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Normality Versus Paracompactness in Locally Compact Spaces

Alan Dow and Franklin D. Tall

Abstract. This note provides a correct proof of the result claimed by the second author that locally compact normal spaces are collectionwise Hausdorff in certain models obtained by forcing with a coherent Souslin tree. A novel feature of the proof is the use of saturation of the non-stationary ideal on ω_1, as well as of a strong form of Chang's Conjecture. Together with other improvements, this enables the consistent characterization of locally compact hereditarily paracompact spaces as those locally compact, hereditarily normal spaces that do not include a copy of ω_1.

1 Introduction

The space of countable ordinals is locally compact, normal, but not paracompact. The question of what additional conditions make a locally compact normal space paracompact has a long history. At least 45 years ago, it was recognized that subparacompactness plus collectionwise Hausdorffness would do (see *e.g.*, [36]), as would perfect normality plus metacompactness [2]. Z. Balogh proved a variety of results under MA_{ω_1} [3] and **Axiom R** [4], and was the first to realize the importance of not including a perfect pre-image of ω_1 (equivalently, the one-point compactification being countably tight [3]). However, he assumed collectionwise Hausdorffness in order to obtain paracompactness. A breakthrough came with S. Watson's proof of the following proposition.

Proposition 1.1 ([46]) *$V = L$ implies locally compact normal spaces are collectionwise Hausdorff, and hence locally compact normal metacompact spaces are paracompact.*

Watson's proof crucially involved the idea of *character reduction*: if one wants to separate a closed discrete subspace of size κ, κ regular, in a locally compact normal space, it suffices to separate κ compact sets, each with an *outer base* of size $\leq \kappa$.

Definition An *outer base* for a set $K \subseteq X$ is a collection $\mathcal{B}$ of open sets including K such that each open set including K includes a member of $\mathcal{B}$.

The use of $V = L$ was to get that normal spaces of character at most $\aleph_1$ are collectionwise Hausdorff [16] and variations on that theme.

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It was known that locally compact normal non-collectionwise Hausdorff spaces could be constructed from MA_{ω_1}, indeed from the existence of a Q-set [36], so it was a big surprise when G. Gruenhage and P. Koszmider proved the following proposition.

Proposition 1.2 ([20]) *MA_{ω_1} implies locally compact, normal, metacompact spaces are $\aleph_1$-collectionwise Hausdorff and (hence) paracompact.*

The next result involving iteration axioms and a positive "normal implies collectionwise Hausdorff" type of result was from [28].

Proposition 1.3 (Larson and Todorcevic [28]) *Let S be a coherent Souslin tree (obtainable from $\diamond$ or a Cohen real). Force $MA_{\omega_1}(S)$,* i.e., *MA_{ω_1} for countable chain condition posets preserving S. Then force with S. In the resulting model, there are no first countable L-spaces, no compact first countable S-spaces, and separable normal first countable spaces are collectionwise Hausdorff.*

The first two statements are consequences of MA_{ω_1} [35]; the last is a consequence of $V = L$, indeed of $2^{\aleph_0} < 2^{\aleph_1}$. Larson and Todorcevic used this combination to solve Katětov's problem. This idea of combining consequences of a iteration axiom with "normal implies collectionwise Hausdorff" consequences of $V = L$ was exploited in [26] in order to prove the consistency, modulo a supercompact cardinal, of *every locally compact perfectly normal space is paracompact*. The large cardinal was later removed, which gives the following theorem.

Theorem 1.4 ([11]) *If ZFC is consistent, then so is ZFC plus every locally compact perfectly normal space is paracompact.*

In the models of [26] and [11], every first countable normal space is collectionwise Hausdorff. This is achieved in two stages. The novel one is the following lemma.

Lemma 1.5 ([26]) *Force with a Souslin tree. Then normal first countable spaces are $\aleph_1$-collectionwise Hausdorff.*

This is obtained by showing that if a normal first countable space is not $\aleph_1$-collectionwise Hausdorff, a generic branch of the Souslin tree induces a generic partition of the unseparated closed discrete subspace that cannot be "normalized"; *i.e.,* there do not exist disjoint open sets about the two halves of the partition. The argument is a blend of the two usual methods of proving "normal implies $\aleph_1$-collectionwise Hausdorff" results, namely those of adjoining Cohen subsets of ω_1 by countably closed forcing [36, 37] and using $\diamond$ for stationary systems on ω_1, a strengthening of $\diamond$ that holds in L [16].

Proposition 1.6 ([16, 36]) *Either force to add $\aleph_2$ Cohen subsets of ω_1, or assume $\diamond$ for stationary systems on ω_1. Then normal spaces of character $\leq \aleph_1$ are $\aleph_1$-collectionwise Hausdorff.*

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Once one has that normal first countable spaces are $\aleph_1$-collectionwise Hausdorff, it is easy to obtain full collectionwise Hausdorffness by starting with L as the ground model and following [16]. However, if a supercompact cardinal is involved, then instead of L we need to follow the method of [26], based on [37]. Namely, first make the supercompact indestructible under countably closed, forcing [29], and then perform an Easton extension, adding κ^+ Cohen subsets of each regular κ, before forcing with the Souslin tree.

In order to extend the theorems about locally compact normal spaces being paracompact beyond the realm of first countability, one first needs to get that locally compact normal spaces are collectionwise Hausdorff. In [39], the second author claimed to have done so, in the model of [26]. The key was to force a closed discrete subspace in a locally compact normal space to expand to a discrete collection of compact sets with countable outer bases and then apply the methods of [26]. Unfortunately, the expansion argument was flawed. A corrected argument is presented below, but at the cost of using a stronger iteration axiom (but not a larger large cardinal).

With the conclusion of [39] restored, papers [27, 38, 40] are re-instated. We shall then proceed to improve the results of the latter two.

2 $\mathrm{PFA}(S)[S]$ and the Role of ω_1

Definition $\mathrm{PFA}(S)$ is the Proper Forcing Axiom (PFA) restricted to those posets that preserve the (Souslinity of the) coherent Souslin tree S. For the definition of coherence, see *e.g.,* [44, Chapter 5]. For a proof that $\diamond$ implies the existence of a coherent Souslin tree, see [24].

"$\mathrm{PFA}(S)[S]$ implies φ" is shorthand for "whenever one forces with a coherent Souslin tree S over a model of $\mathrm{PFA}(S)$, φ holds". "φ holds in a model of form $\mathrm{PFA}(S)[S]$" is shorthand for "there is a coherent Souslin tree S and a model of $\mathrm{PFA}(S)$ such that when one forces with S over that model, φ holds".

For discussion of $\mathrm{PFA}(S)[S]$, see [9, 15, 26, 27, 38, 40, 42, 45].

The following results appear in [27] and [40], respectively.

Theorem 2.1 *There is a model of form* $\mathrm{PFA}(S)[S]$ *in which a locally compact, hereditarily normal space is hereditarily paracompact if and only if it does not include a perfect pre-image of* ω_1.

Theorem 2.2 *There is a model of form* $\mathrm{PFA}(S)[S]$ *in which a locally compact normal space is paracompact and countably tight if and only if its separable closed subspaces are Lindelöf and it does not include a perfect pre-image of* ω_1.

Definition **PPI** is the assertion that every first countable perfect pre-image of ω_1 includes a copy of ω_1.

Lemma 2.3 ([10]) $\mathrm{PFA}(S)[S]$ *implies* **PPI**.

PPI was originally proved from PFA in [5]. Using **PPI**, we are able to weaken "perfect pre-image" to "copy" in the improved version of the first theorem, but provably one cannot in the second theorem.

Theorem 2.4 *There is a model of form* $\mathrm{PFA}(S)[S]$ *in which a locally compact, hereditarily normal space is hereditarily paracompact if and only if it does not include a copy of* ω_1.

Example 1 *There is a locally compact space* X *(indeed a perfect pre-image of* ω_1*) which is normal, does not include a copy of* ω_1*, in which all separable closed subspaces are compact, but* X *is not paracompact.*

It is clear that to establish Theorem 2.4, it suffices to use Theorem 2.1 and apply **PPI** after proving the following theorem.

Theorem 2.5 $\mathrm{PFA}(S)[S]$ *implies a hereditarily normal perfect pre-image of* ω_1 *includes a first countable perfect pre-image of* ω_1.

This is a consequence of the following three lemmas.

Lemma 2.6 *Let* X *be a perfect pre-image of* ω_1*, and suppose separable subspaces of* X *are Lindelöf. Then* X *includes a first countable perfect pre-image of* ω_1.

Lemma 2.7 ([40, 45]) $\mathrm{PFA}(S)[S]$ *implies compact, separable, hereditarily normal spaces are hereditarily Lindelöf.*

Here is the proof of Lemma 2.6.

Proof Let $f: X \to \omega_1$ be perfect and onto. Then X is locally compact, countably compact, but not compact. There is a closed $Y \subseteq X$ such that $f' = f|Y$ is perfect, irreducible, and maps Y onto ω_1. So $Y = \bigcup_{\alpha<\omega_1} f'^{-1}(\{\beta : \beta \leq \alpha\})$. Each $D_\alpha = f'^{-1}(\{\beta : \beta \leq \alpha\})$ is clopen and hence countably compact. It suffices to show D_α is hereditarily Lindelöf, because then points are G_δ and D_α is first countable. But then Y is first countable, since D_α is open. To show D_α is hereditarily Lindelöf, we need only show it is separable. Then $f_\alpha = f'|D_\alpha$ is irreducible, for if there were a proper closed subset A of D_α such that $f'(A) = f'(D_\alpha)$, then f would map $A \cup (Y - D_\alpha)$ onto ω_1, contradicting f's irreducibility. Thus, D_α is separable. ∎

Lemma 2.8 ([33, Section 6.5]) *If* f *is a closed irreducible map of* X *onto* Y *and* E *is dense in* Y*, then* $f^{-1}(E)$ *is dense in* X.

Let us construct the example that constrains the hoped-for improvement of Theorem 2.2. Consider a stationary, co-stationary subset E of ω_1 and its Stone–Čech extension βE. The identity map ι embeds E into the compact space $\omega_1 + 1$. ι extends to $\widehat{\iota}$ mapping βE onto $\omega_1 + 1$; we claim that $\widehat{\iota}$ maps only one element, call it z, of $\beta E \setminus E$ to the point ω_1. The reason is that every real-valued continuous function on E is eventually constant. If there were another such point, say z', let f be a continuous real-valued function sending z to 0 and z' to 1. Let $\alpha \in \omega_1$ be such that for every β and

$\gamma \in E$ greater than α, $f(\beta) = f(\gamma)$. Notice that neither z nor z' is in the closure of any countable subset F of E. The reason is that $\beta F \subseteq \beta E \subseteq \beta\omega_1$ and countable subsets of ω_1 have compact closures included in ω_1. Thus, $f^{-1}(-\frac{1}{2}, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, \frac{3}{2})$ will both contain members of E greater than α, yielding a contradiction.

