J. Aust. Math. Soc. **85** (2008), 377–417
doi:10.1017/S1446788708000657

WEIGHTED ESTIMATES FOR SINGULAR INTEGRAL OPERATORS WITH NONSMOOTH KERNELS AND APPLICATIONS

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(Received 19 November 2006; revised 18 February 2007)

Communicated by J. J. Koliha

Abstract

Let $\mathcal{X}$ be a space of homogeneous type in the sense of Coifman and Weiss. In this paper, two weighted estimates related to A_∞ weights are established for singular integral operators with nonsmooth kernels via a new sharp maximal operator associated with a generalized approximation to the identity. As applications, the weighted $L^p(\mathcal{X})$ and weighted endpoint estimates with general weights are obtained for singular integral operators with nonsmooth kernels, their commutators with BMO $(\mathcal{X})$ functions, and associated maximal operators. Some applications to holomorphic functional calculi of elliptic operators and Schrödinger operators are also presented.

2000 *Mathematics subject classification*: primary 42B20; secondary 47B47, 42B25, 47A30, 43A99.

Keywords and phrases: space of homogeneous type, approximation to the identity, weight, norm inequality, singular integral operator, maximal operator, nonsmooth kernel.

1. Introduction

Let $\mathcal{X}$ be a set endowed with a positive Borel regular measure μ and a quasi-metric d satisfying that there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in \mathcal{X}$,

$$d(x, y) \leq \kappa[d(x, z) + d(z, y)].$$

The triple $(\mathcal{X}, d, \mu)$ is said to be a *space of homogeneous type* in the sense of Coifman and Weiss [3], if μ satisfies the following *doubling condition*: there exists a constant $C \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty;$$

here and in what follows $B(x, r) = \{y \in \mathcal{X} : d(y, x) < r\}$. It is easy to see that the above doubling property implies the following *strong homogeneity property*: there exist positive constants c_0 and n such that for all $\lambda \geq 1$, $r > 0$ and $x \in \mathcal{X}$,

The first author was supported by the NNSF (No. 10671210) of China and the second (corresponding) author was supported by National Science Foundation for Distinguished Young Scholars (No. 10425106) and NCET (No. 04-0142) of China.

$$\mu(B(x, \lambda r)) \leq c_0 \lambda^n \mu(B(x, r)). \tag{1.1}$$

Moreover, there also exist constants $C > 0$ and $N \in [0, n]$ such that for all $x, y \in \mathcal{X}$ and $r > 0$,

$$\mu(B(y, r)) \leq C\left(1 + \frac{d(x, y)}{r}\right)^N \mu(B(x, r)). \tag{1.2}$$

We remark that although that all balls defined by d satisfy the axioms of complete system of neighborhoods in $\mathcal{X}$, and therefore induced a (separated) topology in $\mathcal{X}$, the balls $B(x, r)$ for $x \in \mathcal{X}$ and $r > 0$ need not be open with respect to this topology. However, by a well-known result of Macías and Segovia [8], we know that there exists another quasi-metric $\widetilde{d}$ which is equivalent to d such that the balls corresponding to $\widetilde{d}$ are open in the topology induced by $\widetilde{d}$. Thus, throughout this paper, we always assume that the *balls $B(x, r)$ for $x \in \mathcal{X}$ and $r > 0$ are open*.

Let T be an $L^2(\mathcal{X})$ *bounded linear operator with kernel K* in the sense that for all bounded functions f with bounded support and almost all $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathcal{X}} K(x, y) f(y)\, d\mu(y), \tag{1.3}$$

where K is a measurable function on $\mathcal{X} \times \mathcal{X} \setminus \{(x, y) : x = y\}$. To obtain a weak (1, 1) estimate for certain Riesz transforms, and L^p-boundedness with $p \in (1, \infty)$ of holomorphic functional calculi of linear elliptic operators on irregular domains, Duong and McIntosh [4] introduced singular integral operators with nonsmooth kernels on spaces of homogeneous type via the following generalized approximation to the identity.

DEFINITION 1.1. A family of operators $\{D_t\}_{t>0}$ is said to be an *approximation to the identity*, if for every $t > 0$, D_t can be represented by the kernel a_t in the following sense: for every function $u \in L^p(\mathcal{X})$ with $p \in [1, \infty]$ and almost everywhere $x \in \mathcal{X}$,

$$D_t u(x) = \int_{\mathcal{X}} a_t(x, y) u(y)\, d\mu(y),$$

and the kernel a_t satisfies that for all $x, y \in \mathcal{X}$ and $t > 0$,

$$|a_t(x, y)| \leq h_t(x, y) = \frac{1}{\mu(B(x, t^{1/m}))} s(d(x, y)^m t^{-1}), \tag{1.4}$$

where $m > 0$ is a constant and s is a positive, bounded and decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\delta} s(r^m) = 0 \tag{1.5}$$

for some $\delta > N$ appearing in (1.2).

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Duong and McIntosh [4] proved that if T is an $L^2(\mathcal{X})$-bounded linear operator with kernel K, and satisfies that:

(i) there exists an approximation to the identity $\{D_t\}_{t>0}$ such that the composite operator TD_t with $t>0$ has an associated kernel K_t in the sense (1.3), and there exist positive constants c_1 and C such that for all $y\in\mathcal{X}$ and $t>0$,

$$\int_{d(x,y)\ge c_1 t^{1/m}} |K(x,y)-K_t(x,y)|\,d\mu(x)\le C;$$

then T is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$, that is, there exists a constant $C>0$ such that for any $f\in L^1(\mathcal{X})$ and any $\lambda>0$,

$$\mu(\{x\in\mathcal{X}:\,|Tf(x)|>\lambda\})\le C\lambda^{-1}\|f\|_{L^1(\mathcal{X})}.$$

An $L^2(\mathcal{X})$-bounded linear operator with kernel K satisfying (i) is called a *singular integral operator with nonsmooth kernel*, since K does not enjoy smoothness in space variables. Martell [9] considered the weighted $L^p(\mathcal{X})$ estimate with A_p weights for $p\in(1,\infty)$ and weighted $L^{1,\infty}(\mathcal{X})$ estimates with A_1 weights for T. Here and in what follows, A_p with $p\in[1,\infty]$ is the *weight function class of Muckenhoupt on* $\mathcal{X}$; see, for example, [20] (or [7]) for its definition and properties. To be precise, Martell [9] proved that if T is an $L^2(\mathcal{X})$-bounded linear operator, satisfies (i) and:

(ii) there exists an approximation to the identity $\{\widetilde{D}_t\}_{t>0}$ such that the composite operator $\widetilde{D}_tT$ with $t>0$ has an associated kernel K^t, and there exist positive constants c_2, C and α such that for all $t>0$ and $x,\ y\in\mathcal{X}$ with $d(x,y)\ge c_2t^{1/m}$,

$$|K(x,y)-K^t(x,y)|\le C\frac{1}{\mu(B(x,d(x,y)))}\frac{t^{\alpha/m}}{[d(x,y)]^\alpha};$$

then for any $p\in(1,\infty)$ and $u\in A_p$, T is bounded on $L^p(\mathcal{X},u)$. Moreover, Martell [9] proved that if T is an $L^2(\mathcal{X})$-bounded linear operator, satisfies (ii) and:

(iii) there exists an approximation to the identity $\{D_t\}_{t>0}$ such that the composite operator TD_t with $t>0$ has an associated kernel K_t in the sense (1.3), and there exist positive constants C, c_3 and β such that for all $t>0$ and $x,\ y\in\mathcal{X}$ with $d(x,y)\ge c_3t^{1/m}$,

$$|K(x,y)-K_t(x,y)|\le C\frac{1}{\mu(B(y,d(x,y)))}\frac{t^{\beta/m}}{[d(x,y)]^\beta};$$

then for $u\in A_1$, T is bounded from $L^1(\mathcal{X},u)$ to $L^{1,\infty}(\mathcal{X},u)$. Here and in what follows, $L^p(\mathcal{X},u)$ *means* $L^p(\mathcal{X},u\,d\mu)$.

Now let $u\in A_\infty$ and T be an $L^2(\mathcal{X})$-bounded linear operator satisfying (i) and (ii). It was proved by Martell [9] that for any $p\in(0,\infty)$ and $u\in A_\infty$ and bounded function f with bounded support,

$$\int_{\mathcal{X}}(M(Tf)(x))^p u(x)\,d\mu(x)\le C\int_{\mathcal{X}}(M(|f|^r)(x))^{p/r}u(x)\,d\mu(x), \tag{1.6}$$

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where $r \in (1, \infty)$, $C > 0$ is a constant depending only on $\{D_t\}_{t>0}$, $\{\widetilde{D}_t\}_{t>0}$, r and the weight u. This weighted estimate plays an important role in establishing weighted L^p estimates with A_p weights for T, where $p \in (1, \infty)$. However, as was shown in [10, 11, 13] on Euclidean spaces, to prove weighted estimates with general weights for singular integral operators and their commutators with BMO $(\mathbb{R}^n)$ functions, the inequality (1.6) is not enough. In this paper, we establish two weighted estimates with A_∞ weights, which are more general than (1.6) and are useful in establishing weighted estimates with general weights for singular integral operators and their commutators with BMO $(\mathcal{X})$ functions. To state our results, we first introduce some notation.

