CENTRAL LIMIT THEOREMS FOR INTERCHANGEABLE PROCESSES

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1. Introduction and summary. Let $\{X_n\}$ $(n = 1, 2, \ldots)$ be a stochastic process. The random variables comprising it or the process itself will be said to be interchangeable if, for any choice of distinct positive integers $i_1, i_2, i_3, \ldots, i_k$, the joint distribution of

$$X_{i_1}, X_{i_2}, \ldots, X_{i_k}$$

depends merely on k and is independent of the integers $i_1, i_2, \ldots, i_k$. It was shown by De Finetti **(3)** that the probability measure for any interchangeable process is a mixture of probability measures of processes each consisting of independent and identically distributed random variables. More precisely, let $\mathfrak{F}$ be the class of one-dimensional distribution functions and for each pair of real numbers x and y let

$$\mathfrak{F}(x, y) = \{F \in \mathfrak{F} | F(x) \leqslant y\}.$$

Let $\mathfrak{A}$ be the Borel field of subsets of $\mathfrak{F}$ generated by the class of sets $\mathfrak{F}(x, y)$. Then De Finetti's theorem asserts that for any interchangeable process $\{X_n\}$ there exists a probability measure μ defined on $\mathfrak{A}$ such that

$$P\{B\} = \int_{\mathfrak{F}} P_F\{B\} d\mu\ (F) \tag{1.1}$$

for any Borel measurable set B defined on the sample space of the sequence $\{X_n\}$. Here $P\{B\}$ is the probability of the event B and $P_F\{B\}$ is the probability of the event B computed under the assumption that the random variables X_n are independently distributed with common distribution function F.

Note that for Borel measurable point functions f for which the functional

$$\int_{-\infty}^{\infty} f(x) dF(x)$$

is well-defined,

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is measurable in F. This follows from the fact that it is true for f's that are indicators of half-lines. We then have for any integrable function g on the sample space of the sequence $\{X_n\}$

$$E\{g\} = \int E_F\{g\} d\mu(F)$$

where $E_F\{g\}$ is the expectation of g computed under the assumption that the random variables $\{X_n\}$ are independently distributed with common distribution function F.

In this paper we shall deal only with interchangeable processes having finite first and second moments and consequently shall assume without loss of generality that all such processes have mean zero and variance one. Let $\{X_n\}$ be such a process and for each positive integer n define

$$S_n = \sum_{i=1}^{n} X_i.$$

We shall say that the Central Limit Theorem holds for the process $\{X_n\}$ if for every real number α we have

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = \phi(\alpha),$$

where

$$\phi(\alpha) = \frac{1}{\sqrt{[2\pi]}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} du.$$

In section two, necessary and sufficient conditions for the Central Limit Theorem to hold for an interchangeable process are derived. In section three, we discuss briefly the case of a doubly infinite sequence $\{X_{ni}\}$ where for each n the random variables X_{ni} are an interchangeable process. A number of conditions sufficient for the asymptotic normality of $S_n/\sqrt{n}$, where

$$S_n = \sum_{i=1}^{n} X_{ni}$$

are obtained.

2. The Single Process. Let X_n be an interchangeable process. According to (1.1) we have for every positive integer n

$$P\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = \int_{\mathfrak{F}} P_F\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} d\mu(F). \tag{2.1}$$

For each $F \in \mathfrak{F}$ define $m(F)$ and $\sigma(F)$ by

$$m(F) = \int_{-\infty}^{\infty} x dF(x) \text{ and } \sigma^2(F) = \int_{-\infty}^{\infty} [x - m(F)]^2 dF(x)$$

provided these integrals converge. For every real number m and non-negative number σ let $\mathfrak{F}_{m,\sigma}$ be the set of F for which $m(F) = m$ and $\sigma(F) = \sigma$. It can easily be shown that each such $\mathfrak{F}_{m,\sigma}$ is $\mathfrak{A}$-measurable. Now suppose $F \in \mathfrak{F}_{0,1}$.

