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The complexity of higher Chow groups

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Abstract. Let $X/\mathbb{C}$ be a smooth projective variety. We consider two integral invariants, one of which is the level of the Hodge cohomology algebra $H^*(X,\mathbb{C})$ and the other involving the complexity of the higher Chow groups $\mathrm{CH}^*(X,m;\mathbb{Q})$ for $m \geq 0$. We conjecture that these two invariants are the same and accordingly provide some strong evidence in support of this.

1 Introduction

Let $X/\mathbb{C}$ be a smooth projective variety of dimension n. Associated with X are the Chow groups $\mathrm{CH}^r(X) = \mathrm{CH}_{n-r}(X)$ of algebraic cycles on X of codimension r (resp. dimension $n-r$), modulo rational equivalence. Historically, there are two cycle class maps, namely the fundamental cycle class map $[\ ] : \mathrm{CH}^r(X) \to H^{2r}(X,\mathbb{Z})$ and the Abel–Jacobi map on $\mathrm{CH}^r_{\mathrm{hom}}(X) = \ker[\ ]$, the latter being a certain membrane integral generalizing the classical elliptic integral. It was once thought that $\mathrm{CH}^r(X)$ could be characterized by these two aforementioned maps. However, that myth was debunked by the seminal works of Mumford [11] and Griffiths [3], where both the kernel and image of the Abel–Jacobi map are very complicated in general. These seminal works resulted in a turning point in the subject of algebraic cycles. One point of view is that there should be higher cycle class maps "explaining" the kernel of the previous map, giving rise to a descending filtration on $\mathrm{CH}^r(X,\mathbb{Q}) := \mathrm{CH}^r(X) \otimes \mathbb{Q}$,[1] an idea which originally goes back to Bloch. Such a filtration can be seen as a measure of the "complexity" of Chow groups. Subsequent to this was Beilinson's fortification, that the graded pieces of such a filtration should be described in terms of extension datum involving a conjectural category $\mathcal{MM}(\mathbb{C})$ of mixed motives. Moving toward the latter part of the twentieth century, we have the higher Chow groups $\mathrm{CH}^r(W/k,m)$ invented by Bloch [2] (see Definition 1.3), where W is quasi-projective over a field k, the case $m = 0$ recovers the original Chow groups. As in the case $m = 0$, one conjectures that there should be a descending filtration

$$\mathrm{CH}^r(X/\mathbb{C},m;\mathbb{Q}) = F^0 \supset F^1 \supset \cdots,$$

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[1]The reason for $\mathbb{Q}$-coefficients is the connection of this filtration with motives. Specifically, the Künneth decomposition of the diagonal class involves $\mathbb{Q}$-coefficients.

whose graded pieces Gr_F^ν can be described in terms of extension datum. Of course, this was formulated for smooth projective X over a field k. A generalization of Beilinson's formula is then

$$Gr_F^\nu \mathrm{CH}^r(X/k, m; \mathbb{Q}) \simeq \mathrm{Ext}^\nu_{\mathcal{MM}(k)}(\mathrm{Sp}(k), h^{2r-m-\nu}(X)(r)),$$

where Beilinson's formulation involved the case $m = 0$ and where conjecturally speaking, $h^\bullet(-)(r)$ is motivic cohomology. It is then reasonably clear that the existence of such a filtration (called the conjectural Bloch–Beilinson filtration) is pivotable with regard to issues of complexity. With this in mind, we begin with the following.

Definition 1.1 Let $X/\mathbb{C}$ be a smooth projective variety, with Hodge cohomology $H^*(X, \mathbb{C}) = \oplus_{p,q} H^{p,q}(X)$. We define[2]

$$\mathrm{Level}(H^*(X, \mathbb{C})) = \max\{p - q \mid H^{p,q}(X) \neq 0\}.$$

$$\mathrm{Level}(\mathrm{CH}^*(X, m; \mathbb{Q})) = \max_r \left\{\mathrm{Level}(\mathrm{CH}^r(X, m; \mathbb{Q}))\right\},$$

where $\mathrm{Level}(\mathrm{CH}^r(X, m; \mathbb{Q})) = 0$ for $r < m$, otherwise for $r \geq m$, Level $(\mathrm{CH}^r(X, m; \mathbb{Q})) =$

$$\min\left\{\mu \geq 0 \,\middle|\, \mathrm{CH}^r(X, m; \mathbb{Q}) \to \mathrm{CH}^r(X \backslash Y, m; \mathbb{Q}) \text{ is zero}, \right.$$
$$\left. Y \hookrightarrow X \text{ closed}, \ \mathrm{codim}_X Y = r - \mu - m\right\}.$$

The[3] expected relationship between these invariants is the following conjecture.