A measurable function w is said to be a *weight* if it is nonnegative and locally integrable on $\mathcal{X}$, and a *weight w on $\mathcal{X}$ is said to belong to A_∞* if there exist two positive constants $C_{A_\infty}(w)$ and $\delta_{A_\infty}(w)$ such that for any ball B and any measurable set $E \subset B$,

$$\frac{w(E)}{w(B)} \le C_{A_\infty}(w)\left(\frac{\mu(E)}{\mu(B)}\right)^{\delta_{A_\infty}(w)};$$

here and in what follows, $w(E) = \int_E w(x)\, d\mu(x)$. Let M be the *classical Hardy–Littlewood maximal operator on $\mathcal{X}$* and M^k with $k \in \mathbb{N}$ be the *operator M iterated k times*. Our main results can be stated as follows.

THEOREM 1.2. *Let T be an $L^2(\mathcal{X})$-bounded linear operator with kernel K as in (1.3). Suppose that T satisfies (i) and (ii) above. Then for any $k \in \mathbb{N}$:*

(i) *if $p \in (0, \infty)$ and $u \in A_\infty$, then there exists a constant $C > 0$ depending only on k, p, $C_{A_\infty}(u)$ and $\delta_{A_\infty}(u)$ such that for any bounded function f with bounded support,*

$$\int_{\mathcal{X}} (M^k(Tf)(x))^p u(x)\, d\mu(x) \le C \int_{\mathcal{X}} (M^{k+1} f(x))^p u(x)\, d\mu(x), \tag{1.7}$$

provided that $\|M^k(Tf)\|_{L^p(\mathcal{X},u)} < \infty$;

(ii) *if $l \in \mathbb{N}$ and w is a weight such that $M^l w$ is finite almost everywhere, then for any $p \in (1, \infty)$ and $\varrho > 0$, there exists a constant $C > 0$ depending only on k, p and ϱ such that for any bounded function f with bounded support,*

$$\begin{aligned}\int_{\mathcal{X}} &(M^k(Tf)(x))^p (M^l w(x))^{-\varrho}\, d\mu(x) \\ &\le C \int_{\mathcal{X}} (M^{k+1} f(x))^p (M^l w(x))^{-\varrho}\, d\mu(x). \end{aligned} \tag{1.8}$$

THEOREM 1.3. *Let $u \in A_\infty$ and $k \in \mathbb{N}$, and let Φ be an increasing function on $[0, \infty)$ satisfying the* doubling condition *that there exists a constant $C > 0$ such that for all $t \ge 0$,*

$$\Phi(2t) \le C\Phi(t). \tag{1.9}$$

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Under the hypotheses of Theorem 1.2, there exists a constant $C > 0$ depending only on k, $C_{A_\infty}(u)$ and $\delta_{A_\infty}(u)$ such that for any bounded function f with bounded support,

$$\sup_{\lambda>0} \Phi(\lambda)u(\{x \in \mathcal{X} : M^k(Tf)(x) > \lambda\}) \\ \leq C \sup_{\lambda>0} \Phi(\lambda)u(\{x \in \mathcal{X} : M^{k+1} f(x) > \lambda\}), \tag{1.10}$$

provided that

$$\sup_{0<\lambda<R} \Phi(\lambda)u(\{x \in \mathcal{X} : M^k(Tf)(x) > \lambda\}) < \infty$$

for any $R > 0$.

We remark that when $\mu(\mathcal{X}) < \infty$, the assumption that

$$\sup_{0<\lambda<R} \Phi(\lambda)u(\{x \in \mathcal{X} : M^k(Tf)(x) > \lambda\}) < \infty$$

for any $R > 0$ in Theorem 1.3 automatically holds.

To prove Theorems 1.2 and 1.3, we need a new sharp maximal operator related to an approximation to the identity, which is more general than the sharp maximal operator introduced by Martell [9]. Moreover, we also need its certain weighted A_∞ estimates; see Theorems 2.1 and 2.2 below. It was proved by Duong and Yan [6] that such sharp maximal operators play an important role in the theory of some new BMO-type spaces.

REMARK 1.4. As pointed out by Duong and McIntosh [4], if the kernel K associated with T satisfies a *Hölder continuity estimate*, that is, there exist positive constants C, c and ϵ such that for $x, y, y' \in \mathcal{X}$ satisfying $d(x, y) \geq cd(y, y')$,

$$|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \\ \leq C \frac{1}{\mu(B(y, d(x, y)))} \left[\frac{d(y, y')}{d(x, y)}\right]^\epsilon, \tag{1.11}$$

then there exist two approximations to the identity $\{D_t\}_{t>0}$ and $\{\widetilde{D}_t\}_{t>0}$ such that T satisfies (i) and (ii) above. Thus, the condition that T satisfies (i) and (ii) above is weaker than that K satisfies (1.11). On the other hand, even for the case that K satisfies (1.11) and $(\mathcal{X}, d, \mu)$ is the Euclidean space, both Theorems 1.2 and 1.3 are also new.

Using Theorem 1.2, we can obtain the weighted $L^p(\mathcal{X})$ when $p \in (1, \infty)$ and weak type $(1, 1)$ estimates with general weights for singular integral operators with nonsmooth kernels.

THEOREM 1.5. *Let T be an $L^2(\mathcal{X})$ bounded linear operator with kernel K as in (1.3). Suppose that T satisfies (ii) and (iii). Then for any $p \in (1, \infty)$, there exists a constant $C > 0$ depending only on p such that for any weight w, and any bounded function f with bounded support,*

$$\int_{\mathcal{X}} |Tf(x)|^p w(x)\, d\mu(x) \leq C \int_{\mathcal{X}} |f(x)|^p M^{[2p]+1} w(x)\, d\mu(x); \tag{1.12}$$

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here and in what follows, for a positive number θ, $[\theta]$ *denotes the* biggest integer no more than θ.

THEOREM 1.6. *Let* T *be an* $L^2(\mathcal{X})$ *bounded linear operator with kernel* K *as in (1.3). Suppose that* T *satisfies (ii) and (iii). Then there exists a constant* $C > 0$ *such that for any weight* w, $\lambda > 0$ *and bounded function* f *with bounded support,*

$$\int_{\{x\in\mathcal{X}:\ |Tf(x)|>\lambda\}} w(x)\,d\mu(x) \le C\lambda^{-1}\int_{\mathcal{X}} |f(x)|M^3w(x)\,d\mu(x).$$

As another application of Theorem 1.2, we consider the weighted estimates with general weights for commutators of BMO $(\mathcal{X})$ functions and singular integral operators with nonsmooth kernels. For $b \in \mathrm{BMO}\,(\mathcal{X})$ and T as in Theorem 1.5, define the *commutator* T_b by

$$T_bf(x) = b(x)Tf(x) - T(bf)(x), \tag{1.13}$$

where $x \in \mathcal{X}$ and f is any bounded function with bounded support. Duong and Yan [5] considered the $L^p(\mathcal{X})$-boundedness of T_b, and proved that if T is bounded on $L^2(\mathcal{X})$ and satisfies (i) and (ii), then T_b is bounded on $L^p(\mathcal{X})$ for any $p \in (1, \infty)$. From Theorem 1.2, we can deduce the following conclusions.

THEOREM 1.7. *Let* $b \in \mathrm{BMO}\,(\mathcal{X})$ *and* T_b *be as in (1.13). Under the hypotheses of Theorem 1.5, for any* $p \in (1, \infty)$, *there exists a constant* $C > 0$ *depending only on* p *such that for any weight* w, *and any bounded function* f *with bounded support,*

$$\begin{aligned}\int_{\mathcal{X}} &|T_bf(x)|^p w(x)\,d\mu(x)\\ &\le C\|b\|^p_{\mathrm{BMO}\,(\mathcal{X})}\int_{\mathcal{X}} |f(x)|^p M^{[3p]+1}w(x)\,d\mu(x).\end{aligned} \tag{1.14}$$

THEOREM 1.8. *Let* $b \in \mathrm{BMO}\,(\mathcal{X})$ *and* T_b *be as in (1.13). Under the hypotheses of Theorem 1.6, there exists a constant* $C > 0$ *such that for any weight* w, *any* $\lambda > 0$ *and bounded function* f *with bounded support,*

$$\begin{aligned}\int_{\{x\in\mathcal{X}:\ |T_bf(x)|>\lambda\}} &w(x)\,d\mu(x)\\ &\le C\|b\|_{\mathrm{BMO}\,(\mathcal{X})}\log(2+\|b\|_{\mathrm{BMO}\,(\mathcal{X})})\\ &\quad\times\int_{\mathcal{X}} \frac{|f(x)|}{\lambda}\log\Big(2+\frac{|f(x)|}{\lambda}\Big)M^4w(x)\,d\mu(x).\end{aligned}$$

We remark that Theorem 1.8 is also new, even when $w(x) \equiv 1$.

We also establish some weighted estimates for the following maximal operators. Let T be an $L^2(\mathcal{X})$-bounded linear operator associated with kernel K in the sense of (1.3). For each fixed $\epsilon > 0$, define the *truncated operator* T_ϵ by

$$T_\epsilon f(x) = \int_{d(x,y)\ge\epsilon} K(x, y)f(y)\,d\mu(y),$$

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and the associated *maximal operator* by

$$T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|.$$

Our result concerning the weighted $L^p(\mathcal{X})$ estimate for T^* can be stated as follows.

