnetic waves *in vacuo*.

2. $nmc^2 \gg \frac{1}{2}B^2/\mu_0 \gg nkT$. The density of thermal energy can still be neglected compared with the density of magnetic energy but the kinetic

https://doi.org/10.1017/S0074180900237650 Published online by Cambridge University Press

energy due to the drift in crossed fields exceeds the electric-field energy. The phase velocity is equal to the Alfvén velocity. Looked upon in the present way they are transverse electromagnetic waves, but if we concentrate on the mechanical properties they are longitudinal compression waves.

3. $nkT \gg \frac{1}{2}B^2/\mu_0$. The electromagnetic-energy densities can be neglected besides the corresponding mechanical energies. The electric and magnetic fields are nevertheless necessary for the transfer of momentum in the medium. In this case we get diamagnetic sound waves. They have a phase velocity which is almost the sound velocity in the same gas if it were not ionized. Again it has to be looked upon as a transverse wave viewed from the present treatment but it can as well be accepted as a longitudinal wave (cp. N. Herlofson [8] and van de Hulst [9]).

Under the present assumptions the validity of the deduction is restricted to frequencies which are small compared to the gyro-frequency, but the collision frequency does not enter into consideration.

When we keep to regions where μ is constant we encounter no trouble, but when this is no longer the case the extra term in Maxwell's equations becomes important. This for instance means that $H_{\parallel}$ is no longer continuous at a boundary.

We have introduced **H** and μ in order to be able to use the results conventional electrodynamics offer. Since there exist other methods for solving the present problem it is perhaps suitable to restrict our present method to the case when μ is space-independent.

(1/2) **BH** is actually a pressure. When we have a solid boundary this quantity is no longer treated as an entity since the wall discriminates between two components. The wall can take up the mechanical part but does not react with the magnetic part. In equilibrium then the magnetic field has the same value on both sides of the boundary (Bohr [10]). To take such an experiment as a definition of permeability, i.e. to say that the gas is not diamagnetic, seems to be unrealistic since permeability ought to be a property of a homogeneous medium (here plasma) and not something characterizing a specific boundary value problem (cp. H. Alfvén [3], p. 57, Spitzer [11], p. 27).

https://doi.org/10.1017/S0074180900237650 Published online by Cambridge University Press

REFERENCES

[1] Stratton, J. A. *Electromagnetic Theory* (New York, 1941), p. 2.
[2] Nichols, H. W. and Schelleng, J. C. *Bell Syst. Tech. J.* **4**, 215, 1925.
[3] Alfvén, H. *Cosmical Electrodynamics* (Oxford University Press, 1950).
[4] Åström, E. *Ark. Fys.* **2**, 443, 1950.
[5] Cowling, T. G. *Mon. Not. R. Astr. Soc.* **90**, 140, 1929 and **92**, 407, 1932.
[6] Spitzer, L. *Physics of Fully Ionized Gases*, Interscience Publishers, Inc. (New York, 1956).
[7] Davis, Leverett, Jr. *Phys. Rev.* **101**, 351, 1956.
[8] Herlofson, N. *Nature, Lond.* **165**, 1020, 1950.
[9] van de Hulst, H. C. *Problems of cosmical aerodynamics*, Central Air Documents Office, Dayton, Ohio, 1951.
[10] Bohr, N. *Studier over metallernes elektronteori*, Dissertation, Copenhagen, 1911.
[11] Spitzer, L., Jr. *Astrophys. J.* **116**, 299, 1952.

Discussion

Cowling: I would like to ask about the term $nmv_\perp^2/B^2$; where does it come from? Is it a term arising from the diamagnetism?

Åström: This term comes from two sources. One is the drift velocity perpendicular to the magnetic field caused by its inhomogeneity. The velocity is proportional to $mv_\perp^2$ Further, the momentum of the rotational motion in the magnetic field also gives a term of this kind.

Cowling: If the transverse pressure effect arises as I expect from diamagnetism, care is necessary, or the macroscopic approach may prove misleading. Care is, of course, necessary in a microscopic approach in defining the relation between **B** and **H**, as is evidenced by the lengthy discussions on Lorentz' polarization in the ionosphere. But I am not sure that the difficulties of a macroscopic approach are less than for a microscopic approach.

Spitzer: Dr Åström has made the point that in a hot ionized gas, where collisions are infrequent, one must go to the microscopic picture for a detailed solution. I should like to agree entirely with this result, subject to the proviso that this approach involves difficulties of its own. In particular, the velocity distribution in any particular situation is no longer Maxwellian and must be determined directly from the Boltzmann equation. Problems of this sort are sufficiently complicated to keep many theorists busy for a long time.

There is one result in this area which I should like to report at this time. One may ask how constant is the diamagnetic moment of a gyrating particle. Professor Alfvén showed that the quantity is constant to the first order in an expansion parameter, t, which is essentially the ratio of the Larmor radius to the distance over which the magnetic field changes substantially. Hellwig demonstrated that the diamagnetic moment is constant to the second order in t. Recently Kruskal and Kulsrud at Princeton demonstrated that this quantity is constant to all orders in t. This does not mean that the diamagnetic moment is rigorously constant, but rather that in the asymptotic expansion of the magnetic moment in powers of t, all the coefficients of t are zero. Thus we may conclude that the diamagnetic moment is very constant, indeed!

https://doi.org/10.1017/S0074180900237650 Published online by Cambridge University Press

Swann: Starting from the basic equations in the microscopic form one can derive macroscopic relations by taking averages. It appears that the averaging of the ρu-terms gives rise to a conduction current and a polarization term and a complicated term which has to be subtracted from **B** to give **H**. In order to realize the macroscopic equations one has to average over a macroscopic element of volume of space. One then hopes to be able to formulate simple relations between j and **E**, etc., to complete the equations in usable form. Only if this is the case is there any sense in introducing the macroscopic equations. If this is not possible it is better to proceed directly from the microscopic relations.

Åström: It is true that one has to be careful when treating these problems, but I also think that if one divides the motion of the individual particle into a circular motion and a drift motion one avoids trouble. This does not mean that I want to state that the solution I have presented here is necessarily the best one. This paper has been presented mainly to stress that before one uses some terms one has to define them. In this special case it is about diamagnetism and related quantities.

https://doi.org/10.1017/S0074180900237650 Published online by Cambridge University Press