Our space X will be $\beta E \smallsetminus \{z\}$. Then $\widehat{\iota}|X$ maps X onto ω_1; we claim this map is perfect. By [13, 3.7.16(iii)], it suffices to show that $\widehat{\iota}[\beta X - X] = \beta\omega_1 \smallsetminus \omega_1$. But $\beta\omega_1 = \omega_1 + 1$ and $\beta X = \beta E$, so this just says $\widehat{\iota}(z) = \omega_1$, which we have.

If H, K are disjoint closed subsets of X, then their closures in βE have at most z in common. Thus, their images $\widehat{\iota}[H]$ and $\widehat{\iota}[K]$ cannot overlap in a subspace with a point of E in its closure. Since E is co-stationary, their overlap is countable. Then at least one of them is bounded, and hence compact. It is then easy to pull back disjoint open sets to establish normality.

For any perfect pre-image of ω_1, it is easy to see that separable closed subspaces are compact, since they are included in a pre-image of an initial closed segment of ω_1.

It remains to show that X does not include a copy W of ω_1. A standard $\beta\mathbb{N}$ argument shows that no point in $X - E$ is the limit of a convergent sequence, so the set C of all limits of convergent sequences from W is a subset of E. But C is homeomorphic to ω_1, so cannot be included in a co-stationary E.

There is, however, a satisfactory improvement of Theorem 2.2.

Theorem 2.9 *There is a model of form* PFA$(S)[S]$ *in which a locally compact, normal, countably tight space is paracompact if and only if its separable closed subspaces are Lindelöf, and it does not include a copy of* ω_1.

The proof of Theorem 2.9 is essentially the same as the proof in [40] of our Theorem 2.2. It follows from Theorem 2.2 and the following theorem.

Theorem 2.10 ([11]) PFA$(S)[S]$ *implies a countably tight, perfect pre-image of* ω_1 *includes a copy of* ω_1.

Countably tight, hereditarily normal perfect pre-images of ω_1 are rather special.

Definition Suppose $\pi: X \to \omega_1$. We say $Y \subseteq X$ is *unbounded* if $\pi(Y)$ is unbounded.

Theorem 2.11 PFA$(S)[S]$ *implies that a countably tight, hereditarily normal, perfect pre-image of* ω_1 *is the union of a paracompact space with a finite number of disjoint unbounded copies of* ω_1.

Proof By Theorem 2.10, the perfect pre-image X includes a copy, W_1, of ω_1. If W_1 were bounded, then for some α, $W_1 \subseteq \pi^{-1}([0, \alpha])$. But $\pi^{-1}([0, \alpha])$ is compact, and W_1 – being a countably compact subspace of a countably tight space – is closed in X and hence in $\pi^{-1}([0, \alpha])$. But then W_1 is compact, a contradiction. Since perfect pre-images of locally compact spaces are locally compact, X is locally compact. Since W_1 is closed, $X - W_1$ is open and so is also locally compact. If it is paracompact, we are done; if not, apply Theorem 2.10 to get a copy W_2 of ω_1 included in $X - W_1$. Continuing, the process must end at some finite stage, because of the following theorem.

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Lemma 2.12 ([32, 3.6]) *Let X be a T_5 space, $\pi\colon X \to \omega_1$ continuous, $\pi^{-1}(\{\alpha\})$ countably compact for all $\alpha \in S$, a stationary subset of ω_1. Then X cannot include an infinite disjoint family of closed, countably compact subspaces each with unbounded range.*

Note that the paracompact subspace is the topological sum of $\leq \aleph_1$ σ-compact subspaces. ■

An early version of [10] used the axioms $\sum^-$ (defined in Section 5), **PPI**, and the $\aleph_1$-collectionwise Hausdorffness of first countable normal spaces, as well as 2.11 to obtain "countably compact, hereditarily normal manifolds of dimension > 1 are metrizable" without the $\mathbf{P}_{22}$ axiom used in [10] to get the stronger assertion in which "countably compact" is omitted.

Both of the conditions for paracompactness in Theorem 2.9 are necessary.

Example 2 *ω_1 is locally compact, hereditarily normal, first countable, its separable subspaces are countable, but it is not paracompact.*

Example 3 *Van Douwen's "honest example" [7] is locally compact, normal, first countable, separable, does not include a perfect pre-image of ω_1 (because it has a G_δ-diagonal), but is not paracompact.*

3 Strengthenings of $\mathrm{PFA}(S)[S]$

In addition to "front-loading" a $\mathrm{PFA}(S)[S]$ model in order to get full collectionwise Hausdorffness, it has also been useful to employ strengthenings of $\mathrm{PFA}(S)$ so as to obtain more reflection. For example, in [27] and [40], **Axiom R** is employed.

Definition $\mathcal{C} \subseteq [X]^{<\kappa}$ is *tight* if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from $\mathcal{C}$ and $\omega < \mathrm{cf}(\delta) < \kappa$, $\bigcup\{C_\alpha : \alpha < \beta\} \in \mathcal{C}$.

Axiom R If $\mathcal{S} \subseteq [X]^{<\omega_1}$ is stationary and $\mathcal{C} \subseteq [X]^{<\omega_2}$ is tight and unbounded, then there is a $Y \in \mathcal{C}$ such that $\mathcal{P}(Y) \cap \mathcal{S}$ is stationary in $[Y]^{<\omega_1}$.

Axiom R (due to Fleissner [17]) was obtained by using what is called $\mathrm{PFA}^{++}(S)$ in [27], before forcing with S [27]. $\mathrm{PFA}^{++}(S)$ holds if $\mathrm{PFA}(S)$ is forced in the usual Laver-diamond way. Here we shall use a conceptually simpler principle, $\mathrm{MM}(S)$, which is forced in a more complicated way, but does not require a larger cardinal. The axiom Martin's Maximum was introduced in [18].

Definition Let $\mathcal{P}$ be a partial order such that forcing with $\mathcal{P}$ preserves stationary subsets of ω_1. Let $\mathcal{D}$ be a collection of $\aleph_1$ dense subsets of $\mathcal{P}$. *Martin's Maximum (*MM*)* asserts that for each such $\mathcal{D}$, there is a $\mathcal{D}$-generic filter included in $\mathcal{P}$.

Theorem 3.1 ([18]) *Assume there is a supercompact cardinal. Then there is a revised countable support iteration establishing* MM.

MM(S) is defined analogously to PFA(S); Miyamoto [31] proved that there is a "nice" iteration establishing MM(S) but preserving S. One can then define MM(S)[S] analogously to PFA(S)[S].

In order to obtain a model of PFA(S)[S] in which Theorem 2.4 holds, we need to improve the model of [27] so as to not only have **Axiom R** but also the following properties:

LCN($\aleph_1$) Every locally compact normal space is $\aleph_1$-collectionwise Hausdorff.

We shall prove that MM(S) implies the following:

NSSAT NS_{ω_1} (the non-stationary ideal on ω_1) is $\aleph_2$-saturated.

SCC Strong Chang Conjecture. Let $\lambda > 2^{\aleph_2}$ be a regular cardinal. Let $H(\lambda)$ be the collection of hereditarily $< \lambda$ sets. Let M^* be an expansion of $\langle H(\lambda), \in\rangle$. Let $N \prec M^*$ (*i.e.*, N is an elementary submodel of M^*) be countable. Then there is an N' such that $N \prec N' \prec M^*$, $N' \cap \omega_1 = N \cap \omega_1$, and $|N' \cap \omega_2| = \aleph_1$.

Lemma 3.2 ([45]) PFA(S) *(and hence* MM(S)*) implies* $2^{\aleph_1} = \aleph_2$.

With these, we can modify the proof in [26] that forcing with a Souslin tree makes first countable normal spaces $\aleph_1$-collectionwise Hausdorff to obtain that locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff, and then, if we wish, front-load the model as in [26] to obtain full collectionwise Hausdorffness, using the character reduction method of [46]. More precisely, the crucial new step is the following theorem.

Theorem 3.3 *Suppose there is a model in which there is a Souslin tree S and in which* **NSSAT**, **SCC**, *and* $2^{\aleph_1} = \aleph_2$ *hold. Then S forces that locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff. In particular,* MM(S)[S] *implies* **LCN($\aleph_1$)**.

It will be convenient to consider the following intermediate proposition, which was introduced by Todorcevic in 1987 and implies the three things that we want.

SRP Strong Reflection Principle. Suppose $\lambda \geq \aleph_2$ and $\mathfrak{Z} \subseteq \mathcal{P}_{\omega_1}(\lambda)$ and that for each stationary $T \subseteq \omega_1$,

$$\{\sigma \in \mathfrak{Z} : \sigma \cap \omega_1 \in T\}$$

is stationary in $\mathcal{P}_{\omega_1}(\lambda)$. Then for all $X \subseteq \lambda$ of cardinality $\aleph_1$, there exists $Y \subseteq \lambda$ such that:

(a) $X \subseteq Y$ and $|Y| = \aleph_1$;
(b) $\mathfrak{Z} \cap \mathcal{P}_{\omega_1}(Y)$ includes a set that is closed unbounded in $\mathcal{P}_{\omega_1}(Y)$.

With regard to **SCC**, Shelah [34, XII.2.2, XII.2.5] proves the following lemma.

Lemma 3.4 *If there is a semi-proper forcing P changing the cofinality of $\aleph_2$ to $\aleph_0$, then* **SCC** *holds.*

There are various versions of *Namba forcing*, *e.g.*, two in [34] and one in [25]. All of these change the cofinality of $\aleph_2$ to $\aleph_0$. Larson states in [25, p.142] that his version of Namba forcing preserves stationary subsets of ω_1. In [18], it is shown that a principle, **SR**, implies that any forcing which preserves stationary subsets of ω_1 is semi-proper.

SR is a consequence of MM [18]. **SRP** is stronger than **SR** and so we have the following lemmas.

Lemma 3.5 **SRP** *implies* **SCC**.