THEOREM 1.9. *Let T be an $L^2(\mathcal{X})$-bounded linear operator with kernel K as in (1.3). Suppose that T satisfies (ii) and (iii), and that the approximation to the identity $\{\widetilde{D}_t\}_{t>0}$ that appeared in (ii) above also satisfies that for all $t > 0$ and $x, y \in \mathcal{X}$ with $d(x, y) \le c_2 t^{1/m}$,*

$$|K^t(x, y)| \le C \frac{1}{\mu(B(x, t^{1/m}))}, \tag{1.15}$$

where $C > 0$ is a constant independent of t, x and y. Then for any $p \in (1, \infty)$, there exists a constant $C > 0$ depending only on p such that for any weight w, and any bounded function f with bounded support,

$$\int_{\mathcal{X}} [T^* f(x)]^p w(x)\, d\mu(x) \le C \int_{\mathcal{X}} |f(x)|^p M^{[2p]+2} w(x)\, d\mu(x).$$

Although it is still unclear whether there exists certain weighted endpoint estimate for T^* with general weights, we have the following conclusion, which is new even when $u(x) \equiv 1$.

THEOREM 1.10. *Let $u \in A_1$ and T be as in Theorem 1.9. Then there exists a constant $C > 0$ depending only on the A_1-constant of u such that for any $\lambda > 0$ and any bounded function f with bounded support,*

$$\int_{\{x\in\mathcal{X}:\ T^* f(x)>\lambda\}} u(x)\, d\mu(x) \le C \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log\Big(2 + \frac{|f(x)|}{\lambda}\Big) u(x)\, d\mu(x).$$

REMARK 1.11. By the results of [10, 16], we know that if T is the classical Calderón–Zygmund operator on $\mathbb{R}^n$, then for each fixed $p \in (1, \infty)$, the iterations of the Hardy–Littlewood maximal operator in (1.12) and (1.14) should be $[p] + 1$ and $[2p] + 1$, respectively, which are optimal. Although the singular integral satisfying (i) and (ii) is more singular (its kernel has no regularity in the space variable), it is still unclear whether the iterations $[2p] + 1$ and $[3p] + 1$ in Theorems 1.5 and 1.7 are optimal.

The organization of this paper is as follows. In Section 2, we introduce a new sharp maximal operator and establish its weighted A_∞ estimates. The proofs of Theorems 1.2 and 1.3 are presented in Section 3. Section 4 is devoted to the proofs of Theorems 1.5 and 1.6. In Section 5, we prove Theorems 1.7 and 1.8. The proofs of Theorems 1.9 and 1.10 are given in Section 6. Finally, in Section 7, we present some applications of our results to holomorphic functional calculi of elliptic operators and Schrödinger operators.

We finally make some *conventions*. Throughout this paper, we let $\mathbb{N} = \{1, 2, \ldots\}$ and let C denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line. The *symbol* $f \lesssim g$ means $f \le Cg$, which will be used only in proofs of theorems and lemmas and only when the omitted constant will not be cited later. *Constants with subscripts*, such as c_1, do not change in different occurrences. For a fixed p with $1 \le p < \infty$, p' denotes the *dual exponent* of p, namely, $p' = p/(p-1)$. For any ball $B = B(x, r)$ and $t > 0$, we set $tB = B(x, tr)$.

2. A new sharp maximal operator

In this section, we introduce a new sharp maximal operator associated with an approximation to the identity, which is a generalization of the sharp maximal operator introduced by Martell [9], and establish certain weighted A_∞ estimates related to this new sharp maximal operator and some other maximal operators.

Let k be a nonnegative integer and let V be a measurable set with $\mu(V) < \infty$. For any suitable function f, let $\|f\|_{L(\log L)^k, V}$ be the *Luxemburg norm* of f defined by

$$\|f\|_{L(\log L)^k, V} = \inf\left\{\lambda > 0 : \frac{1}{\mu(V)} \int_V \frac{|f(x)|}{\lambda} \log^k\left(2 + \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}.$$

The *maximal operator* $M_{L(\log L)^k}$ is defined by

$$M_{L(\log L)^k} f(x) = \sup_{B \ni x} \|f\|_{L(\log L)^k, B},$$

where the supremum is taken over all balls containing x. For an approximation to the identity $\{\widetilde{D}_t\}_{t>0}$ and any $f \in L^{p_0}(\mathcal{X})$ with $p_0 \in [1, \infty)$, we define the *sharp maximal operator* $M^\sharp_{\widetilde{D}, L(\log L)^k}$ by

$$M^\sharp_{\widetilde{D}, L(\log L)^k} f(x) = \sup_{B \ni x} \|f - \widetilde{D}_{t_B} f\|_{L(\log L)^k, B},$$

where $t_B = r_B^m$ and r_B is the radius of B. If $k = 0$, we denote $M^\sharp_{\widetilde{D}, L}$ simply by $M^\sharp_{\widetilde{D}}$, which was introduced by Martell [9] and plays an important role in [6].

By (1.4) and (1.5), we can verify that for any $f \in \bigcup_{p=1}^\infty L^p(\mathcal{X})$ and ball B,

$$\sup_{y \in B} |\widetilde{D}_{t_B} f(y)| \le C \inf_{y \in B} Mf(y).$$

It then follows that for all $x \in \mathcal{X}$,

$$M^\sharp_{\widetilde{D}, L(\log L)^k} f(x) \le C M_{L(\log L)^k} f(x). \tag{2.1}$$

Our goal in this section is to prove that, in some sense, the converse of (2.1) is true.

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THEOREM 2.1. *Let k be a nonnegative integer, $p \in (0, \infty)$ and $u \in A_\infty$, $\{\widetilde{D}_t\}_{t>0}$ be an approximation to the identity as in Definition 1.1. There exists a constant $C > 0$ depending only on p, $C_{A_\infty}(u)$ and $\delta_{A_\infty}(u)$ such that for any f with $M_{L(\log L)^k} f \in L^p(\mathcal{X}, u)$ and $f \in L^{p_0}(\mathcal{X})$ for some $p_0 \in (1, \infty)$:*

(a) *if $\mu(\mathcal{X}) = \infty$, then*

$$\|M_{L(\log L)^k} f\|_{L^p(\mathcal{X},u)} \leq C \|M^{\sharp}_{\widetilde{D},L(\log L)^k} f\|_{L^p(\mathcal{X},u)};$$

(b) *if $\mu(\mathcal{X}) < \infty$, then*

$$\begin{aligned}&\|M_{L(\log L)^k} f\|_{L^p(\mathcal{X},u)}\\ &\quad \leq C \|M^{\sharp}_{\widetilde{D},L(\log L)^k} f\|_{L^p(\mathcal{X},u)} + C[u(\mathcal{X})]^{1/p} \|f\|_{L(\log L)^k,\mathcal{X}}.\end{aligned}$$

THEOREM 2.2. *Let Φ be an increasing function on $[0, \infty)$ satisfying (1.9), let k be a nonnegative integer, let $u \in A_\infty$ and let $\{\widetilde{D}_t\}_{t>0}$ be an approximation to the identity as in Definition 1.1. Then there exists a constant $C > 0$ depending only on p, $C_{A_\infty}(u)$ and $\delta_{A_\infty}(u)$ such that for any $f \in L^{p_0}(\mathcal{X})$ with $p_0 \in (1, \infty)$:*

(a) *if $\mu(\mathcal{X}) = \infty$, and for any $R > 0$,*

$$\sup_{0<\lambda<R} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) < \infty,$$

then

$$\begin{aligned}&\sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\})\\ &\quad \leq C \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D},L(\log L)^k} f(x) > \lambda\});\end{aligned}$$

(b) *if $\mu(\mathcal{X}) < \infty$, then*

$$\begin{aligned}&\sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\})\\ &\quad \leq C \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D},L(\log L)^k} f(x) > \lambda\})\\ &\qquad + C u(\mathcal{X}) \Phi(\|f\|_{L(\log L)^k,\mathcal{X}}).\end{aligned}$$

To prove Theorems 2.1 and 2.2, we need some preliminary lemmas.

LEMMA 2.3 [1]. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and $\mathcal{B} = \{\mathcal{B}_\alpha\}_{\alpha\in\Lambda}$ be a family of balls in $\mathcal{X}$ such that $U = \bigcup_{\alpha\in\Lambda} \mathcal{B}_\alpha$ is measurable and $\mu(U) < \infty$. Then there exists a disjoint sequence $\{B(x_j, r_j)\}_j \subset \mathcal{B}$ such that $U \subset \bigcup_j B(x_j, c_4 r_j)$ with c_4 a positive constant depending only on κ. Moreover, for any $\alpha \in \Lambda$, $\mathcal{B}_\alpha$ is contained in some $B(x_j, c_4 r_j)$.*

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LEMMA 2.4. *Let k be a nonnegative integer and $p \in (1, \infty)$. Then there exists a constant $C > 0$ depending only on k and p such that for any weight w,*

$$\int_{\mathcal{X}} (M_{L(\log L)^k} f(x))^p w(x)\, d\mu(x) \le C \int_{\mathcal{X}} |f(x)|^p Mw(x)\, d\mu(x).$$

Moreover, there exists a constant $C > 0$ such that for any weight w and any $\lambda > 0$,

$$\begin{aligned} &\int_{\{x\in\mathcal{X}:\ M_{L(\log L)^k} f(x)>\lambda\}} w(x)\, d\mu(x) \\ &\quad \le C \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log^k\left(2 + \frac{|f(x)|}{\lambda}\right) Mw(x)\, d\mu(x), \end{aligned} \tag{2.2}$$

provided that $\mu(\mathcal{X}) < \infty$, or $\mu(\mathcal{X}) = \infty$ and $f \in L^{p_0}(\mathcal{X})$ for some $p_0 \in (1, \infty)$.