Conjecture 1.2 *For all $m \geq 0$,*

$$\mathrm{Level}(H^*(X, \mathbb{C})) = \mathrm{Level}(\mathrm{CH}^*(X, m; \mathbb{Q})).$$

Based on conjectural assumptions, an outline of a proof of Conjecture 1.2 in the case where $m = 0$ appeared in [8, Corollary 15.64]. In this paper, we provide the full details for all $m \geq 0$; the case $m > 0$ requires some new ingredients which should be of interest to the reader. Finally, we exhibit a class of examples involving complete intersections, based on new ideas from [10].

Before stating our main result, it is important to include some background. We will assume the (Grothendieck amended) general Hodge conjecture (GHC), which is discussed in [8, Chapter 7]. We will also assume that the reader is familiar with the category of $\mathbb{Q}$ mixed Hodge structures MHS, as well as a description of the Abel–Jacobi map in terms of extension classes of MHS. Again, all of this appears in Lewis (*op. cit.*) as well as in [9, Definition 3.11]; however, the latter (viz., extension classes) is explicitly described in [6]. Let $\mathbb{A} \subseteq \mathbb{R}$ be a subring. The Tate twist is given by $\mathbb{A}(r) = (2\pi \mathrm{i})^r \cdot \mathbb{A}$. It is a mixed Hodge structure of pure weight $-2r$ and Hodge type $(-r, -r)$. $H^\bullet(X, \mathbb{A})$ represents Betti cohomology and $H^\bullet(X, \mathbb{A}(r)) := H^\bullet(X, \mathbb{A}) \otimes \mathbb{A}(r)$. For our purposes, we only need real Deligne cohomology in one instance,

[2] Note that $\overline{H^{p,q}(X)} = H^{q,p}(X)$. Hence, this is the same as $|p - q|$.
[3] One can extend this definition to subspaces and subquotients of $\mathrm{CH}^*(X, m; \mathbb{Q})$.

namely $H^m_{\mathcal{D}}(X,\mathbb{R}(m)) \simeq H^{m-1}(X,\mathbb{R}(m-1))$. We only need the following abridged definition of Bloch's higher Chow groups. Let

$$\Delta^m := \mathrm{Sp}\left(\frac{\mathbb{C}[t_0,\ldots,t_m]}{1-\sum_{j=0}^m t_j}\right)$$

be the standard m-simplex. Furthermore, let $z^r(X,m)$ be the codimension r cycles in $X \times \Delta^m$ which meet all faces defined by $\{t_{i_1} = 0,\ldots,t_{i_q} = 0,\ q < n\}$ properly. Associated with $z^r(X,m)$ are ith face maps $\partial_i : z^r(X,m) \to z^r(X,m-1)$, $i = 0,\ldots,m$, defined by $t_i = 0$. One then has the boundary operator $\partial = \sum_{i=0}^m (-1)^i \partial_i : z^r(X,m) \to z^r(X,m-1)$ satisfying $\partial^2 = 0$.

Definition 1.3 $\mathrm{CH}^r(X,m) = H_m^{\partial}(z^r(X,\bullet))$.

As in the case $m = 0$, there is a cycle class map $[\ ] : \mathrm{CH}^r(X,m) \to H^{2r-m}(X,\mathbb{Z}(r))$ with torsion image for $m > 0$ (due to a Hodge theoretic argument). We put $\mathrm{CH}^r(X,m;\mathbb{Q}) = \mathrm{CH}^r(X,m)\otimes\mathbb{Q}$. It is clear that $\mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q}) = \mathrm{CH}^r(X,m;\mathbb{Q})$ for $m > 0$.

Our main result is the following theorem.

Theorem 1.4 *Assume that:*

$\bullet_1$ *The GHC holds.*

$\bullet_2$ *For any smooth projective variety Y defined over $\overline{\mathbb{Q}}$, the Abel–Jacobi map*

$$\Phi_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(Y/\overline{\mathbb{Q}},m;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H^{2r-m-1}(Y(\mathbb{C}),\mathbb{Q}(r)))$$

is injective.