Lemma 3.6 ([31]) MM(S) *implies* **SRP**.

Lemma 3.7 **SRP** *implies* **NSSAT** *and* $2^{\aleph_1} = \aleph_2$.

For the proof of Theorem 2.4 we should also note the following lemma.

Lemma 3.8 **SRP** *implies* **Axiom R**.

Proof We use an equivalent formulation of **SRP** due to Feng and Jech [14].

SRP For every cardinal κ and every $S \subseteq [\kappa]^\omega$, for every regular $\theta > \kappa$, there is a continuous elementary chain $\{N_\alpha : \alpha \in \omega_1\}$ (with N_0 containing some given element of $H(\theta)$, *e.g.*, S) such that for all α, $N_\alpha \cap \kappa \in S$ if and only if there is a countable $M \prec H(\theta)$ such that $N_\alpha \subseteq M$, $M \cap \omega_1 = N_\alpha \cap \omega_1$, and $M \cap \kappa \in S$.

Let $\mathcal{S}$ and $\mathcal{C}$ be as in **Axiom R** with $X = \kappa$. Choose θ sufficiently large so that $\mathcal{S}, \mathcal{C} \in H(\theta)$ and so that $\theta^{\aleph_1} = \theta$. Let $\{\mathcal{S}, \mathcal{C}\} \in N_0$ and let $\{N_\alpha : \alpha \in \omega_1\}$ be as in **SRP**. By induction on $\alpha \in \omega_1$, choose $Y_\alpha \in \mathcal{C} \cap N_{\alpha+1}$ so that $\bigcup(\mathcal{C} \cap N_\alpha) \subseteq Y_\alpha$. Then $\{Y_\alpha : \alpha \in \omega_1\}$ is an increasing chain in $\mathcal{C}$. Therefore, $Y = \bigcup_{\alpha\in\omega_1}(N_\alpha \cap \kappa)$ is in $\mathcal{C}$.

Then $\mathcal{S}^+ = \{M \prec H(\theta) : M \text{ is countable}, M \cap \kappa \in \mathcal{S}\}$ is a stationary subset of $[H(\theta)]^\omega$. This is proved in the same way as [34, Claim 1.12 1), p. 196]. Since $\{N_\alpha : \alpha \in \omega_1\}$ is an element of $H(\theta)$, there is an $M \in \mathcal{S}^+$ such that $\{N_\alpha : \alpha \in \omega_1\} \in M$. Let $M \cap \omega_1 = \delta$. Obviously $M \cap \kappa \in \mathcal{S}$, and, by continuity, $N_\delta \subseteq M$ and $M \cap \omega_1 = N_\delta \cap \omega_1$. It then follows from **SRP** that $N_\delta \in \mathcal{S}$.

This actually proves that $\{\alpha \in \omega_1 : N_\alpha \cap \kappa \in \mathcal{S}\}$ is a stationary subset of ω_1, because we could have put any cub of ω_1 as an element of M. Now assume that $\mathfrak{Z} \subseteq [Y]^\omega$ is a cub of $[Y]^\omega$. Choose a strictly increasing $g : \omega_1 \to \omega_1$ such that for each α, there is a $Z_\alpha \in \mathfrak{Z}$ such that $N_\alpha \cap \kappa \subseteq Z_\alpha \subseteq N_{g(\alpha)}$. If limit δ satisfies that $g(\alpha) < \delta$ for all $\alpha < \delta$, then we have that $N_\delta \cap \kappa \in \mathfrak{Z}$. This finishes the proof that $\mathcal{S} \cap [Y]^\omega$ is stationary. ∎

Theorem 3.9 *Suppose we have a model with a Souslin tree S in which **Axiom R** holds. Then, after forcing with S, **Axiom R** still holds.*

Proof This is an improvement over [27], which required a stronger axiom, Axiom R^{++}, holding in the model. We will use *t.u.b.* as an abbreviation for *tight unbounded*. We must consider two S-names: $\dot{\mathcal{C}}$ and $\dot{\mathcal{X}}$, where $\dot{\mathcal{C}}$ is forced to be a t.u.b. subset of $[\kappa]^{\omega_1}$ and $\dot{\mathcal{X}}$ is forced to be a stationary subset of $[\kappa]^\omega$. Let us assume that some $s_0 \in S$ forces that there is no Y in $\dot{\mathcal{C}}$ such that $\dot{\mathcal{X}} \cap [Y]^\omega$ is stationary. (It would make the discussion below easier if we just assumed that s_0 was the root of S, which one can certainly immediately do if S is a coherent Souslin tree.)

We first show that $\dot{\mathcal{C}}$ includes a t.u.b. $\mathcal{C}$ from the ground model. Simply put, $Y \in \mathcal{C}$ if every $s \in S$ forces that $Y \in \dot{\mathcal{C}}$. It is clear that $\mathcal{C}$ is closed under increasing ω_1-chains. Thus, we just have to show that it is unbounded. Let us enumerate S as $\{s_\alpha : \alpha \in \omega_1\}$.

Fix any $Y_0 \in [\kappa]^{\omega_1}$. By recursion, choose an increasing chain $\{Y_\alpha : \alpha \in \omega_1\}$ so that for each α, $\bigcup\{Y_\beta : \beta < \alpha\} \subseteq Y_\alpha$ and there is an extension s of s_α forcing that $Y_{\alpha+1} \in \dot{\mathcal{C}}$. This we can do, since s_α forces that $\dot{\mathcal{C}}$ is unbounded. Now let Y be the union of the chain $\{Y_\alpha : \alpha \in \omega_1\}$. Note that for each $s \in S$ and each $\beta \in \omega_1$, there is an $\alpha > \beta$ such that s_α is an extension of s. It follows that s forces that $\dot{\mathcal{C}} \cap \{Y_\alpha : \alpha \in \omega_1\}$ is uncountable, hence $s \Vdash Y \in \dot{\mathcal{C}}$.

Now we let $\mathcal{X}$ be the set of $x \in [\kappa]^\omega$ such that there is some $s \in S$ extending s_0 with $s \Vdash x \in \dot{\mathcal{X}}$. It is clear that $\mathcal{X}$ is a stationary subset of $[\kappa]^\omega$, because s_0 forces that $\mathcal{X}$ meets every cub. Now apply **Axiom R** to choose $Y \in \mathcal{C}$ so that $\mathcal{X} \cap [Y]^\omega$ is a stationary subset of Y.

Now we obtain a contradiction (and thus a proof) by showing that there is an extension $s \in S$ of s_0 that forces that $\dot{\mathcal{X}} \cap [Y]^\omega$ is stationary. Let $\{y_\alpha : \alpha \in \omega_1\}$ be an enumeration of Y. Let $\mathcal{E}$ be the set of $\delta \in \omega_1$ such that $x_\delta = \{y_\alpha : \alpha \in \delta\} \in \mathcal{X}$. Notice that $\{\{y_\alpha : \alpha \in \delta\} : \delta \in \omega_1\}$ is a cub in $[Y]^\omega$. Thus it follows that $\mathcal{E}$ is stationary. In fact, if $\mathcal{E}'$ is any stationary subset of $\mathcal{E}$, then $\mathcal{E}'$ is also a stationary subset of $[Y]^\omega$.

For each $\delta \in \mathcal{E}$, choose $s_\delta \in S$ above s_0 so that $s_\delta \Vdash x_\delta \in \dot{\mathcal{X}}$ (as per the definition of $\mathcal{X}$). Now we have a name $\dot{\mathcal{E}} = \{(x_\delta, s_\delta) : \delta \in \omega_1\}$. We prove that there is some $s \in S$ above s_0 that forces that $\dot{\mathcal{E}}$ is stationary. Thus, such an s forces that $\dot{\mathcal{X}} \cap [Y]^\omega$ is stationary, as required.

Let s_0 be on level α_0 of S. There is a $\gamma > \alpha_0$ so that each member of S_γ decides if $\dot{\mathcal{E}}$ is stationary. Also, for each $\bar{s} \in S_\gamma$ that forces $\dot{\mathcal{E}}$ is not stationary, there is a cub $C_{\bar{s}}$ of ω_1 such that $\bar{s}$ forces $C_{\bar{s}}$ is disjoint from $\dot{\mathcal{E}}$. Choose any δ in the intersection of those countably many cubs that is also in $\mathcal{E}$. Clearly if $\bar{s} \in S_\gamma$ is compatible with s_δ, then $C_{\bar{s}}$ did not exist, since $\bar{s} \cap s_\delta$ would force that $\delta \in C_{\bar{s}} \cap \dot{\mathcal{E}}$. This completes the proof, since that element $\bar{s}$ is above s_0 and forces that $\dot{\mathcal{X}} \cap [Y]^\omega$ is stationary. ∎

Corollary 3.10 MM$(S)[S]$ *implies* **Axiom R**.

We next need the following lemma.

Lemma 3.11 (P. Larson) *Suppose*

(i) **NSSAT**, *and*
(ii) *for sufficiently large θ and stationary $E \subseteq \omega_1$, for any $X \in H(\theta)$, there is a model M with $M \cap \omega_1 \in E$, $X \in M$ and $|M \cap \omega_2| = \aleph_1$.*

Then for such M, if $\mathcal{A} = \{A_\alpha : \alpha < \omega_2\}$ is a family of stationary subsets of ω_1, $\mathcal{A} \in M$, $M \cap \omega_1 = \delta$ is in uncountably many A_α, $\alpha \in M$.

Proof It is well known that NS_{ω_1} is $\aleph_1$-complete, since the diagonal union of $\aleph_1$ non-stationary subsets of ω_1 is non-stationary. It follows that $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ is a complete Boolean algebra, because (i) says it satisfies the $\aleph_2$-chain condition. Since it is complete, for each $\alpha < \omega_2$ there is a stationary B_α that is the supremum of $\{A_\beta : \beta \in (\alpha, \omega_2)\}$. Let E be the infimum of the family of B_α's. By saturation, E is really the infimum of an $\aleph_1$-sized family, and so is itself stationary. Given any $\alpha \in \omega_2$, we can find an $\eta(\alpha) > \alpha$ such that the diagonal union of $\{A_\beta : \beta \in (\alpha, \eta(\alpha))\}$ includes E, mod NS_{ω_1}. It follows that there is a cub $C \subseteq \omega_2$ such that for each $\alpha \in C$, there is a

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subset of $\{A_\beta : \beta \in (\alpha, \alpha^+)\}$ of cardinality $\aleph_1$ with diagonal union including E, mod NS_{ω_1}, where α^+ denotes the next element of C after α.