PROOF. The argument is similar to the case of Euclidean spaces; see [11, Lemma 1.6]. For the convenience of the reader, we present some details. It is obvious that $M_{L(\log L)^k}$ is bounded on $L^\infty(\mathcal{X})$. Thus, by an interpolation theorem of Rivière (see [18, Theorem 1.1]), it suffices to prove (2.2). Recall that for any nonnegative integer k, there exists a constant $C_k > 1$ such that for any suitable function h,

$$C_k^{-1} M_{L(\log L)^k} h(x) \le M^{k+1} h(x) \le C_k M_{L(\log L)^k} h(x). \tag{2.3}$$

In fact, the first inequality was proved by Pérez and Wheeden [15, Lemma 8.5], and the second inequality can be proved by the same argument as used in [11, p. 174]; see also [16, (4.7)]. If $f \in L^{p_0}(\mathcal{X})$, the $L^{p_0}(\mathcal{X})$-boundedness of M then states that

$$\mu(\{x \in \mathcal{X}:\ M_{L(\log L)^k} f(x) > \lambda\}) \lesssim \lambda^{-p_0} \|f\|^{p_0}_{L^{p_0}(\mathcal{X})} < \infty. \tag{2.4}$$

For any $\lambda > 0$ and $x \in \mathcal{X}$ with $M_{L(\log L)^k} f(x) > \lambda$, we choose a ball B_x containing x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} \frac{|f(y)|}{\lambda} \log^k\left(2 + \frac{|f(y)|}{\lambda}\right) d\mu(y) > 1.$$

By (2.4) and Lemma 2.3, we obtain a sequence of nonoverlapping balls $\{B_j\}_j$ such that

$$\{x \in \mathcal{X}:\ M_{L(\log L)^k} f(x) > \lambda\} \subset \bigcup_j c_4 B_j,$$

and, for all j,

$$\frac{1}{\mu(B_j)} \int_{B_j} \frac{|f(x)|}{\lambda} \log^k\left(2 + \frac{|f(x)|}{\lambda}\right) d\mu(x) > 1.$$

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Therefore,

$$\begin{aligned}
&\int_{\{x\in\mathcal{X}:\ M_{L(\log L)^k} f(x)>\lambda\}} w(x)\,d\mu(x)\\
&\quad\le \sum_j \int_{c_4 B_j} w(x)\,d\mu(x)\\
&\quad\lesssim \sum_j \mu(B_j)\frac{1}{\mu(c_4(B_j))}\int_{c_4 B_j} w(x)\,d\mu(x)\\
&\quad\lesssim \sum_j \mu(B_j)\inf_{y\in B_j} Mw(y)\\
&\quad\lesssim \int_{\mathcal{X}} \frac{|f(y)|}{\lambda}\log^k\Big(2+\frac{|f(y)|}{\lambda}\Big) Mw(y)\,d\mu(y),
\end{aligned}$$

which completes the proof of Lemma 2.4. □

LEMMA 2.5. *Let k be a nonnegative integer. There exists a constant $c_5>0$ depending only on k such that for any measurable set V with $\mu(V)<\infty$, function f supported on V and belonging to $L^{p_0}(\mathcal{X})$ with $p_0\in(1,\infty)$, and $\lambda>0$,*

$$\begin{aligned}
&\mu(\{x\in\mathcal{X}:\ M_{L(\log L)^k} f(x)>\lambda\})\\
&\quad\le c_5\mu(V)\frac{\|f\|_{L(\log L)^k,V}}{\lambda}\log^k\Big(2+\frac{\|f\|_{L(\log L)^k,V}}{\lambda}\Big).
\end{aligned}$$

PROOF. By homogeneity, we may assume that $\|f\|_{L(\log L)^k,V}=1$, which means that

$$\int_V |f(x)|\log^k(2+|f(x)|)\,d\mu(x)\le\mu(V).$$

It follows from Lemma 2.4 with $w\equiv 1$ that

$$\begin{aligned}
&\mu(\{x\in\mathcal{X}:\ M_{L(\log L)^k} f(x)>\lambda\})\\
&\quad\lesssim \int_V \frac{|f(x)|}{\lambda}\log^k\Big(2+\frac{|f(x)|}{\lambda}\Big)\,d\mu(x)\\
&\quad\lesssim \frac{1}{\lambda}\log^k(2+\lambda^{-1})\int_V |f(x)|\log^k(2+|f(x)|)\,d\mu(x)\\
&\quad\lesssim \frac{1}{\lambda}\log^k(2+\lambda^{-1})\mu(V),
\end{aligned}$$

which completes the proof of Lemma 2.5. □

On the maximal operators $M_{L(\log L)^k}$ and $M^{\sharp}_{\widetilde{D},L(\log L)^k}$, we have the following good-λ inequality.

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LEMMA 2.6. *Let $\{\widetilde{D}_t\}_{t>0}$ be an approximation to the identity as in Definition 1.1. Then there exists a constant $c_6 > 1$ which depends on $\{\widetilde{D}_t\}_{t>0}$ such that for all $\lambda > 0$, all functions $f \in L^{p_0}(\mathcal{X})$ with some $p_0 \in (1, \infty)$, all balls B such that $M_{L(\log L)^k} f(x_0) \le \lambda$ for some $x_0 \in B$, and any fixed $\eta \in (0, 1)$, there exists a constant $\gamma > 0$, which depends on η, but is independent of f, λ and B, such that*

$$\mu(\{x \in B:\ M_{L(\log L)^k} f(x) > c_6\lambda,\ M^{\sharp}_{\widetilde{D},L(\log L)^k} f(x) \le \gamma\lambda\}) \le \eta\mu(B).$$

PROOF. We follow the argument used in the proof of [9, Proposition 4.1]. Let $M^c_{L(\log L)^k}$ be the *operator* defined by

$$M^c_{L(\log L)^k} h(x) = \sup_{r>0} \|h\|_{L(\log L)^k, B(x,r)}$$

for any $x \in \mathcal{X}$ and suitable function h. It is easy to verify that there exists a constant $c_7 \in (0, 1)$ such that for any $x \in \mathcal{X}$,

$$c_7 M_{L(\log L)^k} h(x) \le M^c_{L(\log L)^k} h(x) \le M_{L(\log L)^k} h(x).$$

Let B be as in the assumption of the lemma, let r_B be its radius, and let $t_0 = (4\kappa^2 r_B)^m$. Set

$$G_\lambda = \{x \in B:\ M_{L(\log L)^k} f(x) > c_6\lambda,\ M^{\sharp}_{\widetilde{D},L(\log L)^k} f(x) \le \gamma\lambda\},$$

where $c_6 > 1$ will be determined later. If $G_\lambda = \emptyset$, there is nothing to prove. Thus, we may assume that there exists a point $x_{G_\lambda} \in G_\lambda$ such that

$$M^{\sharp}_{\widetilde{D},L(\log L)^k} f(x_{G_\lambda}) \le \gamma\lambda.$$

It is obvious that for any $x \in G_\lambda$, there exists $r_x > 0$ such that

$$\|f\|_{L(\log L)^k, B(x,r_x)} > c_7 c_6 \lambda.$$

Choose c_6 such that $c_7 c_6 > 1$. Then $x_0 \notin B(x, r_x)$ (otherwise $M_{L(\log L)^k} f(x_0) > \lambda$), so $r_x \le 2\kappa r_B$ and $B(x, r_x) \subset 4\kappa^2 B$. Therefore, for any $x \in G_\lambda$,

$$M_{L(\log L)^k}(f\chi_{4\kappa^2 B})(x) > c_7 c_6 \lambda. \tag{2.5}$$

As was pointed out in [9, p. 122], for each $y \in 4\kappa^2 B$,

$$|\widetilde{D}_{t_0}(f\chi_{16\kappa^3 B})(y)| \lesssim Mf(x_0) \lesssim \lambda$$

and

$$|\widetilde{D}_{t_0}(f\chi_{\mathcal{X}\setminus 16\kappa^3 B})(y)| \lesssim Mf(x_0) \lesssim \lambda.$$

This, in turn, implies that for a certain constant $c_8 > 0$,

$$\begin{aligned} M_{L(\log L)^k}((\widetilde{D}_{t_0} f)\chi_{4\kappa^2 B})(x) &\le M_{L(\log L)^k}(\widetilde{D}_{t_0}(f\chi_{16\kappa^3 B})\chi_{4\kappa^2 B})(x) \\ &\quad + M_{L(\log L)^k}(\widetilde{D}_{t_0}(f\chi_{\mathcal{X}\setminus 16\kappa^3 B})\chi_{4\kappa^2 B})(x) \\ &\le c_8\lambda, \end{aligned}$$

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and so, for $x \in G_\lambda$,

$$\begin{aligned} M_{L(\log L)^k}(f\chi_{4\kappa^2 B})(x) &\leq M_{L(\log L)^k}((f - \widetilde{D}_{t_0} f)\chi_{4\kappa^2 B})(x) \\ &\quad + M_{L(\log L)^k}((\widetilde{D}_{t_0} f)\chi_{4\kappa^2 B})(x) \\ &\leq M_{L(\log L)^k}((f - \widetilde{D}_{t_0} f)\chi_{4\kappa^2 B})(x) + c_8\lambda. \end{aligned}$$

Now we take c_6 such that $c_7 c_6 = c_8 + 1$. The estimate (2.5) tells us that

$$G_\lambda \subset \{x \in B : M_{L(\log L)^k}((f - \widetilde{D}_{t_0} f)\chi_{4\kappa^2 B})(x) > \lambda\}.$$