$\bullet_3$ *Either $m \le 2$, or for a given $m \ge 3$, there exists a smooth projective variety $B/\mathbb{C}$ of dimension $m-1$ and a class $\gamma \in H^{m-1}(B,\mathbb{R}(m-1))$ with (Hodge component) $\gamma^{m-1,0} \ne 0$ in the image of the real regulator map $r_{\mathcal{D}} : \mathrm{CH}^m(B,\mathbb{Q}(m))\otimes\mathbb{R} \to H^m_{\mathcal{D}}(B,\mathbb{R}(m)) \simeq H^{m-1}(B,\mathbb{R}(m-1))$.*

Then, for any smooth projective $X/\mathbb{C}$,

$$\mathrm{Level}(H^*(X,\mathbb{C})) = \mathrm{Level}(\mathrm{CH}^*(X,m;\mathbb{Q})).$$

2 More background

(i) To re-iterate and unless otherwise stated, we assume that $X/\mathbb{C}$ is a smooth projective variety of dimension n.

(ii) A full explanation of the Hodge conjecture, the GHC, and the hard Lefschetz conjecture appear in [8, Chapters 7 and 15]. For the reader with pressing obligations, here is an abridged description. Corresponding to a projective embedding $X \subset \mathbb{P}^N$ is a hyperplane class H_X and corresponding operator $L_X = H_X \cup : H^\bullet(X,\mathbb{Q}) \to H^{\bullet+2}(X,\mathbb{Q})$. Iterating this is the hard Lefschetz theorem: for $i \le n$, $L_X^{n-i} : H^i(X,\mathbb{Q}) \to H^{2n-i}(X,\mathbb{Q})$ is an isomorphism. The hard Lefschetz conjecture states that the inverse map $H^{2n-i}(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is algebraic cycle induced. For a complete clarification of cycle induced, the reader can consult [8, Chapter 7]. The reader may find it helpful to know that the Hodge conjecture implies the hard Lefschetz conjecture [8, Chapter 15].

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(iii) $\{N^pH^i(X,\mathbb{Q})\}_{p\ge 0}$ is the coniveau filtration as defined in [8, Chapter 7]. Succinctly,

$$N^pH^i(X,\mathbb{Q}) = \ker\left(H^i(X,\mathbb{Q}) \to \varinjlim_{V\subset X} H^i(X\backslash V,\mathbb{Q})\right),$$

as V runs through all codimension $\ge p$ closed algebraic subsets of X. This can be compared to another filtration,

$$F_H^pH^i(X,\mathbb{Q}) = \text{maximal Hodge structure contained in } F^pH^i(X,\mathbb{C})\cap H^i(X,\mathbb{Q}).$$

Indeed, one knows that $N^pH^i(X,\mathbb{Q}) \subseteq F_H^pH^i(X,\mathbb{Q})$ and the GHC states that the inclusion is an equality (*op. cit.*).

(iv) Given a family of varieties $\{X_t\}_{t\in S}$, where S is a base variety, a general member of this family refers to an X_t, $t\in U$, where $U\subset S$ is a nonempty Zariski open subset, determined by certain "generic" properties, such as X_t nonsingular.

3 Proof of Theorem 1.4

We introduce our first key ingredient.

Theorem 3.1 [7] *Let X be a projective algebraic manifold. Assume the following:*

(i) The hard Lefschetz conjecture holds.

(ii) Either $m\le 2$, or for a given $m\ge 3$, there exists a projective algebraic manifold B of dimension $m-1$ and a class $\gamma\in H^{m-1}(B,\mathbb{R}(m-1))$ with $\gamma^{m-1,0}\ne 0$ in the image of the regulator map $r_{\mathcal{D}}:\mathrm{CH}^m(B,m;\mathbb{Q})\otimes\mathbb{R}\to H^m_{\mathcal{D}}(B,\mathbb{R}(m))\simeq H^{m-1}(B,\mathbb{R}(m-1))$. Then,

$$\mathrm{Level}\big(N^{r-\mu}H^{2r-\mu-m}(X,\mathbb{Q})\big) = \mu - m \Rightarrow \mathrm{Level}\big(\mathrm{CH}^r(X,m;\mathbb{Q})\big) \ge \mu - m.$$

Corollary 3.2 *Let us assume the GHC and Theorem 3.1. Then,*

$$\mathrm{Level}(H^*(X,\mathbb{C})) \le \mathrm{Level}(\mathrm{CH}^*(X,m;\mathbb{Q})).$$