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Then it follows from the Central Limit Theorem for a sequence of independent and identically distributed random variable that

$$\lim_{n\to\infty} P_F\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = \phi(\alpha).$$

Consequently, we see from (2.1) and the Lebesgue bounded convergence theorem that the process $\{X_n\}$ will satisfy the Central Limit Theorem if $\mu(\mathfrak{F}_{o,1}) = 1$. We shall show that this condition is also necessary. To do this, let $\mathfrak{F}'$ be the subset of $\mathfrak{F}$ for which $m(F)$ and $\sigma(F)$ exist and are finite. Again, it is easily seen that $\mathfrak{F}'$ is $\mathfrak{A}$-measurable and from the existence of the first and second moments of the process $\{X_n\}$ it follows that $\mu(\mathfrak{F}') = 1$. An easy computation shows that

$$\lim_{n\to\infty} P_F\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\}$$

exists for each $F \in \mathfrak{F}'$ and depends only on $m(F)$, $\sigma(F)$, and α. If we denote the limiting function by $f[m(F), \sigma(F), \alpha]$ we find

$$f[m(F), \sigma(F), \alpha] = \begin{cases} 0 \text{ if } m(F) > 0, \\ 1 \text{ if } m(F) < 0, \\ 0 \text{ if } m(F) = 0, \sigma(F) = 0, \alpha < 0, \\ 1 \text{ if } m(F) = 0, \sigma(F) = 0, \alpha \geqslant 0, \\ \phi\left(\dfrac{\alpha}{\sigma(F)}\right) \text{ if } m(F) = 0, \sigma(F) > 0. \end{cases} \tag{2.2}$$

Also, let $\mathfrak{F}_o$ be the set of $F \in \mathfrak{F}$ for which $m(F) = 0$ and $\mathfrak{F}_{o,+}$ $(\mathfrak{F}_{o,o})$ be the set of $F \in \mathfrak{F}_o$ for which $\sigma(F) > 0$, $(\sigma(F) = 0)$. Now, if we again employ (2.1) and the Lebesgue bounded convergence theorem we find that if the Central Limit Theorem is to hold we must have

$$\phi(\alpha) = \int_{\mathfrak{F}'} f[m(F), \sigma(F), \alpha]\, d\mu(F). \tag{2.3}$$

Let $\alpha > 0$. If we use (2.2), (2.3) and the fact that $\phi(-a) = 1 - \phi(\alpha)$ we find that

$$2\phi(\alpha) - 1 = \mu(\mathfrak{F}_{o,o}) + \int_{\mathfrak{F}_{o,+}} \left(2\phi\left[\frac{\alpha}{\sigma(F)}\right] - 1\right) d\mu(F). \tag{2.4}$$

On letting α approach infinity in (2.4) we have $\mu(\mathfrak{F}_o) = 1$.

Now let $G(\sigma)$ be the distribution function on the real line defined by

$$G(\sigma) = \begin{cases} 0 & \text{for } \sigma < 0 \\ \mu(F|m(F) = 0, \sigma(F) \leqslant \sigma) & \text{for } \sigma \geqslant 0. \end{cases}$$

Then we may write (2.3) in the form

$$\phi(\alpha) = \int_o^\infty f[o, \sigma, \alpha] dG(\sigma). \tag{2.5}$$

If we put $\alpha = 0$ in (2.5) and use (2.2) we see that $G(0) = 0$. Thus we have

$$\phi(\alpha) = \int_{o+}^{\infty} \phi\left(\frac{\alpha}{\sigma}\right) dG(\sigma) = \frac{1}{\sqrt{[2\pi]}} \int_{o}^{\infty} \int_{-\infty}^{\alpha/\sigma} e^{-\frac{1}{2}u^2}\, du dG(\sigma). \tag{2.6}$$

Differentiating both sides of (2.6) with respect to α and setting $\alpha = 1$, we have

$$e^{-\frac{1}{2}} = \int_{o}^{\infty} \frac{1}{\sigma} e^{-\frac{1}{2}\sigma^2}\, dG(\sigma). \tag{2.7}$$

But the integrand of the right hand side of (2.7) achieves a unique maximum at $\sigma = 1$ where its value is $e^{-\frac{1}{2}}$. Thus, we see that (2.7) holds if and only if $G(\sigma)$ has all of its mass concentrated at the point $\sigma = 1$.

We summarize in

LEMMA 1. *If $\{X_n\}$ is an interchangeable process with mean zero and variance one the Central Limit Theorem holds if and only if* $\mu(\mathfrak{F}_{o,1}) = 1$.