By Lemma 2.5, we finally obtain

$$\begin{aligned} \mu(G_\lambda) &\leq c_5 \mu(4\kappa^2 B) \frac{\|f - \widetilde{D}_{t_0} f\|_{L(\log L)^k, 4\kappa^2 B}}{\lambda} \\ &\quad \times \log^k\Big(2 + \frac{\|f - \widetilde{D}_{t_0} f\|_{L(\log L)^k, 4\kappa^2 B}}{\lambda}\Big) \\ &\leq c_5 \mu(4\kappa^2 B) \frac{M^\sharp_{\widetilde{D},\, L(\log L)^k} f(x_{G_\lambda})}{\lambda} \log^k\Big(2 + \frac{M^\sharp_{\widetilde{D}, L(\log L)^k} f(x_{G_\lambda})}{\lambda}\Big) \\ &\leq (4\kappa^2)^n c_5 c_0 \mu(B) \gamma \log^k(2 + \gamma), \end{aligned}$$

where c_0 is the constant in (1.1). Taking

$$\gamma \in (0, \min\{1/2, \eta/((4\kappa^2)^n c_5 c_0 \log^k 3)\})$$

then completes the proof of Lemma 2.6. □

Proof of Theorem 2.1. Let c_6 be as in Lemma 2.6. For $\eta \in (0, 1)$ which will be chosen later, let γ be the corresponding constant as in Lemma 2.6. For each fixed $\lambda > 0$, set

$$H_\lambda = \{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}$$

and

$$F_\lambda = \{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > c_6\lambda, M^\sharp_{\widetilde{D}, L(\log L)^k} f(x) \leq \gamma\lambda\}.$$

It is easy to see that H_λ is an open set. Moreover, applying Lemma 2.5, we know that when $\mu(\mathcal{X}) < \infty$,

$$\mu(H_\lambda) \leq c_5 \mu(\mathcal{X}) \frac{\|f\|_{L(\log L)^k, \mathcal{X}}}{\lambda} \log^k\Big(2 + \frac{\|f\|_{L(\log L)^k, \mathcal{X}}}{\lambda}\Big).$$

Let $\lambda_{f,\mathcal{X}} = 0$ if $\mu(\mathcal{X}) = \infty$, and

$$\lambda_{f,\mathcal{X}} = (c_5 \log^k 3 + 2)\|f\|_{L(\log L)^k, \mathcal{X}}$$

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if $\mu(\mathcal{X}) < \infty$. It is obvious that if $\mu(\mathcal{X}) < \infty$ and $\lambda > \lambda_{f,\mathcal{X}}$, then $\mu(H_\lambda) < \mu(\mathcal{X})$. On the other hand, by $f \in L^{p_0}(\mathcal{X})$ with $p_0 \in (1, \infty)$ and Lemma 2.4, when $\mu(\mathcal{X}) = \infty$, we still have $\mu(H_\lambda) < \infty = \mu(\mathcal{X})$. Thus, we always have $\mathcal{X} \setminus H_\lambda \neq \emptyset$. For each fixed $x \in H_\lambda$, denote by ρ_x the distance between x and $\mathcal{X} \setminus H_\lambda$. Let c_4 be as in Lemma 2.3. Obviously, we can assume that $c_4 \geq 1$. It is also easy to see that $\rho_x > 0$ and $H_\lambda = \bigcup_{x \in H_\lambda} B(x, \rho_x/(2c_4))$. Applying Lemma 2.3, we find a sequence of nonoverlapping balls $\{B(x_j, \rho_j/(2c_4))\}_j$ such that $H_\lambda = \bigcup_j B_j$ and $\widetilde{B_j} \cap (X \setminus H_\lambda) \neq \emptyset$, where $B_j = B(x_j, 4\rho_j/5)$ and $\widetilde{B_j} = B(x_j, 5\rho_j/4)$. It follows from Lemma 2.6 that, for all j,

$$\mu(\{x \in \widetilde{B_j} : M_{L(\log L)^k} f(x) > c_6\lambda, M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) \leq \gamma\lambda\}) \leq \eta\mu(\widetilde{B_j}),$$

and so, for $u \in A_\infty$,

$$\begin{aligned} &u(\{x \in \widetilde{B_j} : M_{L(\log L)^k} f(x) > c_6\lambda, M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) \leq \gamma\lambda\}) \\ &\quad \leq C_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} u(\widetilde{B_j}). \end{aligned}$$

A straightforward computation then leads to that

$$\begin{aligned} u(F_\lambda) &\leq \sum_j u(\{x \in \widetilde{B_j} : M_{L(\log L)^k} f(x) > c_6\lambda, M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) \leq \gamma\lambda\}) \\ &\leq C_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} \sum_j u(\widetilde{B_j}) \\ &\leq C_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} \sum_j u(B(x_j, \rho_j/(2c_4))) \\ &\leq C_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} u(H_\lambda). \end{aligned} \tag{2.6}$$

We can now conclude the proof of Theorem 2.1. If $\mu(\mathcal{X}) = \infty$, integrating the last inequality then yields

$$\begin{aligned} &\|M_{L(\log L)^k} f\|^p_{L^p(\mathcal{X},u)} \\ &\quad = c_6^p p \int_0^\infty \lambda^{p-1} u(H_{c_6\lambda})\, d\lambda \\ &\quad \leq c_6^p p \int_0^\infty \lambda^{p-1} u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) > \gamma\lambda\})\, d\lambda \\ &\qquad + c_6^p p \int_0^\infty \lambda^{p-1} u(F_\lambda)\, d\lambda \\ &\quad \leq C\gamma^{-p} \|M^{\sharp}_{\widetilde{D}, L(\log L)^k} f\|^p_{L^p(\mathcal{X},u)} + CC_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} \|M_{L(\log L)^k} f\|^p_{L^p(\mathcal{X},u)}. \end{aligned}$$

On the other hand, for the case that $\mu(\mathcal{X}) < \infty$, again by (2.6),

$$\begin{aligned}
&\|M_{L(\log L)^k} f\|^p_{L^p(\mathcal{X},u)} \\
&\quad = c_6^p p \int_{\lambda_{f,\mathcal{X}}}^{\infty} \lambda^{p-1} u(H_{c_6\lambda})\, d\lambda + c_6^p p \int_0^{\lambda_{f,\mathcal{X}}} \lambda^{p-1} u(H_{c_6\lambda})\, d\lambda \\
&\quad \le C\gamma^{-p} \|M^{\sharp}_{\widetilde{D},\, L(\log L)^k} f\|^p_{L^p(\mathcal{X},u)} + C C_{A_\infty}(u) \eta^{\delta_{A_\infty}(u)} \|M_{L(\log L)^k} f\|^p_{L^p(\mathcal{X},u)} \\
&\qquad + C u(\mathcal{X}) \lambda^p_{f,\mathcal{X}}.
\end{aligned}$$

Choosing η such that $C C_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} = 1/2$ gives us the desired conclusion, which completes the proof of Theorem 2.1. □

PROOF OF THEOREM 2.2. Using the same notation as in the proof of Theorem 2.1. Let $\eta \in (0, 1)$ which is a constant and will be determined later. For each fixed $\lambda > \lambda_{f,\mathcal{X}}$, by the inequality (2.6), we then have

$$\begin{aligned}
&u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > c_6\lambda\}) \\
&\quad \le u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > c_6\lambda,\ M^{\sharp}_{\widetilde{D},\, L(\log L)^k} f(x) \le \gamma\lambda\}) \\
&\qquad + u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D},\, L(\log L)^k} f(x) > \gamma\lambda\}) \\
&\quad \le C C_{A_\infty}(u) \eta^{\delta_{A_\infty}(u)} u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\qquad + u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) > \gamma\lambda\}).
\end{aligned}$$

If $\mu(\mathcal{X}) = \infty$, the last inequality, via (1.9) together with a trivial computation, tells us that, for any $R > 0$,

$$\begin{aligned}
&\sup_{0<\lambda<R} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\quad = \sup_{0<\lambda<R/c_6} \Phi(c_6\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > c_6\lambda\}) \\
&\quad \le C C_{A_\infty}(u) \eta^{\delta_{A_\infty}(u)} \sup_{0<\lambda<R} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\qquad + C \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) > \gamma\lambda\}).
\end{aligned}$$

On the other hand, if $\mu(\mathcal{X}) < \infty$, we have that, for any $R > c_6\lambda_{f,\mathcal{X}}$,

$$\begin{aligned}
&\sup_{c_6\lambda_{f,\mathcal{X}}<\lambda<R} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\quad = \sup_{\lambda_{f,\mathcal{X}}<\lambda<R/c_6} \Phi(c_6\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > c_6\lambda\}) \\
&\quad \le C C_{A_\infty}(u) \eta^{\delta_{A_\infty}(u)} \sup_{0<\lambda<R} \Phi(\lambda) u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\qquad + C \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D}, L(\log L)^k} f(x) > \gamma\lambda\}),
\end{aligned}$$

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which, in turn, shows that

$$\begin{aligned}
&\sup_{0<\lambda<R} \Phi(\lambda)u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\quad \leq \sup_{0<\lambda\leq c_6\lambda_{f,\mathcal{X}}} \Phi(\lambda)u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\qquad + \sup_{c_6\lambda_{f,\mathcal{X}}<\lambda<R} \Phi(\lambda)u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\quad \leq CC_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} \sup_{0<\lambda<R} \Phi(\lambda)u(\{x \in \mathcal{X} : M_{L(\log L)^k} f(x) > \lambda\}) \\
&\qquad + C\sup_{\lambda>0} \Phi(\lambda)u(\{x \in \mathcal{X} : M^{\sharp}_{\widetilde{D},L(\log L)^k} f(x) > \gamma\lambda\}) + C\Phi(\lambda_{f,\mathcal{X}})u(\mathcal{X}).
\end{aligned}$$

Choosing η such that $CC_{A_\infty}(u)\eta^{\delta_{A_\infty}(u)} = 1/2$ then leads to the desired estimates, which completes the proof of Theorem 2.2. □

3. Proofs of Theorems 1.2 and 1.3

We begin with some lemmas.