Proof Let $\ell = \mathrm{Level}(H^*(X,\mathbb{C}))$. Then, for some i, we have

$$\mathrm{Level}(H^i(X,\mathbb{C})) = \ell = q-p,\ \text{where } p+q=i,\ q\ge p,\ F^pH^i(X,\mathbb{C}) = H^i(X,\mathbb{C}).$$

Now, for fixed $m\ge 0$, we need only find (r,μ) such that $2r-\mu-m=i$ and $p=r-\mu$. Solving for (r,μ) gives $r=i+m-p$ and $\mu=r-p=\ell+m$. By Theorem 3.1, it follows that $\mathrm{Level}(\mathrm{CH}^*(X,m;\mathbb{Q}))\ge\ell$ and we are done. ■

Next, let $V/\overline{\mathbb{Q}}$ be a smooth quasi-projective variety defined over $\overline{\mathbb{Q}}$. Based on a Bloch–Beilinson conjecture assumption in the case $m=0$, it is conjectured in [5] that:

Conjecture 3.3 *The Abel–Jacobi map*

$$\Phi_{r,m}:\mathrm{CH}^r_{\mathrm{hom}}(V/\overline{\mathbb{Q}},m;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H^{2r-m-1}(V/\mathbb{C},\mathbb{Q}(r)))$$

is injective.

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Remark 3.4 One can show [5] using a weight filtered spectral sequence together with the Hodge conjecture, that one need to only verify Conjecture 3.3 for smooth projective $V/\overline{\mathbb{Q}}$.

Now we need to establish the reverse inequality, viz.,

Theorem 3.5 *Let us assume the GHC and Conjecture 3.3. Then, for any integer $m \geq 0$,*

$$\mathrm{Level}(H^*(X,\mathbb{C})) \geq \mathrm{Level}(\mathrm{CH}^*(X,m;\mathbb{Q})).$$

Proof Using [1] together with the GHC and Conjecture 3.3, there is for any smooth projective variety $X/\mathbb{C}$ a descending filtration $\{F^\nu \mathrm{CH}^r(X,m;\mathbb{Q})\}_{\nu=0}^r$ with:

$\bullet_0$ $F^0 = \mathrm{CH}^r(X,m;\mathbb{Q})$.

$\bullet_1$ $F^1 = \mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q})$ for $m=0$, $F^0 = F^1 = \mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q})$ for $m \geq 1$.

$\bullet_2$ $F^2 \subseteq \ker \Phi_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(X/\mathbb{C},m;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(X,\mathbb{Q}(r)))$.

$\bullet_3$ $F^{\nu_1}\mathrm{CH}^{r_1}(X,m_1;\mathbb{Q}) \smile F^{\nu_2}\mathrm{CH}^{r_2}(X,m_2;\mathbb{Q}) \subset F^{\nu_1+\nu_2}\mathrm{CH}^{r_1+r_2}(X,m_1+m_2;\mathbb{Q})$.[4]

$\bullet_4$ $\{F^\nu\}_{\nu\geq 0}$ is functorial with respect to correspondences between smooth projective varieties.

$\bullet_5$ Factorization through the Grothendieck motive: Namely, if

$$[\Delta_X] = \bigoplus_{p+q=2n} [\Delta_X(p,q)] \in H^{2n}(X\times X,\mathbb{Q})$$

is the Künneth decomposition with $[\Delta_X(p,q)] \in H^p(X,\mathbb{Q}) \otimes H^q(X,\mathbb{Q})$ algebraic, then $[\Delta_X(p,q)]$ acts on the graded pieces $Gr_F^\nu \mathrm{CH}^r(X,m;\mathbb{Q})$ and in particular

$$[\Delta_X(2n-2r+\nu+m, 2r-\nu-m)]_* Gr_F^k \mathrm{CH}^r(X,m;\mathbb{Q}) = \delta_{\nu,k}\cdot \mathrm{Id}.$$

[Explanation. Any correspondence induced by $Z \in \mathrm{CH}^\bullet(X\times X;\mathbb{Q})$ on $Gr_F^\nu \mathrm{CH}^r(X.m;\mathbb{Q})$ depends only on the cohomology class $[Z] \in H^{2\bullet}(X\times X,\mathbb{Q})$. This is immediate from the previous $\bullet$'s.]