The condition of the lemma is not very practical since in general it is rather difficult to compute the measure μ associated with a given interchangeable process $\{X_n\}$. However, we shall show that the condition of Lemma 1 is equivalent to a simple condition on the moments of the process. Suppose then that the condition of Lemma 1 holds. Then we have for $i \neq j$

$$\begin{aligned} E\{X_i X_j\} &= \int_{\mathfrak{F}_{o,1}} m^2(F) d\mu(F) = 0, \\ E\{X_i^2 X_j^2\} &= \int_{\mathfrak{F}_{o,1}} [E_F\{X^2\}]^2 d\mu(F) = 1. \end{aligned} \tag{2.8}$$

Conversely, suppose (2.8) holds for $i \neq j$. Then $E\{[X_i^2 - 1][X_j^2 - 1]\} = 0$ and we obtain

$$\begin{aligned} &\int_{\mathfrak{F}} m^2(F) d\mu(F) = 0, \\ &\int_{\mathfrak{F}} [E_F\{X^2 - 1\}]^2 d\mu(F) = 0. \end{aligned} \tag{2.9}$$

But clearly (2.9) implies that $\mu(\mathfrak{F}_{o,1}) = 1$. Thus, we have

THEOREM 1. *Let $\{X_n\}$ be an interchangeable process with mean zero and variance one. Then the Central Limit Theorem holds for the process if and only if for $i \neq j$*

$$E\{X_i X_j\} = 0 \quad \text{and} \quad E\{[X_i^2 - 1][X_j^2 - 1]\} = 0.$$

We can rephrase the conditions of the theorem by saying that X_i and X_j as well as X_i^2 and X_j^2 must have covariance zero (be uncorrelated) for $i \neq j$.

Several remarks of interest can be made concerning such processes. In the first place, let $\{X_n\}$ be a sequence of independent and identically distributed random variables with mean m, variance σ^2, and finite third moment. Then, it was shown by Berry **(1)** and Esseen **(4)** that

(2.10) $$\left| P\left\{ \frac{S_n - nm}{\sqrt{[n\sigma^2]}} \leqslant \alpha \right\} - \phi(\alpha) \right| \leqslant \frac{c}{\sqrt{n}} \frac{E\{|X - m|^3\}}{\sigma^3}$$

where c is a universal constant. It is simple to verify that if the Central Limit Theorem holds for an interchangeable process $\{X_n\}$ with finite third moment, then the Berry-Esseen bound still applies.

Secondly, consider an interchangeable process which is generated by a mixture over a family of one-dimensional distributions with the property that each distribution in the family is completely determined by specifying its mean and variance. The Normal distributions, the Poisson distributions and the Binomial distributions furnish examples of such families. But in such a case it follows easily from Lemma 1 that if the Central Limit Theorem holds, the mixture must be concentrated at a single distribution of the family. Consequently, we find that if the Central Limit Theorem holds for such a process, the random variables must be independent and identically distributed.

We observe that if $\{X_n\}$ $(n = 1, 2, \ldots)$ is an interchangeable process with $EX_1 = 0$, $EX_1^2 = 1$, $EX_1X_2 = \rho$ (necessarily non-negative) and such that every finite subset has a joint normal distribution,

$$\sum_{i=1}^{n} X_i$$

will be normal with mean 0 and variance $n + n(n-1)\rho$. From Theorem 1 the normalization $1/\sqrt{n}$ will suffice only if $\rho = 0$ (the X_n are independent). However, if $\rho > 0$,

$$\frac{1}{n}\sum_{i=1}^{n} X_i$$

has a limiting normal distribution with mean 0 and variance ρ.

Finally, we consider again a sequence $\{X_n\}$ of independent and identically distributed random variables. Then, if $f(x)$ is any bounded measurable function and the sequence $\{Y_n\}$ is defined by $Y_n = f(X_n)$ it follows that the process $\{Y_n\}$ satisfies the Central Limit Theorem. However this is not, in general, true for interchangeable processes. For suppose $f(X)$ and $g(X)$ are bounded measurable functions and let $h(X) = f(X) + g(X)$. Then, if the Central Limit Theorem is to hold for the process $h(X_n)$ we find from Theorem 1 that we must have

(2.11) $$E\{h(X_i)\, h(X_j)\} = [E\{h(X)\}]^2, \qquad i \neq j.$$

Now, from the interchangeability of the process $\{X_n\}$ it follows that

$$E\{f(X_i)\, g(X_j)\} = E\{f(X_j)\, g(X_i)\}$$

for all i and j. Using this we can expand the left side of (2.11) to obtain that

$$E\{f(X_i)\, g(X_j)\} = E\{f(X_i)\}\, E\{g(X_j)\}.$$

Since f and g are arbitrary, it follows that the random variables X_n are independently distributed.

3. Sequences of interchangeable processes. For each positive integer n, let $\{X_{ni}, i = 1, 2, \ldots, \}$ be an interchangeable process with mean zero, variance one, and finite absolute third moment. If we let μ_n denote the measure on $\mathfrak{A}$ occurring in the representation (1.1), it is clear that we must have $\mu_n\ (F|E_F\{|X|^3\} < \infty) = 1$ for every positive integer n.