LEMMA 3.1. *Let S be a sublinear operator which is bounded from $L^{p_0}(\mathcal{X})$ to $L^{p_0,\infty}(\mathcal{X})$ for certain $p_0 \in (1, \infty)$ and from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$. Then for any nonnegative integer k, there exists a constant $C > 0$ depending only on k such that for any measurable sets V_1 and V_2 with $\mu(V_1) \leq \mu(V_2) < \infty$, and function f supported on V_1,*

$$\|Sf\|_{L(\log L)^k, V_2} \leq C\|f\|_{L(\log L)^{k+1}, V_1}.$$

PROOF. By homogeneity, we may assume that $\|f\|_{L(\log L)^{k+1}, V_1} = 1$, which means that

$$\int_{V_1} |f(x)| \log^{k+1}(2 + |f(x)|)\, d\mu(x) \leq \mu(V_1).$$

For each fixed $\lambda > 0$, set $\Omega_\lambda = \{x \in \mathcal{X} : |f(x)| > \lambda^{(1-1/p_0)/2}\}$. Decompose f into

$$f(x) = f(x)\chi_{\Omega_\lambda}(x) + f(x)\chi_{\mathcal{X}\setminus\Omega_\lambda}(x) = f_1(x) + f_2(x).$$

A trivial computation leads to that

$$
\begin{aligned}
&\int_{V_2} |Sf(x)| \log^k(2+|Sf(x)|)\, d\mu(x) \\
&\quad \le \int_0^1 \mu(\{x \in V_2 : |Sf(x)| > \lambda\})(\lambda \log^k(2+\lambda))'\, d\lambda \\
&\qquad + \int_1^\infty \mu(\{x \in \mathcal{X} : |Sf_1(x)| > \lambda/2\})(\lambda \log^k(2+\lambda))'\, d\lambda \\
&\qquad + \int_1^\infty \mu(\{x \in \mathcal{X} : |Sf_2(x)| > \lambda/2\})(\lambda \log^k(2+\lambda))'\, d\lambda \\
&\quad \lesssim \mu(V_2) + \int_1^\infty \int_{V_1} \frac{|f_1(x)|}{\lambda}\, d\mu(x)(\lambda \log^k(2+\lambda))'\, d\lambda \\
&\qquad + \int_1^\infty \int_{V_1} \left(\frac{|f_2(x)|}{\lambda}\right)^{p_0} d\mu(x)(\lambda \log^k(2+\lambda))'\, d\lambda.
\end{aligned}
$$

The fact that $\|f_2\|_{L^\infty(\mathcal{X})} \le \lambda^{(1-1/p_0)/2}$ now tells us that

$$
\int_1^\infty \int_{V_1} \left(\frac{|f_2(x)|}{\lambda}\right)^{p_0} d\mu(x)(\lambda \log^k(2+\lambda))'\, d\lambda \lesssim \mu(V_1).
$$

On the other hand,

$$
\begin{aligned}
&\int_1^\infty \int_{V_1} \frac{|f_1(x)|}{\lambda}\, d\mu(x)(\lambda \log^k(2+\lambda))'\, d\lambda \\
&\quad \le \int_{V_1} |f(x)| \int_1^{|f(x)|^{2/(1-1/p_0)}} \lambda^{-1}(\lambda \log^k(2+\lambda))'\, d\lambda\, d\mu(x) \\
&\quad \lesssim \int_{V_1} |f(x)| \int_1^{|f(x)|^{2/(1-1/p_0)}} \lambda^{-1} \log^k(2+\lambda)\, d\lambda\, d\mu(x) \\
&\quad \lesssim \int_{V_1} |f(x)| \int_1^{|f(x)|^{2/(1-1/p_0)}} (\log^{k+1}(2+\lambda))'\, d\lambda\, d\mu(x) \\
&\quad \lesssim \int_{V_1} |f(x)| \log^{k+1}(2+|f(x)|)\, d\mu(x) \\
&\quad \lesssim \mu(V_1).
\end{aligned}
$$

Combining the estimates above leads to that

$$
\int_{V_2} |Sf(x)| \log^k(2+|Sf(x)|)\, d\mu(x) \lesssim \mu(V_1) + \mu(V_2) \lesssim \mu(V_2)
$$

and our desired conclusion follows directly. □

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REMARK 3.2. We point out that in Lemma 3.1, if $\mathcal{X} = \mathbb{R}^n$, $V_1 = V_2$ is a ball, and S is the Hardy–Littlewood maximal operator, then Lemma 3.1 was obtained by Stein [19]. In fact, in this case, Stein proved that for any $k \geq 0$, ball $B \subset \mathbb{R}^n$ and function f supported on B, $Mf \log^k(2 + Mf) \in L^1(B)$ if and only if

$$|f| \log^{k+1}(2 + |f|) \in L^1(B).$$

LEMMA 3.3. *Let w be a locally integrable function such that Mw is finite almost everywhere. For $\nu > 0$, set $u(x) = (Mw(x))^{-\nu}$ for $x \in \mathcal{X}$. Then there exists a constant $C > 0$ depending only on ν such that for any ball B and measurable set $E \subset B$,*

$$\frac{u(E)}{u(B)} \leq C \frac{\mu(E)}{\mu(B)}.$$

Namely, $u \in A_\infty$ with $C_{A_\infty}(u) \leq C$ and $\delta_{A_\infty}(u) = 1$.

PROOF. The argument is similar to that used in the proof of [13, Lemma 3]. We first notice that following the argument in the case of Euclidean spaces (see, for example, [7, pp. 158–160]) shows that $(Mw)^{\nu/(\nu+1)}$ is an A_1 weight with A_1 constant C_ν depending only on ν. Namely, for any ball B,

$$\frac{1}{\mu(B)} \int_B (Mw(x))^{\nu/(\nu+1)} \, d\mu(x) \leq C_\nu \operatorname*{ess\,inf}_{x \in B} (Mw(x))^{\nu/(\nu+1)}.$$

Note that, for any fixed ball B, it follows from Hölder's inequality that

$$\left[\int_B (Mw(x))^{-\nu} \, d\mu(x) \right]^{1/(\nu+2)} \left[\int_B (Mw(x))^{\nu/(\nu+1)} \, d\mu(x) \right]^{(\nu+1)/(\nu+2)} \geq \mu(B).$$

Therefore,

$$\begin{aligned} \sup_{x \in B} u(x) &= \left[\inf_{x \in B} (Mw(x))^{\nu/(\nu+1)} \right]^{-\nu-1} \\ &\leq C_\nu \left[\frac{1}{\mu(B)} \int_B (Mw(x))^{\nu/(\nu+1)} \, d\mu(x) \right]^{-\nu-1} \\ &\leq \frac{C_\nu}{\mu(B)} \int_B u(x) \, d\mu(x). \end{aligned}$$

If $E \subset B$ is a measurable set, then

$$\frac{u(E)}{u(B)} \leq \frac{\mu(E) \sup_{x \in B} u(x)}{u(B)} \leq C_\nu \frac{\mu(E)}{\mu(B)},$$

which completes the proof of Lemma 3.3. □

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PROOF OF THEOREM 1.2. First, we claim that for any $f \in L^{p_0}(\mathcal{X})$ with $p_0 \in (1, \infty)$, and any $x \in \mathcal{X}$,

$$M^{\sharp}_{\widetilde{D}, L(\log L)^k}(Tf)(x) \lesssim M_{L(\log L)^{k+1}} f(x). \tag{3.1}$$

In fact, for each fixed $x \in \mathcal{X}$ and any fixed ball B containing x, denote the radius of B by r_B. We decompose f as

$$f(y) = f(y)\chi_{c_9 B}(y) + f(y)\chi_{\mathcal{X}\setminus c_9 B}(y) = f_1(y) + f_2(y),$$

where $c_9 = \kappa(c_2 + 2\kappa + 1)$. Recall that the operator T is bounded on $L^2(\mathcal{X})$, and is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$ (see [4]), it follows from Lemma 3.1 that

$$\begin{aligned}\|Tf_1\|_{L(\log L)^k, B} &\lesssim \|Tf_1\|_{L(\log L)^k, c_9 B}\\ &\lesssim \|f_1\|_{L(\log L)^{k+1}, c_9 B}\\ &\lesssim M_{L(\log L)^{k+1}} f(x).\end{aligned}$$

Set

$$\mathrm{F}_0 = \int_{2c_9 B} |Tf_1(z)|\, d\mu(z),$$

and

$$\mathrm{F}_j = \int_{2^{j+1}c_9 B\setminus 2^j c_9 B} |Tf_1(z)|\, d\mu(z)$$

for $j \in \mathbb{N}$. Another application of Lemma 3.1 yields that for all $j \geq 0$,

$$\mathrm{F}_j \lesssim \mu(2^{j+1}B)\|f_1\|_{L\log L, c_9 B} \lesssim 2^{jn}\mu(B) M_{L\log L} f(x).$$

Note that if $y \in B$ and $z \notin 2^j c_9 B$ for some positive integer j, then $d(y, z) \geq 2^{j-1} r_B$, and so by (1.4) and (1.2),

$$h_{t_B}(y, z) \leq \frac{1}{\mu(B(y, r_B))} s(2^{m(j-1)}) \lesssim \frac{1}{\mu(B)} s(2^{m(j-1)}).$$