Now let M be an elementary submodel of a suitable $H(\theta)$, with $\langle A_\alpha : \alpha < \omega_2 \rangle$, E, and $C \in M$ and $\delta = M \cap \omega_1 \in E$, $|M \cap \omega_2| = \aleph_1$. We claim that δ is an element of uncountably many A_α, $\alpha \in M$.

Since the cub C divides ω_2 into $\aleph_2$ disjoint intervals, $C \cap M$ divides $\omega_2 \cap M$ into $\aleph_1$ disjoint intervals. Choose any one of these intervals J. There is a family $\mathcal{F}_J = \{F_\gamma : \gamma < \omega_1\}$ in M consisting of A_α's indexed in the interval J, with diagonal union including E, mod NS_{ω_1}. Then there is a cub D_J in M disjoint from $E \setminus \nabla \mathcal{F}_J$. $D_J \cap M$ is unbounded in M, so $\delta = M \cap \omega_1 \in D_J$, so $\delta \notin E \setminus \nabla \mathcal{F}_J$. Then $\delta \in \nabla \mathcal{F}_J$, so $\delta \in F_\gamma$ for some $\gamma \in M \cap \omega_1$, and therefore δ is in some A_ξ with $\xi \in J$. ■

Let us note for future reference that **SCC** implies Lemma 3.11(ii). To see this, expand $H(\theta)$ to M^* containing X. There is a cub $\mathcal{C}$ of countable elementary submodels N of M^*. Then $\{N \cap \omega_1 : N \in \mathcal{C}\}$ is cub in ω_1, and so there is an $N \in \mathcal{C}$ with $N \cap \omega_1 \in E$. By **SCC** we can then enlarge N to the desired $N' = M$.

We shall finish the proof that $\mathrm{MM}(S)[S]$ implies $\mathbf{LCN}(\aleph_1)$ in Section 4, but first let us note another advantage of stating $\mathrm{MM}(S)[S]$ as a hypothesis is that we can often avoid front-loading to get collectionwise Hausdorffness, since **Axiom R** provides enough reflection.

Theorem 3.12 $\mathrm{MM}(S)[S]$ *implies that a locally compact, hereditarily normal space is hereditarily paracompact if and only if it does not include a copy of* ω_1.

Proof As usual, we can assume the space does not include a perfect pre-image of ω_1. The proof for that case in [40] uses P-ideal Dichotomy, $\sum$, $\mathbf{LCN}(\aleph_1)$, and **Axiom R**. We can get all of these from $\mathrm{MM}(S)[S]$. (Todorcevic [45] proved that $\mathrm{PFA}(S)[S]$ implies P-ideal Dichotomy; a proof was published in [9].) ■

Similar considerations enable us to prove the following theorem.

Theorem 3.13 $\mathrm{MM}(S)[S]$ *implies a locally compact, normal, countably tight space is paracompact if and only if its separable closed subspaces are Lindelöf, and it does not include a copy of* ω_1.

We thank Paul Larson for Lemma 3.11 and several discussions concerning the material in this section. Next, we need to do some topology.

4 Getting Locally Compact Normal Spaces Collectionwise Hausdorff

Lemma 4.1 *Let X be a locally compact normal space and suppose Y is a closed discrete subspace of X of size $\aleph_1$. Then there is a locally compact normal space X' with a closed discrete subspace Y' of size $\aleph_1$, such that if Y' is separated in X', then Y is separated in X, but each point in Y' has character $\leq \aleph_1$.*

Proof By Watson's character reduction technique [46], there is a discrete collection of compact subsets of X, $\mathcal{K} = \{K_y : y \in Y\}$, such that $y \in K_y$, and each K_y has character $\le \aleph_1$. Let X' be the quotient of X obtained by collapsing each K_y to a point y'. This collapse is a perfect map, so it preserves normality and local compactness, and it is clear that $\{y' : y' \in Y\}$ is separated if and only if $\{K_y : y \in Y\}$ is separated, and that Y is separated if $\{K_y : y \in Y\}$ is. ∎

Lemma 4.2 *Suppose X is a locally compact normal space with Lindelöf number* $\aleph_1$ *and an uncountable closed discrete subspace. Then there is a continuous image of X of weight* $\aleph_1$ *enjoying the same properties.*

Proof Let $\mathcal{U}$ be an open cover of X of size $\aleph_1$ with each member of $\mathcal{U}$ a cozero set with compact closure. Without loss of generality, assume that for each $x \in X$ there is a $U \in \mathcal{U}$ such that $x \in U$ and U meets at most one element of a given closed discrete set D of size $\aleph_1$. Also without loss of generality, assume that $\mathcal{U}$ is closed under finite intersections. For each $U \in \mathcal{U}$, let $f_U : X \to [0,1]$ with $U = f_U^{-1}((0,1])$. Define an equivalence relation on X by letting $x_0 \sim x_1$ if $f_U(x_0) = f_U(x_1)$ for all $U \in \mathcal{U}$. Let $X/\sim$ be the quotient set, with $\pi : X \to X/\sim$ the projection. Topologize $X/\sim$ by taking as base all sets of form $\pi(U)$, $U \in \mathcal{U}$. Then $X/\sim$ is $\mathrm{T}_{3\frac{1}{2}}$ and of weight $\le \aleph_1$. To see the former, consider X as embedded in $[0,1]^{C^*(X)}$ by $e(x) = (f(x))_{f \in C^*(X)}$. Let $p : [0,1]^{C^*(X)} \to [0,1]^{\{f_U : U \in \mathcal{U}\}}$ be given by $(x_f)_{f \in C^*(X)} \to (x_{f_U})_{U \in \mathcal{U}}$, *i.e.*, p projects onto only those coordinates in $C^*(X)$ which are f_U's. Then $X/\sim = p \circ e(X)$.

The projection map π is closed; for let $F \subseteq X$ be closed and suppose $y \in \overline{\pi[F]}$. We claim that $y \in \pi[F]$. We have $y \in \pi[U]$ for some $U \in \mathcal{U}$; note $\pi^{-1}(\pi[U]) = U$ for if $\pi(x) \in \pi[U]$, $x \sim x_0$ for some $x_0 \in U$. Then $f_V(x) = f_V(x_0)$ for every $V \in \mathcal{U}$. But $U = f_U^{-1}((0,1])$. Thus $f_U(x) = f_U(x_0) \in (0,1]$, which implies $x \in U$. So $\overline{U} = \overline{\pi^{-1}(\pi[U])}$ is compact. Suppose $y \notin \pi[F]$. Then $y \notin \pi[F \cap \overline{U}]$, which is compact. Then $\pi[U] \setminus \pi[F \cap \overline{U}]$ is a neighborhood of y disjoint from $\pi[F]$.

Since π is closed and X is normal, $X/\sim$ is normal. It is clear that $\pi[D]$ is closed discrete. By continuity, $\pi[\overline{U}] \subseteq \overline{\pi[U]}$; $\pi[\overline{U}]$ is a closed set including $\pi[U]$, so $\pi[\overline{U}] = \overline{\pi[U]}$, so $X/\sim$ is covered by open sets with compact closures, so it is locally compact. ∎

Lemma 4.3 *In any model obtained by forcing with a Souslin tree S, any locally compact normal space with a dense Lindelöf subspace has countable extent.*

Proof Suppose X_0 is a locally compact normal space with an uncountable closed discrete subspace, which we can conveniently label as ω_1, and a dense Lindelöf subspace L. Via normality, we can find a closed subspace X_1 with ω_1 in its interior that is covered by $\aleph_1$-many open sets with compact closures. Without loss of generality, we can assume $X_1 = \overline{\operatorname{int} X_1}$. Then L is dense in $\operatorname{int} X_1$, so $L \cap (\operatorname{int} X_1)$ is dense in X_1. Then $L \cap X_1$ is a dense Lindelöf subspace of X_1.

Thus, without loss of generality, we may as well assume our original space X_0 has a cover by $\aleph_1$-many open sets, each with compact closure. Without loss of generality, we can assume each is a cozero set and indeed is σ-compact. By Lemma 4.2, there is a continuous image of X_0, call it X, which is also locally compact, normal, has an uncountable closed discrete subspace, and has weight $\aleph_1$. Since both density and Lindelöfness are preserved by continuous functions, X also has a dense Lindelöf subspace. Thus, it suffices to find a contradiction for the special case in which the weight of our space is $\aleph_1$.

For $\delta \in \omega_1$ and a cub $C \subseteq \omega_1$, let $\delta^+(C)$ denote the minimum element of C greater than δ. Without loss of generality, we can assume our cubs only consist of limit ordinals. For a cub C, we use $\mathrm{Fix}(C)$ to denote the set $\{\delta \in C : \text{order-type}(C \cap \delta) = \delta\}$. Let S_δ be the δ-th level of the Souslin tree.

As usual, we work in the ground model and fix names $\dot{\mathcal{B}} = \{\dot{B}_\alpha : \alpha \in \omega_1\}$ for a base of X consisting of open sets with compact closures. It is convenient to assume that $\{\dot{B}_n : n \in \omega\}$ is forced to have dense union. Again, we let ω_1 label a closed discrete subspace and let $\{\dot{U}(\alpha, \xi) : \xi \in \omega_1\}$ be a subset of $\dot{\mathcal{B}}$ forced to be a local base at α. Without loss of generality, assume each B_n is disjoint from the closed discrete set ω_1. Fix a cub C_0 such that for each $\delta \in C_0$ and each $s \in S_\delta$, s decides all equations of the form $\dot{B}_\alpha \cap \dot{B}_\beta = \varnothing$, for $\alpha, \beta < \delta$. Also assume that for each $s \in S_\delta$ $(\delta \in C_0)$ and each $\xi, \beta \in \delta$, there is an $\alpha \in \delta$ such that s forces that $\dot{U}(\xi, \beta) = \dot{B}_\alpha$.

It is convenient to assume that S is ω-branching (specifying any infinite maximal antichain above each element would serve the same purpose). We can use $C_1 = \mathrm{Fix}(C_0)$ to define a partition $\dot{f}$ of ω_1 so that for each $\xi \in \omega_1$ and each $s \in S_{\xi^+(C_1)}$, $s^\frown j$ forces that $\dot{f}(\xi) = j$. (We list the immediate successors of s as $\{s^\frown j : j \in \omega\}$.) Now we choose two (names of) functions $\dot{h}_1$ and $\dot{h}_2$ witnessing normality as follows:

(a) for each $j \in \omega$ and each $i \in 2$, let $\dot{W}^i_j = \bigcup\{\dot{U}(\xi, \dot{h}_i(\xi)) : \xi \in \dot{f}^{-1}(j)\}$;
(b) $\{\dot{W}^1_j : j \in \omega\}$ is a discrete family;
(c) the closure of $\dot{W}^2_j$ is included in $\dot{W}^1_j$.