By techniques paralleling those employed in the previous section we obtain

LEMMA 2. *Suppose that for every $\epsilon > 0$,*

(i) $$\lim_{n\to\infty} \mu_n\left(F|\ |m(F)| < \frac{\epsilon}{\sqrt{n}}\right) = 1,$$

(ii) $$\lim_{n\to\infty} \mu_n(F|\ |\sigma(F) - 1| < \epsilon) = 1,$$

(iii) $$\lim_{n\to\infty} \mu_n\left(F|\ \frac{E_F\{|X - m(F)|^3\}}{\sigma^3(F)} < \epsilon\sqrt{n}\right) = 1.$$

Then for every real number α

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = \phi(\alpha).$$

For each integer n let $E_n\{\quad\}$ stand for the expectation of the quantity between the brackets computed with respect to the distribution of the nth process. With this notation we have

THEOREM 2. *For each positive integer n let $\{X_{ni}; i = 1, 2, \ldots\}$ be an interchangeable process with mean zero, variance one, and finite absolute third moment. If*

(i) $$E_n\{X_{n1}X_{n2}\} = o\left(\frac{1}{n}\right),$$

(ii) $$\lim_{n\to\infty} E_n\{X_{n1}^2X_{n2}^2\} = 1,$$

(iii) $$E_n\{|X_{n1}|^3\} = o(\sqrt{n}),$$

then for every real number α

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = \phi(\alpha).$$

The theorem is obtained by showing that conditions (i), (ii), and (iii) of Lemma 2 are satisfied.

The conditions of Theorem 2 are not necessary. However, it is of some interest to remark that in a certain sense these conditions are the best of their kind. Given any one of the three conditions, one can find an interchangeable process with mean zero, unit variance, and finite third moment which narrowly violates the condition, satisfies the remaining two conditions and for which the Central Limit Theorem is not valid.

A somewhat different limit theorem can be obtained in the following way. For each positive integer n let $\{X_{ni}\}$ be an interchangeable process with mean zero, unit variance, and mixing measure μ_n. Define the distribution function $F_n(m)$ by

$$F_n(m) = \mu_n\{F | \sqrt{n}\, m(F) \leqslant m\} \text{ for } n = 1, 2, \ldots .$$

Let $F(m)$ be an arbitrary distribution. Then we have

THEOREM 3. *If*

(i) $$\lim_{n\to\infty} F_n(m) = F(m)$$

at every continuity point m of F,

(ii) $$\lim_{n\to\infty} E_n\{X_{n,1}X_{n,2}\} = 1,$$

and

(iii) $$E_n\{|X_{n,1}|^3\} = o(\sqrt{n}),$$

then

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = \int_{-\infty}^{\infty} \phi(\alpha - m)dF(m), \qquad -\infty < \alpha < \infty.$$

The result is obtained by making use of the Helly-Bray theorem.

We note that the limiting distribution obtained in Theorem 3 is the convolution of ϕ and F. Consequently, it may be regarded as the distribution of the sum of two independent random variables, one of which is normal with mean zero and variance one and the other with distribution F. It follows from a theorem of Cramer (2) that the limit distribution is normal if and only if $F(m)$ is a normal distribution. Now, let $N_{a,b}(\alpha)$ be the normal distribution with mean a and variance b, that is,

$$N_{a,b}(\alpha) = \frac{1}{\sqrt{[2\pi b]}} \int_{-\infty}^{\alpha} e^{-(x-a)^2/2b}\, dx.$$

Further, for $k = 1, 2, \ldots$, let a_k denote the kth moment of

$$N_{a_1,b}(\alpha).$$

Then we can give a criterion for normality of the limiting distribution in terms of moment conditions on the process as follows:

COROLLARY. *If condition* (i) *in Theorem* 3 *is replaced by the condition*

$$\lim_{n\to\infty} E_n\{n^{k/2}X_{n1}X_{n2}\ldots X_{nk}\} = a_k \qquad k = 1, 2, \ldots$$

then

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \leqslant \alpha\right\} = N_{a_1,1+b}(\alpha).$$

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References

1. A. C. Berry, *The accuracy of the Gaussian approximation to the sum of independent variates,* Trans. Amer. Math. Soc., *49* (1941), 122–136.
2. Harold Cramer, *Random variables and probability distributions,* Cambridge Tracts in Mathematics, *136* (Cambridge, 1937).
3. Bruno De Finetti, *La prévision, ses lois logiques, ses sources subjectives,* Annales de l'Institut Henri Poincaré, *7* (1937), 1–68.
4. Carl-Gustav Esseen, *Fourier analysis of distribution functions,* Acta Math., *77* (1944), 1–125.

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