Thus, for any $y \in B$,

$$\begin{aligned}|\widetilde{D}_{t_B}(Tf_1)(y)| &\leq \int_{2c_9 B} h_{t_B}(y, z)|Tf_1(z)|\, d\mu(z)\\ &\quad + \sum_{j=1}^{\infty} \int_{2^{j+1}c_9 B\setminus 2^j c_9 B} h_{t_B}(y, z)|Tf_1(z)|\, d\mu(z)\\ &\lesssim \frac{1}{\mu(B)} F_0 + \frac{1}{\mu(B)} \sum_{j=1}^{\infty} s(2^{m(j-1)}) F_j\\ &\lesssim M_{L\log L} f(x) + M_{L\log L} f(x) \sum_{j=1}^{\infty} s(2^{m(j-1)}) 2^{jn}\\ &\lesssim M_{L\log L} f(x).\end{aligned}$$

This, in turn, implies that

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$$\|\widetilde{D}_{t_B}(Tf_1)\|_{L(\log L)^k,B} \lesssim M_{L(\log L)^{k+1}}f(x).$$

We now turn our attention to Tf_2. Since for $x,\ y \in B$ and $z \in \mathcal{X} \setminus c_9B$,

$$d(x,z) \geq (c_2+2\kappa)r_B \quad \text{and} \quad d(y,z) \geq (c_2+2\kappa)r_B.$$

Thus, for any $y \in B$, an argument used in the proof of [9, Proposition 5.4] gives us that

$$\begin{aligned}|Tf_2(y) - \widetilde{D}_{t_B}(Tf_2)(y)| &\leq \int_{\substack{d(y,z)\geq(c_2+2\kappa)r_B\\ d(x,z)\geq(c_2+2\kappa)r_B}} |K(y,z)-K^{t_B}(y,z)||f(z)|\,d\mu(z)\\ &\lesssim Mf(x).\end{aligned}$$

Combining the estimates for Tf_1 and Tf_2 then leads to the estimate (3.1).

Let $\mu(\mathcal{X}) < \infty$. We claim that

$$\|f\|_{L(\log L)^k,\mathcal{X}} \leq \inf_{x\in\mathcal{X}} M_{L(\log L)^k}f(x). \tag{3.2}$$

In fact, for any $\epsilon > 0$, we take some point $x_0 \in \mathcal{X}$ such that

$$M_{L(\log L)^k}f(x_0) < \lambda_\epsilon = \inf_{x\in\mathcal{X}} M_{L(\log L)^k}f(x) + \epsilon.$$

Let B be a ball containing x_0. Then $\|f\|_{L(\log L)^k,2^jB} < \lambda_\epsilon$, for any $j \in \mathbb{N}$, which means that

$$\frac{1}{\mu(2^jB)}\int_{2^jB}\frac{|f(x)|}{\lambda_\epsilon}\log^k\left(2+\frac{|f(x)|}{\lambda_\epsilon}\right)d\mu(x) \leq 1.$$

Letting $j \to \infty$ then shows that

$$\frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}}\frac{|f(x)|}{\lambda_\epsilon}\log^k\left(2+\frac{|f(x)|}{\lambda_\epsilon}\right)d\mu(x) \leq 1,$$

which implies that $\|f\|_{L(\log L)^k,\mathcal{X}} \leq \lambda_\epsilon$ and leads to (3.2).

We can now prove (1.7). If $\mu(\mathcal{X}) = \infty$, (1.7) follows from the inequality (2.3), Theorem 2.1 and the estimate (3.1).

On the other hand, if $\mu(\mathcal{X}) < \infty$, again by the inequality (2.3), Theorem 2.1, the estimate (3.1), Lemma 3.1 and the estimate (3.2), we can deduce that

$$\begin{aligned}&\int_{\mathcal{X}}|M^k(Tf)(x)|^p u(x)\,d\mu(x)\\ &\quad\lesssim \int_{\mathcal{X}}(M^{k+1}f(x))^p u(x)\,d\mu(x) + \|Tf\|^p_{L(\log L)^{k-1},\mathcal{X}}u(\mathcal{X})\\ &\quad\lesssim \int_{\mathcal{X}}(M^{k+1}f(x))^p u(x)\,d\mu(x) + \|f\|^p_{L(\log L)^{k},\mathcal{X}}u(\mathcal{X})\\ &\quad\lesssim \int_{\mathcal{X}}(M^{k+1}f(x))^p u(x)\,d\mu(x) + \left(\inf_{x\in\mathcal{X}} M^{k+1}f(x)\right)^p u(\mathcal{X})\\ &\quad\lesssim \int_{\mathcal{X}}(M^{k+1}f(x))^p u(x)\,d\mu(x),\end{aligned}$$

which verifies (1.7).

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The inequality (1.8) is an easy consequence of (1.7) and Lemma 3.3. In fact, for each fixed $\epsilon > 0$, it follows from Lemma 3.3 that $(M^l(w+\epsilon))^{-\varrho}$ is an A_∞ weight, and both $C_{A_\infty}((M^l(w+\epsilon))^{-\varrho})$ and $\delta_{A_\infty}((M^l(w+\epsilon))^{-\varrho})$ depend only on ϱ. By the $L^p(\mathcal{X})$-boundedness for $p \in (1, \infty)$ of T (see [9]) and M, we then have

$$\begin{aligned}\int_{\mathcal{X}} (M^k(Tf)(x))^p (M^l(w+\epsilon)(x))^{-\varrho}\, d\mu(x) &\le \epsilon^{-\varrho} \int_{\mathcal{X}} (M^k(Tf)(x))^p\, d\mu(x)\\ &\lesssim \epsilon^{-\varrho} \|f\|^p_{L^p(\mathcal{X})} < \infty.\end{aligned}$$

This together with (1.7) implies that, for any $\epsilon > 0$,

$$\begin{aligned}&\int_{\mathcal{X}} (M^k(Tf)(x))^p (M^l(w+\epsilon)(x))^{-\varrho}\, d\mu(x)\\ &\quad\lesssim \int_{\mathcal{X}} (M^{k+1} f(x))^p (M^l(w+\epsilon)(x))^{-\varrho}\, d\mu(x).\end{aligned}$$

An application of the monotonic convergence theorem yields (1.8) and then completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. If $\mu(\mathcal{X}) = \infty$, (1.10) follows from the inequality (2.3), Theorem 2.2 together with the assumption that, for any $R > 0$,

$$\sup_{0<\lambda<R} \Phi(\lambda) u(\{x \in \mathcal{X} :\ M^k(Tf)(x) > \lambda\}) < \infty,$$

and the inequality (3.1).

If $\mu(\mathcal{X}) < \infty$, again by the inequality (2.3), Theorem 2.2, the inequality (3.1) and Lemma 3.1, we finally obtain

$$\begin{aligned}&\sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} :\ M^k(Tf)(x) > \lambda\})\\ &\quad\lesssim \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} :\ M^{\sharp}_{\widetilde{D}, L(\log L)^{k-1}}(Tf)(x) > \lambda\})\\ &\qquad + u(\mathcal{X}) \Phi(\|Tf\|_{L(\log L)^{k-1}, \mathcal{X}})\\ &\quad\lesssim \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} :\ M^{k+1} f(x) > \lambda\}) + u(\mathcal{X}) \Phi(\|f\|_{L(\log L)^k, \mathcal{X}})\\ &\quad\lesssim \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} :\ M^{k+1} f(x) > \lambda\}),\end{aligned}$$

where in the last inequality, we have used the inequality (3.2) and employed the fact that

$$\begin{aligned}\Phi(\|f\|_{L(\log L)^k, \mathcal{X}}) &\le \Phi\Big(\inf_{x\in\mathcal{X}} M_{L(\log L)^k} f(x)\Big)\\ &\lesssim (u(\mathcal{X}))^{-1} \sup_{\lambda>0} \Phi(\lambda) u(\{x \in \mathcal{X} :\ M^{k+1} f(x) > \lambda\}).\end{aligned}$$

This finishes the proof of Theorem 1.3. □

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4. Proofs of Theorems 1.5 and 1.6

We begin with the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. We may assume that $M^{[2p]+1}w$ is finite almost everywhere, otherwise there is nothing to prove. By duality it suffices to show that for any $p \in (1, \infty)$, any weight w and any bounded function f with bounded support,

$$\int_{\mathcal{X}} |Tf(x)|^{p'}(M^{[2p]+1}w(x))^{1-p'}\,d\mu(x) \lesssim \int_{\mathcal{X}} |f(x)|^{p'}(w(x))^{1-p'}\,d\mu(x). \quad (4.1)$$

Recall that, for $k \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathcal{X}} &(M^k f(x))^{p'}(M^{[kp]+1}w(x))^{1-p'}\,d\mu(x)\\ &\lesssim \int_{\mathcal{X}} |f(x)|^{p'}(w(x))^{1-p'}\,d\mu(x).\end{aligned} \quad (4.2)$$

In fact, for the Euclidean space, this was proved in [10] and [12]; for the space of homogeneous type, see [16]. The estimate (4.1) then follows from (1.8) with $k = 1$ and $l = [2p] + 1$ and (4.2) immediately, which completes the proof of Theorem 1.5. □