Choose any countable elementary submodel M with all the above as members of M, such that $\delta = M \cap \omega_1$ is an element of C_1. We know that there is a name of an integer $\dot{J}_\delta$ satisfying that it is forced that $\dot{U}(\delta, 0) \cap \dot{W}_j$ is empty for all $j \geq \dot{J}_\delta$. Choose any $s \in S$ of height at least $\delta^+(C_1)$ that decides a value J for $\dot{J}_\delta$. Let $\bar{s} = s \restriction \delta^+(C_1)$. Notice that $\bar{s}$ decides the truth value of the equation $\dot{U}(\delta, 0) \cap \dot{B}_\alpha = \varnothing$, for all $\alpha \in M$. For each $n, j \in \omega$, s and hence by elementarity $s \restriction \delta$ forces that the closure of $\dot{W}^2_j \cap \dot{B}_n$ is included in $\dot{W}^1_j$. By elementarity and compactness, this implies there is a finite $\dot{F}_{j,n} \subseteq \delta$ such that $s \restriction \delta$ forces that $\dot{W}^2_j \cap \dot{B}_n \subseteq \bigcup\{\dot{B}_\eta : \eta \in \dot{F}_{j,n}\} \subseteq \dot{W}^1_j$. But now $\bar{s}$ forces that $\dot{U}(\delta, 0) \cap (\bigcup\{\dot{B}_\eta : \eta \in \dot{F}_{j,n}\})$ is empty for all n and all $j \geq J$ ($\bar{s}$ decides this and cannot contradict what s decides).

On the other hand, fix any $j \geq J$ and consider what $\bar{s}^\frown j$ is forcing. This forces that $\dot{f}(\delta) = j$ and that $\delta \in W^2_j$, and so δ is in the closure of the union of the sequence $\{\dot{U}(\delta, 0) \cap (\bigcup\{\dot{B}_\eta : \eta \in F_{j,n}\}) : n \in \omega\}$. This is a contradiction. ∎

Corollary 4.4 *In any model obtained by forcing with a Souslin tree, if X is locally compact normal, D is a closed discrete subspace of X of size $\aleph_1$ and $\{U_\alpha : \alpha \in \omega_1\}$ are open sets with compact closures, then for any countable $T \subseteq \omega_1$, $\overline{\bigcup\{U_\alpha : \alpha \in T\}} \cap D$ is countable.*

Proof We have that $\bigcup\{\overline{U}_\alpha : \alpha \in T\}$ is dense in $\overline{\bigcup\{U_\alpha : \alpha \in T\}}$, which is locally compact normal. ■

Getting back to the proof of Theorem 3.3, let us assume we are in a model of **NSSAT**, **SCC**, and $2^{\aleph_1} = \aleph_2$ and that we have an S-name $\dot{X}$ for a locally compact normal space, with a closed discrete subspace labeled as ω_1, with each of its points having character $\aleph_1$. Let us note that it follows from character reduction and Lemma 1.5 that if there is a discrete expansion of ω_1 into compact G_δ's, then ω_1 will have a separation. In fact, even more, it is shown in [39, Theorem 12] that if ω_1 is forced to have an expansion by compact G_δ's that is σ-discrete, then ω_1 will be separated. Since our proof is by contradiction, we will henceforth assume that it is forced (by the root of S) that there is no expansion of ω_1 into a σ-discrete family of compact G_δ's.

For each $\xi, \alpha \in \omega_1$, let $\dot{U}(\xi, \alpha)$ be the name of the α-th neighbourhood from a local base for ξ with $\dot{U}(\xi, 0)$ forced to have compact closure. Corollary 4.4, and the fact that S is ccc, ensure that for each $\delta \in \omega_1$, every element of S forces that

$$\omega_1 \cap \overline{\bigcup\{\dot{U}(\xi,0) : \xi < \delta\}}$$

is bounded by γ for some $\gamma \in \omega_1$. Therefore, there is a cub C_0 such that without loss of generality, we can assume that each of the following is forced by each element of S:

(a) for each $\delta \in C_0$, $\omega_1 \cap \overline{\bigcup\{\dot{U}(\xi,0) : \xi < \delta\}}$ is included in $\delta^+(C_0)$;
(b) for all $\beta \neq \xi$ in ω_1, $\beta \notin \dot{U}(\xi, 0)$;
(c) for all $\xi, \alpha \in \omega_1$ $\dot{U}(\xi,\alpha) \subseteq \dot{U}(\xi, 0)$ and $\dot{U}(\xi, 0)$ has compact closure;
(d) for each limit $\delta \in \omega_1$, the sequence $\{\dot{U}(\xi,\alpha) : \alpha < \delta\}$ is a *regular filter*, *i.e.*, each finite intersection of these includes the closure of another.

For an S-name $\dot{h}$ of a function from ω_1 to ω_1, let $\dot{U}(\xi, \dot{h})$ stand for $\dot{U}(\xi, \dot{h}(\xi))$. For limit δ, let $\dot{Z}(\xi,\delta)$ denote the S-name of the compact G_δ equal to $\bigcap\{\dot{U}(\xi,\alpha) : \alpha < \delta\}$. For a cub C and ordinal ξ, we also use $\dot{Z}(\xi, C)$ as an abbreviation for $\dot{Z}(\xi, \xi^+(C))$.

Fix an enumeration $\{C_\gamma : \gamma \in \omega_2\}$ for a base for the cubs on ω_1 (each containing only limit ordinals), chosen so that C_0 is as above and for $0 < \lambda \in \omega_2$, $C_\lambda \subseteq \operatorname{Fix}(C_0)$ and $C_\lambda \smallsetminus \operatorname{Fix}(C_\gamma)$ is countable for all $0 \leq \gamma < \lambda$. We can do this by taking diagonal intersections, since $2^{\aleph_1} = \aleph_2$.

For each $\delta \in C_0$, let $\beta(\delta) = \delta^+(C_0)$. Since $\dot{Z}(\xi, C_\gamma) \subseteq \dot{U}(\xi, C_\gamma)$ for all $\xi \in \omega_1$ for all $\delta \in C_\gamma$, $\beta(\delta) < \delta^+(C_\gamma)$, and so it is forced that

$$\overline{\bigcup\{\dot{Z}(\xi, C) : \xi < \delta\}} \cap \omega_1 \subseteq \beta(\delta).$$

We can also assume that for all cubs $C \subseteq C_0$, there is an S-name $\dot{A}$ that is forced to be a stationary subset of $\operatorname{Fix}(C)$ satisfying:

$$(\forall s \in S)(\forall \delta)\, s \Vdash \Big(\delta \in \dot{A} \Rightarrow (\exists \alpha \in [\delta, \beta(\delta)])\, (\alpha \in \overline{\bigcup\{\dot{Z}(\xi, C) : \xi < \delta\}}\,)\Big).$$

The reason we can make this assumption is that we are assuming there is no σ-discrete expansion of ω_1 by compact G_δ's. If, in the extension, the set

$$A = \{\delta : \overline{\bigcup\{\dot{Z}(\xi, C) : \xi < \delta\}} \nsubseteq \delta\}$$

were not stationary, then there would be a $\lambda \in \omega_2$ such that $A \cap C_\lambda$ is empty. Since the cub C_λ divides ω_1 into countable pieces, we see that we can expand the points in ω_1 into a σ-discrete collection of compact G_δ's.

For each $\lambda \in \omega_2$, let $\dot{A}_\lambda$ denote the name of the stationary set just described for $C = C_\lambda$. For any $B \subseteq \omega_1$, we will write

$$\alpha \in \left\langle \dot{Z}(\xi, C) : \xi < \delta \right\rangle'$$

to mean that α is a limit point of that sequence of sets.

Fix any function $e: S \to \omega$ with the property that for all $\delta \in \omega_1$, $e \upharpoonright S_\delta$ is one-to-one. For an ordinal $\gamma \in \omega_2$, we use $\dot{f}_\gamma$ for the S-name of the function from ω_1 into ω given by the property that each $s \in S_{\xi^+(C_\gamma)}$ forces that $\dot{f}_\gamma(\xi) = e(s)$. Thus, $\dot{f}_\gamma$ partitions ω_1 into a discrete collection of countably many closed subsets. Then let $\{\dot{W}(\gamma, n) : n \in \omega\}$ be a discrete collection of open sets separating the $\dot{f}_\gamma^{-1}(n)$'s. Fix $n \in \omega$. By normality, there is an open $\dot{V}_n$ such that S forces $\dot{f}_\gamma^{-1}(n) \subseteq \dot{V}_n \subseteq \overline{\dot{V}}_n \subseteq \dot{W}(\gamma, n)$. For each $\xi \in \dot{f}_\gamma^{-1}(n)$, there is an $\alpha_\xi \in \omega_1$ such that S forces $\dot{U}(\xi, \alpha_\xi) \subseteq \dot{V}_n$. Let $\zeta_n(\gamma) \in \omega_2$ be such that for $\xi \in \dot{f}_\gamma^{-1}(n)$, $\xi < \rho \in C_{\zeta_n(\gamma)}$ implies $\alpha_\xi < \rho$. Then S forces $\{\dot{Z}(\xi, C_{\zeta_n(\gamma)}) : \xi \in \dot{f}_\gamma^{-1}(n)\} \subseteq \dot{V}_n$. We then can find a $C_{\zeta(\gamma)}$ included in each $C_{\zeta_n(\gamma)}$ such that for every $n \in \omega$, S forces $\{\dot{Z}(\xi, C_{\zeta(\gamma)}) : \zeta \in \dot{f}_\gamma^{-1}(n)\} \subseteq \dot{V}_n$. Thus,

$$\overline{\bigcup\{\dot{Z}(\xi, C_{\zeta(\xi)}) : \xi \in f_\gamma^{-1}(n)\}} \subseteq \dot{W}(\gamma, n).$$

By recursion on $\gamma \in \omega_2$, we can choose $\zeta(\gamma) \geq \gamma$ as above so that the sequence $\{\zeta(\gamma) : \gamma \in \omega_2\}$ is strictly increasing. For each γ, we have the S-name $\dot{A}_{\zeta(\gamma)}$ as above. It is immediate that $A_\gamma = \{\delta : (\exists s \in S) s \Vdash \delta \in \dot{A}_{\zeta(\gamma)}\}$ is a stationary set. In other words, $\delta \in A_\gamma$ implies there is some $s \in S$ and $\eta \in [\delta, \beta(\delta)]$ such that $s \Vdash \eta \in \langle \dot{Z}(\xi, C_{\zeta(\gamma)}) : \xi \in \delta \rangle'$.