$\bullet_6$ $F^{r+1} = 0$.

Note that if we view $\mathrm{CH}^n(X\times X;\mathbb{Q})$ as the ring of correspondences on X, under composition, then by the above $\bullet$'s, $F^1\mathrm{CH}^n(X\times X;\mathbb{Q})$ is a nilpotent two-sided ideal of $\mathrm{CH}^n(X\times X;\mathbb{Q})$. One has a Künneth decomposition of idempotents:

$$[\Delta_X] = \bigoplus_{p+q=2n} [\Delta_X(p,q)] \in \frac{\mathrm{CH}^n(X\times X;\mathbb{Q})}{F^1\mathrm{CH}^n(X\times X;\mathbb{Q})} \subset H^{2n}(X\times X,\mathbb{Q}).$$

As Beilinson observed, such a decomposition lifts to an idempotent decomposition

$$\Delta_X = \bigoplus_{p+q=2n} \Delta_X(p,q) \in \mathrm{CH}^n(X\times X;\mathbb{Q}),$$

called a Chow–Künneth decomposition in the sense of [12]. Let $\ell = \mathrm{Level}(H^*(X,\mathbb{C}))$. One can easily argue that $\ell \leq n$. Note that

$$\mathrm{CH}^r(X,m;\mathbb{Q}) = \bigoplus_{p+q=2n} \Delta_X(p,q)_*\mathrm{CH}^r(X,m;\mathbb{Q}).$$

[4] Not specifically stated in [1], but proved in [4].

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It will suffice, via Proposition 3.6, to determine the level of $\Delta_X(p,q)_*\mathrm{CH}^r(X,m;\mathbb{Q})$, for a fixed algebraic cohomological representative $\Delta_X(p,q)$ of $[\Delta_X(p,q)]$.

Case $p \le \ell$. Then $q = 2n-p \ge n$. By the hard Lefschetz theorem, $H^q(X,\mathbb{Q}) = L_X^{n-p}H^p(X,\mathbb{Q})$; hence, we may assume that the support $|\Delta_X(p,q)| \subset X \times V$, where $\mathrm{codim}_X V = n-p$. It follows that

$$\mathrm{Level}(\Delta_X(p,q)_*\mathrm{CH}^r(X,m;\mathbb{Q})) \le (r-n-m)+p \le p \le \ell,$$

where by dimension reasons alone we use the fact that $\mathrm{CH}^{r>n+m}(X,m) = 0$.

Case $\ell < p \le n$. Thus, $p = \ell + k$, where $k \ge 1$. Note that $\mathrm{Level}(H^p(X,\mathbb{Q})) =: \ell' \le \ell$. Replacing ℓ by ℓ', we can write $p = \ell' + k$. From Hodge theory, k has to be even. Specifically, $\ell' = p - 2M$, where M is maximal subject to $F^M \cap H^p(X) = H^p(X)$. Thus, $k = 2M$. It follows that $H^p(X,\mathbb{Q}) = F^{k/2} \cap H^p(X,\mathbb{Q}) = N^{k/2}H^p(X,\mathbb{Q})$, where the latter equality is due to the GHC. Thus, we can assume that $|\Delta_X(p,q)| \subset Y \times V$, where $\mathrm{codim}_X V = n-p+k/2$ and $\mathrm{codim}_X Y = k/2$. However, $\mathrm{CH}^r(Y,m;\mathbb{Q}) = 0$ for $r > n-k/2+m$. Accordingly, $r \le (n-k/2)+m$. Thus, $\mu \le p-k = \ell' \le \ell$.

Case $p > n$. Thus, $q < n$ and again by hard Lefschetz $L_X^{n-q}H^q(X,\mathbb{Q}) = H^p(X,\mathbb{Q})$. Let us first assume that $q \le \ell \le n$. Then $|\Delta_X(p,q)| \subset V \times X$, where $\mathrm{codim}_X V = n-q$. However, $\mathrm{CH}^r(V,m;\mathbb{Q}) = 0$ for $r > q+m$. Thus, we can assume that $r \le q+m$. We already know that from the definition of level, $r-\mu-m \ge 0$; hence,