By **SCC** and Theorem 3.11 we can assume that there is an elementary submodel M of some $\langle H(\theta), \{\langle \gamma, \zeta(\gamma), A_\gamma \rangle : \gamma \in \omega_2\} \rangle$, with $M \cap \omega_1 = \delta < \omega_1$, $|M \cap \omega_2| = \aleph_1$, and an uncountable $\{\gamma_\alpha : \alpha \in \omega_1\} \subseteq M \cap \omega_2$, so that $\delta \in A_{\gamma_\alpha}$ for all $\alpha \in \omega_1$.

For each $\alpha \in \omega_1$ choose $s_\alpha \in S$, $\eta_\alpha \in [\delta, \beta(\delta)]$ such that

$$s_\alpha \Vdash \eta_\alpha \in \left\langle \dot{Z}(\xi, C_{\zeta(\gamma_\alpha)}) : \xi \in \delta \right\rangle'.$$

We may assume s_α is on a level at least as high as $\delta^+(C_{\gamma_\alpha})$. We can also assume that if $\alpha < \beta \in \omega_1$, then $\gamma_\alpha < \gamma_\beta$. We can also assume that the height of s_α is less than the height of s_β, for $\alpha < \beta$, so that $\{s_\alpha : \alpha \in \omega_1\}$ is an uncountable subset of S. Therefore, there is an $\eta \in [\delta, \beta(\delta)]$ such that $L = \{\alpha : \eta_\alpha = \eta\}$ is uncountable. Also, as is well known for Souslin trees, there is an $\bar{s} \in S$, such that $\{s_\alpha : \alpha \in L\}$ includes a dense subset of $\{s \in S : \bar{s} < s\}$. By passing to an uncountable subset, we can assume that $\bar{s} < s_\alpha$ for all $\alpha \in L$ and that $\bar{s}$ is on a level above δ. Similarly, we can assume that for all $\xi, \rho < \delta$, $\bar{s}$ has decided the statement

$$\dot{U}(\eta, 0) \cap \dot{Z}(\xi, \rho) \neq \varnothing \quad \text{for all } \xi, \rho < \delta.$$

Now choose any $\alpha \in L$ (*e.g.*, the least one), and then choose an infinite sequence $\{\beta_l : l \in \omega\} \subseteq L \setminus (\alpha + 1)$ so that the $s_{\beta_l} \upharpoonright \delta^+(C_{\gamma_\alpha})$ for $l \in \omega$ are all distinct. For each l, let $e(s_{\beta_l} \upharpoonright \delta^+(C_{\gamma_\alpha})) = n_l$.

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Main Claim $\bar{s} \Vdash (\forall l \in \omega)(\dot{W}(\gamma_\alpha, n_l) \cap \dot{U}(\eta, 0) \neq 0)$.

Once this claim is proved we are done, because we then have that $\bar{s}$ forces that $\dot{U}(\eta, 0)$ cannot have compact closure, because it meets infinitely many members of the discrete family $\{\dot{W}(\gamma_\alpha, n) : n \in \omega\}$.

To prove the claim, first note that there is a tail of $C_{\zeta(\gamma_{\beta_l})} \cap \delta$ included in $C_{\zeta(\gamma_\alpha)}$. To see this, recall $C_{\zeta(\gamma_\alpha)} \smallsetminus \mathrm{Fix}(C_{\zeta(\gamma_{\beta_l})})$ is countable, so some tail of $\mathrm{Fix}(C_{\zeta(\gamma_{\beta_l})})$ is included in $C_{\zeta(\gamma_\alpha)}$. By elementarity, since γ_α and γ_β are in M, a tail of $\mathrm{Fix}(C_{\zeta(\gamma_{\beta_l})}) \cap M$ is included in $C_{\zeta(\gamma_\alpha)} \cap M$, so a tail of $\mathrm{Fix}(C_{\zeta(\gamma_{\beta_l})}) \cap \delta$ is included in $C_{\zeta(\gamma_\alpha)}$.

Since there is a tail of $C_{\zeta(\gamma_{\beta_l})} \cap \delta$ included in $C_{\zeta(\gamma_\alpha)}$, $\dot{Z}(\xi, C_{\zeta(\gamma_{\beta_l})}) \subseteq \dot{Z}(\xi, C_{\zeta(\gamma_\alpha)})$ for each $\xi < \delta$ (at least on a tail, which is all that matters for limits above δ). Then s_{β_l} forces that η is a limit of the sequence

$$\left\langle \dot{Z}(\xi, C_{\zeta(\gamma_\alpha)}) : \xi \in \delta \text{ and } \dot{f}_{\gamma_\alpha}(\xi) = n_l \right\rangle.$$

Of course this means that s_{β_l} forces that $\dot{U}(\eta, 0)$ meets $\dot{Z}(\xi, C_{\zeta(\gamma_\alpha)})$ for cofinally many $\xi < \delta$ such that $s_{\beta_l} \restriction \gamma_\alpha \Vdash \dot{f}_{\gamma_\alpha}(\xi) = n_l$. But $\bar{s}$ has already decided the value of $\dot{f}_{\gamma_\alpha} \restriction \delta$, and $\bar{s}$ already forces $\dot{U}(\eta, 0) \cap \dot{Z}(\xi, C_{\zeta(\gamma_\alpha)}) \neq \varnothing$ whenever s_{γ_β} does. In particular then, $\bar{s}$ forces there is a ξ with $\dot{f}_{\gamma_\alpha}(\xi) = n_l$ (and so $\dot{Z}(\xi, C_{\zeta(\gamma_\alpha)}) \subseteq \dot{W}(\gamma_\alpha, n_l)$) and $\dot{U}(\eta, 0) \cap \dot{Z}(\xi, C_\zeta(\gamma_\alpha)) \neq \varnothing$. ∎

For the record, let us state what we have accomplished.

Theorem 4.5 MM$(S)[S]$ *implies* ***LCN***$(\aleph_1)$.

Corollary 4.6 *There is a model of* MM$(S)[S]$ *in which* ***LCN*** *holds;* i.e., *every locally compact normal space is collectionwise Hausdorff.*

Proof As mentioned after Proposition 1.6 and after Lemma 3.2, we can front-load a model of MM$(S)[S]$ as we did for a model of PFA$(S)[S]$ in [26] to get that for character $\kappa \geq \aleph_1$, normal spaces of character $\leq \kappa$ are κ-collectionwise Hausdorff, and then use character reduction to obtain **LCN**. ∎

5 Large Cardinals and the MOP

In [11] we showed that large cardinals are not required to obtain the consistency of every locally compact perfectly normal space is paracompact. It is interesting to see which other PFA$(S)[S]$ results can be obtained without large cardinals. The standard method used was pioneered by Todorcevic in [43] and given several applications in [8], all in the context of PFA results. In the context of PFA$(S)[S]$, it is referred to in [45] and actually carried out in [10] for a version of P-ideal Dichotomy and for **PPI**. It is routine to get additionally that such models are of form MA$_{\omega_1}(S)[S]$ by interleaving additional forcing. In [11] we pointed out that such methods can give models in which in addition the following holds.

Σ^-(sequential) In a compact sequential space, each locally countable subspace of size $\aleph_1$ is σ-discrete.

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A modification of such a proof produces a model in which the following proposition (see [15]) holds:

$\sum$(sequential) Let X be a compact sequential space. Let $Y \subseteq X$, $|Y| = \aleph_1$. Suppose $\{W_\alpha\}_{\alpha\in\omega_1}$, $\{V_\alpha\}_{\alpha\in\omega_1}$ are open subsets of X such that:

(a) $W_\alpha \subseteq \overline{W_\alpha} \subseteq V_\alpha$,
(b) $|V_\alpha \cap Y| \leq \aleph_0$,
(c) $Y \subseteq \bigcup\{W_\alpha : \alpha \in \omega_1\}$.

Then Y is σ-closed discrete in $\bigcup\{W_\alpha : \alpha \in \omega_1\}$.

Without the parenthetical "sequential", $\sum^-$ and $\sum$ refer to the corresponding propositions obtained by replacing "sequential" by "countably tight", which follow from their sequential versions if one has the following.

Moore-Mrówka Every compact countably tight space is sequential.

It follows easily from **Moore-Mrówka** that locally compact countably tight spaces are sequential. A proof of **Moore-Mrówka** from $\mathrm{PFA}(S)[S]$ is sketched in [45] and the author remarks that, by the usual methods, large cardinals are not necessary. Thus, one can obtain a model of $\mathrm{MA}_{\omega_1}(S)[S]$ in which, for example, both **PPI** and $\sum$ hold, without the need for large cardinals. Working in such a model, we can establish the following proposition, the conclusion of which was proved from $\mathrm{PFA}(S)[S]$ in [45] and asserted to be obtainable without large cardinals.

Theorem 5.1 *If ZFC is consistent, it is consistent to additionally assume that locally compact, hereditarily normal, separable spaces are hereditarily Lindelöf.*

Proof We work in a model as above. Let X be such a space. By Lemma 4.3, X has countable spread, since its open subspaces are also locally compact, normal, and separable. So does its one-point compactification X^*, which hence is countably tight [1]. If X were not hereditarily Lindelöf, it would include a right-separated subspace $\{x_\alpha : \alpha \in \omega_1\}$. Let $\{V_\alpha : \alpha \in \omega_1\}$ be open sets witnessing right-separation. Let $x_\alpha \in W_\alpha \subseteq \overline{W_\alpha} \subseteq V_\alpha$, with W_α open and $\overline{W_\alpha}$ compact. Applying $\sum$ to X^*, we see that $\{x_\alpha : \alpha \in \omega_1\}$ is σ-closed discrete in $W = \bigcup\{W_\alpha : \alpha \in \omega_1\}$. But W is locally compact, separable, and hereditarily normal, so this contradicts Lemma 4.3. ∎

Also without large cardinals we obtain the following theorem.

Theorem 5.2 *If ZFC is consistent, it is consistent to additionally assume that each hereditarily normal perfect pre-image of ω_1 includes a copy of ω_1.*

Proof Take a model of $\mathrm{MA}_{\omega_1}(S)[S] + \sum$+**PPI**. By Theorem 5.1, we have Lemma 2.7. Lemmas 2.6, 2.7, and **PPI** give the conclusion. ∎

We also have the following theorem.

Theorem 5.3 *If ZFC is consistent, it is consistent to assume that every locally compact, first countable, hereditarily normal space with Lindelöf number $\leq \aleph_1$ not including a copy of ω_1 is paracompact.*

Proof We use the model of Theorem 5.2. In [40] the second author asserted the following, but under PFA$(S)[S]$ instead of MM$(S)[S]$, which we now see should have been used.

Lemma 5.4 MM$(S)[S]$ *implies that if X has Lindelöf number* $\leq \aleph_1$ *and is locally compact, normal, and does not include a perfect pre-image of* ω_1, *then X is paracompact.*

In addition to the topological properties mentioned, the proof used Σ and that the space was $\aleph_1$-collectionwise Hausdorff. For the purposes of Theorem 5.3, however, we get $\aleph_1$-collectionwise Hausdorff just from the Souslin forcing, since the space is first countable. ∎

MM$(S)[S]$ is also relevant for questions concerning the Baireness of $C_k(X)$, for locally compact X (see [21, 30, 41]).

Definition A *moving off collection* for a space X is a collection $\mathcal{K}$ of non-empty compact sets such that for each compact L, there is a $K \in \mathcal{K}$ disjoint from L. A space satisfies the Moving Off Property (MOP) if each moving off collection includes an infinite subcollection with a discrete open expansion.

Definition $C_k(X)$, for a space X, is the collection of all continuous real-valued functions on X, considered as a subspace of the compact-open topology on the Cartesian power $X^{\mathbb{R}}$.

Theorem 5.5 ([21]) *A locally compact space X satisfies the MOP if and only if* $C_k(X)$ *is Baire,* i.e., *satisfies the Baire Category Theorem.*

Lemma 5.6 ([21, 30]) *Locally compact, paracompact spaces satisfy the MOP.*

Theorem 5.7 MM$(S)[S]$ *implies that normal spaces satisfying the MOP are paracompact if they are:*

(i) *locally compact, countably tight, and hereditarily normal, or*
(ii) *first countable and hereditarily normal, or*
(iii) *locally compact, countably tight with Lindelöf number* $\leq \aleph_1$, *or*
(iv) *first countable, with Lindelöf number* $\leq \aleph_1$, *or*
(v) *locally compact, countably tight, and countable sets have Lindelöf closures.*

Proof These all follow easily from Theorems 2.10 and 3.13, and **Moore-Mrówka**, using the following lemmas.

Lemma 5.8 ([22]) *In a sequential space, countably compact subspaces are closed.*

Lemma 5.9 ([21, 30]) *Countably compact spaces satisfying the MOP are compact.*

Lemma 5.10 ([21, 30]) *First countable spaces satisfying the MOP are locally compact.*

Lemma 5.11 ([3]) *The one-point compactification of a locally compact space X is countably tight if and only if X does not include a perfect pre-image of* ω_1.

If they have the MOP, sequential spaces do not include copies of ω_1, so (i) follows from Theorem 3.12. (ii) follows from (i) plus Lemma 5.10. (iii) follows from Lemma 5.4 plus Theorem 2.10. (iv) follows from (iii) plus 5.10. (v) follows from Theorems 3.13 and 2.10 and Balogh's Lemma above. ■

In the special case of a space with the MOP, we have the following theorem.

Theorem 5.12 *If ZFC is consistent, then it is consistent to additionally assume that first countable normal spaces satisfying the MOP and with Lindelöf number $\leq \aleph_1$ are paracompact.*

Proof Such a space is locally compact and does not include a perfect pre-image of ω_1. ■

MA_{ω_1} gives counterexamples for the conclusions of Theorems 5.7 and 5.12. See *e.g.,* [41].

Theorem 5.13 *If ZFC is consistent, then it is consistent to assume that first countable hereditarily normal, locally connected spaces satisfying the MOP are paracompact.*

Proof The extra ingredient is that the local connectedness will enable us to decompose the space into a sum of pieces with Lindelöf number $\leq \aleph_1$.

Definition A space X is of Type I if $X = \bigcup\{X_\alpha : \alpha \in \omega_1\}$, where each X_α is open, $\alpha < \beta$ implies $\overline{X_\alpha} \subseteq X_\beta$, and each X_α is Lindelöf.

In [40, p. 104], it was shown that, assuming $\sum$ and hereditary $\aleph_1$-collectionwise Hausdorffness for a locally compact hereditarily normal space not including a perfect pre-image of ω_1, the closure of a Lindelöf subspace is Lindelöf. Then we quote:

Lemma 5.14 ([12]) *If X is locally compact, locally connected, and countably tight, then X is a topological sum of Type I spaces if and only if every Lindelöf subspace of X has Lindelöf closure.*

Since a topological sum of paracompact spaces is paracompact, this will complete the proof of the theorem. ■

Problem 1 Without large cardinals, is there a model in which both $\sum$ and $\mathbf{LCN}(\aleph_1)$ hold?

Problem 2 ([21]) Is there in ZFC a locally compact, normal, non-paracompact space with the MOP?

We conjecture that the answer is positive. Large cardinals would be necessary to refute the existence of such a space, since an example can be constructed from the failure of the Covering Lemma for the Core Model K, which entails the consistency of measurable cardinals. We thank Peter Nyikos for referring us to Good [19], where that failure is used to construct a locally compact, locally countable, normal, non-paracompact space X on $\kappa^+ \times \omega$, where κ^+ is the successor of a singular strong limit cardinal of countable cofinality, such that the spaces $X_\alpha = \alpha \times \omega$ are metrizable for all $\alpha \in \kappa^+$. It follows that closed subspaces of X of size $\leq 2^{\aleph_0}$ are locally compact and metrizable, so satisfy the MOP by Lemma 5.6. On the other hand, we have the following theorem.

Theorem 5.15 ([41]) *If a Hausdorff space Z is locally countable, locally compact, and closed subspaces of* $\leq 2^{\aleph_0}$ *have the MOP, then Z has the MOP.*

It follows that X has the MOP. ■

With $\mathrm{MM}(S)[S]$ we have the following theorem.

Theorem 5.16 $\mathrm{MM}(S)[S]$ *implies that if X is normal, locally compact, locally countable, and closed subspaces of size* $\leq 2^{\aleph_0}$ *are metrizable, then X is metrizable.*

Proof By Theorem 5.15, X has the MOP. By Theorem 5.7, to get that X is paracompact, it suffices to show that countable subspaces of X have Lindelöf closures. But if Y is a countable subset of X, $|\overline{Y}| \leq 2^{\aleph_0}$ and hence is separable metrizable and hence Lindelöf. Once we have X paracompact, it follows that X is a topological sum of σ-compact subspaces. But each of these has size $\leq 2^{\aleph_0}$ and so is metrizable.

Good's example also shows that the conclusion of Theorem 3.13 requires large cardinals. The example is already known to be locally compact, normal and non-paracompact. It does not include a copy of ω_1 (or even a perfect pre-image of ω_1), since it is locally countable and hence first countable, but satisfies the MOP. Separable subspaces are of size $\leq 2^{\aleph_0}$ and hence metrizable, so they are Lindelöf. ■

Axiom R precludes stationary non-reflecting sets of ω-cofinal ordinals in ω_2, and hence the locally compact, $\aleph_1$-collectionwise Hausdorff ladder system space built on such a set; we can therefore pose the following problem.

Problem 3 *Does* $\mathrm{MM}(S)[S]$ *imply* ***LCN****? Indeed, does* MM *imply locally compact* $\aleph_1$*-collectionwise Hausdorff spaces are collectionwise Hausdorff?*

6 Examples

A question left open in [26] is whether, as was shown for adjoining $\aleph_2$ Cohen subsets of ω_1 in [36], forcing with a Souslin tree would make normal spaces of character $\aleph_1$ $\aleph_1$-collectionwise Hausdorff. We will show that the answer is negative by showing the following theorem.

Theorem 6.1 $\mathrm{MA}_{\omega_1}(S)[S]$ *implies that there is a normal non-*$\aleph_1$*-collectionwise Hausdorff space of character* $\aleph_1$.

Proof Let $S \subseteq 2^{<\omega_1}$ be a coherent Souslin tree. Fix a family $\{a_s : s \in S\} \subseteq [\omega]^\omega$ so that for $s < t \in S$, $a_t \subseteq^* a_s$ and for each $\gamma \in \omega_1$, $\{a_s : s \in S_\gamma\}$ is pairwise disjoint.

For each limit $\delta \in \omega_1$, let $L_\delta \in \delta^\omega$ be a strictly increasing function with range cofinal in δ consisting of successor ordinals. For $a \subseteq \omega$, let $L[a] = \{L_\delta(n) : n \in a\}$. The generic g for S will enable us to define the required topology on the set ω_1. We declare each successor ordinal to be isolated. For each limit δ, the neighborhood filter for δ will be $\{(L_\delta[a_s] \cup \{\delta\}) \smallsetminus F : s \in g, F \text{ a finite subset of } \omega_1 \smallsetminus \{\delta\}\}$. The set C_0 of limit ordinals is then a closed discrete set. By pressing down, we see that C_0 cannot be separated. It remains to show that the space is normal. It suffices to show that if f is an S-name of a function from C_0 to 2, then there is a neighborhood assignment $\{\dot{U}_\delta : \delta \in C_0\}$ and a cub C_1, such that for each $\alpha < \delta$ with $\delta \in C_1$, S forces that if $\dot{f}(\alpha) \neq \dot{f}(\delta)$, then $\dot{U}(\alpha)$ and $\dot{U}(\delta)$ are disjoint.

There is a cub $C_1 \subseteq C_0$ so that for all $\delta \in C_1$ and $\alpha < \delta_1$ each $s \in S_\delta$ decides the value of $\dot{f}(\alpha)$. For each $\delta \in C_0$, let δ^+ denote the minimal element of C_1 above δ, and choose a function $f_\delta\colon \omega \to 2$ so that for each $s \in S_{\delta^+}$ and each $n \in a_s$, s forces $\dot{f}(\delta) = f_\delta(n)$. We will define an integer n_δ such that the value of $\dot{U}_\delta$ is forced by $s \in S_{\delta^+}$ to equal $\{\delta\} \cup L_\delta[a_s \smallsetminus n_\delta]$. The sequence of functions $\{f_\delta : \delta \in C_0\}$ will be in the $\mathrm{MA}_{\omega_1}(S)$ model.