$$\mu \le r-m \le (q+m)-m = q \le \ell.$$

Finally, we now assume that $\ell < q < n$. Thus, by the GHC,

$$H^q(X,\mathbb{Q}) = N^{k/2}H^q(X,\mathbb{Q}),$$

where as in the previous case we have $k > 0$ is an even integer. Thus,

$$|\Delta_X(p,q)| \subset V \times Y,$$

where $\mathrm{codim}_X V = n-q+k/2$ (because $L_X^{n-q}H^q(X,\mathbb{Q}) = H^p(X,\mathbb{Q})$) and $\mathrm{codim}_X Y = k/2$. Thus, $\mathrm{CH}^r(V,m;\mathbb{Q}) = 0$ for $r > q+m-k/2$. Therefore, we can assume that $r \le q+m-k/2$. From the definition of level, we have $k/2 \le r-\mu-m$. Hence,

$$\mu \le r-m-k/2 \le q-k = \ell. \qquad \blacksquare$$

Now for the details on how to reduce to the graded pieces $Gr_F^{\bullet}\mathrm{CH}^r(X,m;\mathbb{Q})$. What we have is the following. Consider another Künneth decomposition

$$[\Delta] = \bigoplus_{p+q=2n}[\Delta(p,q)'],$$

where $\Delta(p,q)'$ are algebraic. Thus,

$$\Delta' := \oplus_{p+q=2n}\Delta(p,q)' \sim_{\mathrm{hom}} \Delta.$$

We are reduced to the following proposition.

Proposition 3.6 *Let* $\Xi \in \mathrm{CH}^n_{\mathrm{hom}}(X\times X;\mathbb{Q})$. *Then,*

$$\mathrm{Level}\big(\Xi_*\mathrm{CH}^r(X,m;\mathbb{Q})\big) \le \ell.$$

Restatement. For any cycle $\xi \in \mathrm{CH}^n(X \times X; \mathbb{Q})$, the statement $\mathrm{Level}(\xi_* \mathrm{CH}^\bullet(X, m; \mathbb{Q})) \leq \ell$ depends only on the cohomology class of ξ.

Proof Let $F^\nu := F^\nu \mathrm{CH}^r(X, m; \mathbb{Q})$. Note that $F^r = Gr_F^r \mathrm{CH}^r(X, m; \mathbb{Q})$. Since $\mathrm{CH}^n_{\mathrm{hom}}(X \times X; \mathbb{Q}) = F^1 \mathrm{CH}^n(X \times X; \mathbb{Q})$, then $\Xi_* F^r = 0$. Consider the short exact sequence

$$0 \to F^r \to F^{r-1} \to Gr_F^{r-1} \to 0.$$

Then $\Xi_* F^{r-1} \mapsto 0 \in Gr_F^{r-1}$; hence, $\Xi_* F^{r-1} \subset F^r$, where we know that $\mathrm{Level}(F^r) \leq \ell$. Proceeding by downward induction, consider the short exact sequence

$$0 \to F^\nu \to F^{\nu-1} \to Gr_F^\nu \to 0.$$

By induction, $\mathrm{Level}(F^\nu) \leq \ell$ and $\Xi_* F^{\nu-1} \mapsto 0 \in Gr_F^{\nu-1}$. Thus, $\mathrm{Level}(\Xi_* F^{\nu-1}) \leq \ell$, and this completes the induction step. ∎

4 An explicit example

Let $X/\mathbb{C} \subset \mathbb{P}^{n+r}$ be a general complete intersection of multidegree $(d_1, \ldots, d_r)$, let $\Omega_X(k) := \{\mathbb{P}^k\text{'s} \subset X\}$ be its Fano variety of k-planes, and suppose $X = Z \cap \mathbb{P}^{n+r}$, where $Z \subset \mathbb{P}^{n+r+1}$ is a general complete intersection of multidegree $(d_1, \ldots, d_r)$. The geometric properties of such complete intersections are explored elsewhere ([10], [8, Chapter 13]). In particular, $\Omega_X(k)$ is smooth. Set $\delta := (k+1)(n+r-k) - \sum_{j=1}^r \binom{d_j+k}{k}$, $l = k(n+1+r-k) + r - \sum_{j=1}^r \binom{d_j+k}{k}$ and assume $\delta \geq n - 2k \geq 0$. Observe that $\delta = n - 2k + l$. Indeed, $\delta = \dim \Omega_X(k)$ and through a generic point of Z passes an l-dimensional family of $\mathbb{P}^k$'s. Let Ω_Z be the subvariety of $\Omega_Z(k)$ obtained by the intersection of l general hyperplane sections. By Bertini's theorem, and a dimension count, Ω_Z is smooth and irreducible of dimension $n+1-k$. If we set $\Omega_X = \Omega_Z \cap \Omega_X(k)$, then Ω_X is smooth of pure dimension $n - 2k$. There is the following diagram [10]:

$$\begin{tikzcd}
P(X) \arrow[rr, "\pi_X"] \arrow[dr, "j"] \arrow[ddd, "\rho_X"'] & & X \arrow[dr, hook, "i"] & \\
& \tilde{X} \arrow[ur, "\pi"] \arrow[dr, "j_1"] \arrow[dd, "\rho"] & & Z \\
& & P(Z) \arrow[ur, "\pi_Z"] \arrow[dl, "\rho_Z"] & \\
\Omega_X \arrow[r, hook, "i_0"] & \Omega_Z & &
\end{tikzcd}, \tag{1}$$

where $P(X)$ and $P(Z)$ are $\mathbb{P}^k$-bundles, $\tilde{X} = \pi_Z^{-1}(X)$, and all the maps depicted are natural projections, except i_0, i, j which are inclusions.

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Proposition 4.1 [8, Corollary 13.42] *Assume that $\delta \geq n - 2k \geq 0$. Then the cylinder homomorphism*

$$\Phi_* := \pi_{X,*} \circ \rho_X^* : H^{n-2k}(\Omega_X, \mathbb{Q}) \to H^n(X, \mathbb{Q})$$

is surjective.

Set $Y = \pi_X(P(X))$. Then $Y \subset X$ is a subvariety of codimension k. We conclude the following.

Corollary 4.2 *The natural map $H_n(Y, \mathbb{Q}) \to H_n(X, \mathbb{Q})$ is surjective. Hence, cycles on X are supported on a subvariety of codimension k.*

This corollary gives an instance of the GHC.

Corollary 4.3

$$F^k \cap H^n(X, \mathbb{Q}) = H^n(X, \mathbb{Q}) = N^k H^n(X, \mathbb{Q}).$$

Next, let us write $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$, for generic choice of $\{F_1, \ldots, F_r\}$, and $W = V(F_1, \ldots, F_{r-1}) \subset \mathbb{P}^{n+r}$, with inclusion $j : X \hookrightarrow W$.

Theorem 4.4 [10] *Assume that $\delta \geq n - 2k \geq 0$. Then the cylinder map*

$$\Phi_{X,*} : \mathrm{CH}^{r-k}(\Omega_X, m; \mathbb{Q}) \to \mathrm{CH}^r(X, m; \mathbb{Q}) / j^* \mathrm{CH}^r(W, m; \mathbb{Q})$$

is surjective.

Now, observe that $r > n + m - k \Rightarrow r - k > n - 2k + m = \dim \Omega_X + m$; hence, $\mathrm{CH}^{r-k}(\Omega_X, m; \mathbb{Q}) = 0$ by dimension reasons alone. Thus, we can assume that $r \leq n + m - k$. Choose k such that $\mathrm{Level}(H^*(X, \mathbb{Q})) = n - 2k$ and assume that $\delta \geq n - 2k$. Furthermore, put

$$\nu := \mathrm{Level}\big(\mathrm{CH}^r(X, m; \mathbb{Q}) / j^* \mathrm{CH}^r(W, m; \mathbb{Q})\big).$$

By definition of ν, observe that $\nu \leq r - \mathrm{codim}_X Y - m = r - k - m$. However, $r \leq n + m - k$, and hence $\nu \leq n - 2k$. Finally, W is of a lower order (viz., multidegree) than X and one can argue via an inductive argument, that $\mathrm{Level}(j^* \mathrm{CH}^r(W, m; \mathbb{Q})) \leq n - 2k$. From Corollaries 3.2 and 4.3 and Theorem 3.1, we deduce the following corollary.

Corollary 4.5 *If X is a general complete intersection satisfying $\delta \geq n - 2k \geq 0$, where $k \geq 0$ is given such that $\mathrm{Level}(H^*(X, \mathbb{Q})) = n - 2k$, then*

$$\mathrm{Level}(H^*(X, \mathbb{C})) \geq \mathrm{Level}(\mathrm{CH}^*(X, m; \mathbb{Q})),$$

and we have an equality in the case $m < 3$.

Proof The only remaining issue involves replacing the assumption in Theorem 3.1(i) by Proposition 4.1. The details of this can be found in [7, Section 5]. ■

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