Let $\mathcal{Q}$ be the poset of partial functions h from ω_1 into 2 such that

$$h =^* \bigcup\{f_\delta \circ L_\delta^{-1} : \delta \in H\},$$

for some $H \in [C_0]^{<\omega}$. $\mathcal{Q}$ is ordered by extension. We claim that in ZFC, $\mathcal{Q}$ is ccc. If so, there will be a generic for $\aleph_1$ dense subsets of $\mathcal{Q}$ in a model of $\mathrm{MA}_{\omega_1}(S)$. Let $\mathcal{H} = \{(h_\alpha, H_\alpha) : \alpha \in \omega_1\}$ be a subset of $\mathcal{Q} \times [C_0]^{<\omega}$, where $h_\alpha = \bigcup\{f_\delta \circ L_\delta^{-1} : \delta \in H_\alpha\}$. Choose any countable elementary submodel M with $\mathcal{Q}$ and $\mathcal{H}$ in M. Let $\delta = M \cap \omega_1$ and $H_\delta \cap M = H$ and $H_\delta \smallsetminus M = \{\delta_i : i < l\}$. We can assume that $\delta_0 = \delta$ and then choose $\alpha_0 \in M$ so that $H \subseteq \alpha_0$ and $L_{\delta_i} \cap \delta \subseteq \alpha_0$, for $0 < i < l$. Notice that $h_\delta \restriction \alpha$ is an element of M, for all $\alpha \in M$. In M, recursively choose $\alpha_0 < \alpha_1 < \cdots$ so that $h_{\alpha_{n+1}} \restriction \alpha_n = h_\delta \restriction \alpha_n$ and $\mathrm{dom}(h_{\alpha_{n+1}}) \subseteq \alpha_{n+2}$. With $\beta = \sup_n \alpha_n < \delta$, we have that there is an $n \in \omega$ such that $h_\delta \restriction \beta = h_\delta \restriction \alpha_n$. It follows that $h_\delta \restriction \alpha_n \subseteq h_{\alpha_{n+1}}$, and so h_δ and $h_{\alpha_{n+1}}$ are compatible members of $\mathcal{Q}$.

$\mathrm{MA}_{\omega_1}(S)$ implies there is a generic for $\mathcal{Q}$ that adds a function h from ω_1 to 2 that mod finite extends $f_\delta \circ L_\delta^{-1}$, for all $\delta \in C_0$. Now define n_δ to be chosen so that h actually extends $f_\delta \circ L_\delta^{-1}[\omega \smallsetminus n_\delta]$. Suppose $\alpha < \delta$, with $\delta \in C_1$, and let $s \in S_{\delta^+}$. Then s forces that $f_\delta \circ L_\delta^{-1} = \dot{f}_\delta$ on a_s, and similarly, $s \restriction \alpha^+$ forces that $f_\alpha \circ L_\alpha^{-1} = \dot{f}_\alpha$ on $a_{s \restriction \alpha^+}$. Also, h agrees with $f_\delta \circ L_\delta^{-1}$ on $a_s \smallsetminus n_\delta$ and with $f_\alpha \circ L_\alpha^{-1}$ on $a_{s\restriction\alpha^+} \smallsetminus n_\alpha$. Thus, if $\beta \in L_\delta[a_s \smallsetminus n_\delta] \cap L_\alpha[a_{s\restriction\alpha^+} \smallsetminus n_\alpha]$, then $h(\beta) = \dot{f}(\alpha) = \dot{f}(\delta)$. This completes the proof that the space is normal. ∎

The strategy attempted in [39] was to expand a closed discrete subspace of a locally compact normal space to a discrete collection of compact G_δ's. There are limitations on such an approach, given by the following example.

Theorem 6.2 *$\mathrm{MA}_{\omega_1}(S)[S]$ implies there is a locally compact space of character $\aleph_1$ that includes a normalized closed discrete set that does not have a normalized discrete expansion by compact G_δ's.*

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Proof We modify the previous example. Let $\mathcal{A}_s$ denote the Boolean subalgebra of $\mathcal{P}(\omega)$ generated by $[\omega]^{<\omega} \cup \{a_s : s \in S\}$. In the forcing extension by S, let x_g denote the member of the Stone space $\mathcal{S}(\mathcal{A}_s/\mathrm{FIN})$ containing $\{a_s : s \in g\}$.

In the forcing extension, our space has the base set $(\omega_1 \setminus C_0) \cup (C_0 \times \mathcal{S}(\mathcal{A}_s))$. The points of $\omega_1 \setminus C_0$ are isolated. For each $\delta \in C_0$ and $x \in \mathcal{S}(\mathcal{A}_s/\mathrm{FIN})$, a neighborhood of (δ, x) must include $U_\delta(a) = L_\delta[a] \cup (\{\delta\} \times a^*)$ for some $a \in x$, where $a^* = \{p \in \mathcal{S}(\mathcal{A}_S) : a \in p\}$. Notice that $U_\delta(a)$ is disjoint from $\{\gamma\} \times \mathcal{S}(\mathcal{A}_s/\mathrm{FIN})$, for all $\gamma \neq \delta$. It follows immediately that the sequence $D = \{(\gamma, x_g) : \delta \in C_0\}$ is a closed discrete subset. It also follows from the proof of the normality of the previous example that D is normalized.

Now we show that D does not have a normalized discrete expansion by compact G_δ's, indeed by any G_δ's. Assume that $\{\dot{Z}_\delta : \delta \in C_0\}$ is a sequence of S-names so that $\dot{Z}_\delta$ is forced to be a G_δ containing (δ, x_g). There is a cub C_1 such that for each $\alpha \in C_0$ and each $s \in S_{\alpha^+}$ (again, α^+ is the minimal element of C_1 above α), s forces that $\dot{Z}_\alpha$ contains $\{\alpha\} \times a_s^*$. Since S is ccc, the cub C_1 can be chosen to be a member of the $\mathrm{MA}_{\omega_1}(S)$ model.

We use C_1 to define a partition of C_0: for each $\alpha \in C_0$, we define $\dot{f}(\alpha)$ to equal the value $g(\alpha^+)$ (*i.e.*, the element of S_{α^+} that g picks). Thus, if δ is a limit of C_1 and $s \in S_\delta$, then s forces a value for $\dot{f} \restriction \delta$. Then a potential normalizing expansion would consist of a sequence $\{\dot{n}_\alpha : \alpha \in C_0\}$ of S-names of integers for which $L_\alpha[a_{g\restriction\alpha^+} \setminus \dot{n}_\alpha] \cup (\{\alpha\} \times a^*_{g\restriction\alpha^+})$ is an open neighborhood of $\dot{Z}_\alpha$. There is a cub $C_2 \subseteq C_1$ so that for each $\delta \in C_2$ and each $s \in S_\delta$, s forces a value on $\dot{n}_\alpha$ for all $\alpha < \delta$. We can choose any $s_0 \in g$ so that s_0 forces that $L_\alpha[a_{g\restriction\alpha^+} \setminus \dot{n}_\alpha] \cap L_\delta[a_{g\restriction\delta^+} \setminus \dot{n}_\delta]$ is empty whenever $\dot{f}(\alpha) \neq \dot{f}(\delta)$. Working in $V[g]$, we prove there is a stationary E satisfying that $L_\delta[a_{g\restriction\delta^+}] \cap \bigcup\{L_\alpha[a_{g\restriction\alpha^+} \setminus \dot{n}_\alpha] : \alpha \in \delta\}$ is infinite, for all $\delta \in E$. If not, then there would be an assignment $\langle m_\delta : \delta \in C\rangle$ (for some cub C) so that $L_\delta[a_{g\restriction\delta^+} \setminus m_\delta]$ would be disjoint from $\bigcup\{L_\alpha[a_{g\restriction\alpha^+} \setminus \dot{n}_\alpha] : \alpha \in \delta\}$, for all $\delta \in C$. Pressing down, we would arrive at a contradiction.

Let $\dot{E}$ denote the S-name of the stationary set whose existence was shown in the previous paragraph. Choose any s above s_0 and any $\delta \in C_2$ such that s forces that $\delta \in \dot{E}$. Without loss of generality, the height of s is $\geq \delta^+$, but note that $s\restriction\delta$ forces a value on $\dot{n}_\alpha$, for all $\alpha < \delta$. This means that $s\restriction\delta^+$ forces that $\delta \in \dot{E}$, since it will also decide the value of $L_\delta[a_{g\restriction\delta^+}]$. We also have that $s\restriction\delta$ forces a value on $\dot{f}\restriction\delta$ and so we can choose a value $e \in \{0,1\}$ so that $s\restriction\delta$ forces that $L_\delta[a_{s\restriction\delta^+}]$ intersected with $\{L_\alpha[a_{s\restriction\alpha^+} \setminus \dot{n}_\alpha] : \alpha < \delta \text{ and } \dot{f}(\alpha) = e\}$ is infinite. We now have a contradiction, since $s\restriction\delta^+ \cup \{(\delta^+, 1-e)\}$ forces that the assigned neighborhood of δ must meet the assigned neighborhood of α, for some $\alpha < \delta$ with $\dot{f}(\alpha) = e \neq \dot{f}(\delta)$. ∎

7 Point-countable Type

There is another normal-implies-collectionwise-Hausdorff result holding in L for which we do not know whether it holds in our $\mathrm{MM}(S)[S]$ model.

Definition A space is of *point-countable type* if each point is a member of a compact subspace that has a countable outer neighbourhood base.

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Spaces of point-countable type simultaneously generalize locally compact and first countable spaces, and V=L implies normal spaces of point-countable type are collectionwise Hausdorff [46].

Problem 4 *Does* MM$(S)[S]$ *imply normal spaces of point-countable type are* $\aleph_1$*-collectionwise Hausdorff?*

The usual arguments would show that if so, in our front-loaded model of MM$(S)[S]$, normal spaces of point-countable type would be collectionwise Hausdorff.

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Department of Mathematics and Statistics, University of North Carolina, Charlotte, North Carolina 28223, USA
e-mail: adow@uncc.edu

Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4
e-mail: f.tall@math.utoronto.ca