PROOF OF THEOREM 1.6. We assume that $M^3w(x)$ is finite almost everywhere. Note that if $\mu(\mathcal{X}) < \infty$, the inequality (3.2) with $k = 0$ shows that

$$w(\mathcal{X})/\mu(\mathcal{X}) \le \inf_{x\in\mathcal{X}} Mw(x);$$

then for $\lambda \le \|f\|_{L^1(\mathcal{X})}(\mu(\mathcal{X}))^{-1}$, it is easy to see that

$$\begin{aligned}\int_{\{x\in\mathcal{X}:\ |Tf(x)|>\lambda\}} w(x)\,d\mu(x) &\le w(\mathcal{X})\\ &\le \frac{1}{\lambda}\frac{w(\mathcal{X})}{\mu(\mathcal{X})}\|f\|_{L^1(\mathcal{X})}\\ &\le \frac{1}{\lambda}\int_{\mathcal{X}} |f(x)|\,Mw(x)\,d\mu(x).\end{aligned}$$

For each fixed bounded function f with bounded support, let $\tau_{\mathcal{X}} = 0$ if $\mu(\mathcal{X}) = \infty$ and $\tau_{\mathcal{X}} = \|f\|_{L^1(\mathcal{X})}\,(\mu(\mathcal{X}))^{-1}$ if $\mu(\mathcal{X}) < \infty$. For $\lambda > \tau_{\mathcal{X}}$, applying the Calderón–Zygmund decomposition to f at level λ (see [1]), we obtain a sequence of balls $\{B_j\}_j$ with pairwise disjoint interiors and a constant $c_{10} \ge 1$ such that:

(I) for any $x \in \mathcal{X} \setminus \bigcup_j c_{10}B_j$ and any ball B centered at x,

$$\frac{1}{\mu(B)}\int_B |f(x)|\,d\mu(x) \le \lambda;$$

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(II) for all j,

$$\frac{1}{\mu(c_{10}B_j)}\int_{c_{10}B_j}|f(y)|\,d\mu(y)\le\lambda\le\frac{1}{\mu(B_j)}\int_{B_j}|f(y)|\,d\mu(y).$$

As in [1, p. 146], if we set $V_1=c_{10}B_1\setminus\bigcup_{l\ge 2}B_l$, and for $j\ge 2$,

$$V_j=c_{10}B_j\Big\backslash\bigg(\bigcup_{l=1}^{j-1}V_l\bigcup\bigcup_{i\ge j+1}B_i\bigg),$$

it then follows that $B_j\subset V_j\subset c_{10}B_j$, $\bigcup_j V_j=\bigcup_j c_{10}B_j$ and $\{V_j\}_j$ are mutually disjoint. Define the functions g and a_j by

$$g(x)=f(x)\chi_{\mathcal{X}\setminus\bigcup_j V_j}(x)+\sum_j\bigg[\frac{1}{\mu(V_j)}\int_{V_j}f(y)\,d\mu(y)\bigg]\chi_{V_j}(x),$$

and

$$a_j(x)=\bigg[f(x)-\frac{1}{\mu(V_j)}\int_{V_j}f(y)\,d\mu(y)\bigg]\chi_{V_j}(x).$$

Let $E_\lambda=\bigcup_j\vartheta B_j$ with $\vartheta=c_{10}(\kappa+1)\kappa(c_2+1)$ and $a=\sum_j a_j$. By the doubling condition,

$$\begin{aligned}w(E_\lambda)&\lesssim\sum_j\frac{w(\vartheta B_j)}{\mu(\vartheta B_j)}\mu(B_j)\lesssim\sum_j\inf_{x\in B_j}Mw(x)\mu(B_j)\\&\lesssim\lambda^{-1}\sum_j\int_{B_j}|f(y)|\,Mw(y)\,d\mu(y)\lesssim\lambda^{-1}\int_{\mathcal{X}}|f(y)|\,Mw(y)\,d\mu(y).\end{aligned}$$

The proof of Theorem 1.6 can be reduced to proving that

$$w(\{x\in\mathcal{X}\setminus E_\lambda:\,|Tg(x)|>\lambda/2\})\lesssim\lambda^{-1}\int_{\mathcal{X}}|f(x)|\,M^3w(x)\,d\mu(x)\qquad(4.3)$$

and

$$w(\{x\in\mathcal{X}\setminus E_\lambda:\,|Ta(x)|>\lambda/2\})\lesssim\lambda^{-1}\int_{\mathcal{X}}|f(x)|\,M^3w(x)\,d\mu(x).\qquad(4.4)$$

We first prove (4.3). A trivial computation along with Theorem 1.5 with $p=4/3$

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gives

$$\begin{aligned}
&w(\{x \in \mathcal{X} \setminus E_\lambda : |Tg(x)| > \lambda/2\}) \\
&\quad \lesssim \lambda^{-4/3} \int_{\mathcal{X}} |Tg(x)|^{4/3} w(x) \chi_{\mathcal{X} \setminus E_\lambda}(x)\, d\mu(x) \\
&\quad \lesssim \lambda^{-4/3} \int_{\mathcal{X}} |g(x)|^{4/3} M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(x)\, d\mu(x) \\
&\quad \lesssim \lambda^{-1} \int_{\mathcal{X} \setminus \bigcup_j V_j} |f(x)|\, M^3 w(x)\, d\mu(x) \\
&\qquad + \lambda^{-1} \sum_j \int_{V_j} |g(x)|\, M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(x)\, d\mu(x).
\end{aligned}$$

Following an argument similar to the case of Euclidean spaces (see [10, p. 159]), we can verify that, for any $x \in c_{10} B_j$,

$$M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(x) \lesssim \inf_{y \in c_{10} B_j} M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(y). \tag{4.5}$$

Therefore,

$$\begin{aligned}
&\sum_j \int_{V_j} |g(x)|\, M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(x)\, d\mu(x) \\
&\quad \lesssim \sum_j \int_{V_j} |f(x)|\, d\mu(x) \inf_{y \in B_j} M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(y) \\
&\quad \lesssim \lambda \sum_j \mu(c_{10} B_j) \inf_{y \in B_j} M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(y) \\
&\quad \lesssim \lambda \sum_j \mu(B_j) \inf_{y \in B_j} M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(y) \\
&\quad \lesssim \sum_j \int_{B_j} |f(x)|\, d\mu(x) \inf_{y \in B_j} M^3(w\chi_{\mathcal{X} \setminus E_\lambda})(y) \\
&\quad \lesssim \int_{\mathcal{X}} |f(x)|\, M^3 w(x)\, d\mu(x),
\end{aligned}$$

which implies the estimate (4.3).

To prove (4.4), let r_j be the radius of B_j and $t_j = r_j^m$ for each fixed j. Write

$$\begin{aligned}
&w(\{x \in \mathcal{X} \setminus E_\lambda : |Ta(x)| > \lambda/2\}) \\
&\quad \le w\Bigg(\Bigg\{x \in \mathcal{X} \setminus E_\lambda : \Bigg|T\Bigg(\sum_j (a_j - D_{t_j} a_j)\Bigg)(x)\Bigg| > \lambda/4\Bigg\}\Bigg) \\
&\qquad + w\Bigg(\Bigg\{x \in \mathcal{X} \setminus E_\lambda : \Bigg|T\Bigg(\sum_j D_{t_j} a_j\Bigg)(x)\Bigg| > \lambda/4\Bigg\}\Bigg) \\
&\quad = I_1 + I_2.
\end{aligned}$$

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As it was pointed out in [4], we know that

$$T\Big(\sum_j (a_j - D_{t_j} a_j)\Big)(x) = \sum_j T(a_j - D_{t_j} a_j)(x).$$

On the other hand, a straightforward computation shows that for any fixed j,

$$\begin{aligned}
&\int_{\mathcal{X}} |a_j(y)|\, M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y) \\
&\quad \le \frac{1}{\mu(V_j)} \int_{V_j} |f(x)|\, d\mu(x) \int_{c_{10}B_j} M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y) \\
&\qquad + \int_{V_j} |f(y)|\, M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y) \\
&\quad \le \inf_{y\in c_{10}B_j} M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\mu(c_{10}B_j) \frac{1}{\mu(B_j)} \int_{c_{10}B_j} |f(x)|\, d\mu(x) \\
&\qquad + \int_{V_j} |f(y)|\, M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y) \\
&\quad \lesssim \lambda \inf_{y\in c_{10}B_j} M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\mu(B_j) \\
&\qquad + \int_{V_j} |f(y)|\, M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y) \\
&\quad \lesssim \int_{V_j} |f(y)|\, M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \lambda^{-1} \sum_j \int_{\mathcal{X}\setminus\vartheta B_j} \big|T(a_j - D_{t_j}a_j)(x)\big|\, w(x)\chi_{\mathcal{X}\setminus E_\lambda}(x)\, d\mu(x) \\
&= \lambda^{-1} \sum_j \int_{\mathcal{X}} |a_j(y)| \\
&\quad \times \bigg\{ \int_{\mathcal{X}\setminus\vartheta B_j} |K(x,y) - K_{t_j}(x,y)|\, w(x)\chi_{\mathcal{X}\setminus E_\lambda}(x)\, d\mu(x) \bigg\}\, d\mu(y) \\
&\lesssim \lambda^{-1} \sum_j \int_{\mathcal{X}} |a_j(y)|\, M(w\chi_{\mathcal{X}\setminus E_\lambda})(y)\, d\mu(y) \\
&\lesssim \lambda^{-1} \int_{\mathcal{X}} |f(y)| Mw(y)\, d\mu(y),
\end{aligned}$$

where in the second-to-last inequality, we have invoked an estimate in [9, Remark 5.9], which states that for some constant $C > 0$ depending only on the approximation to the identity $\{D_t\}_{t>0}$ such that for any weight v,

$$\int_{d(x,y)\ge c_3 t^{1/m}} |K(x,y) - K_{t_j}(x,y)|\, v(x)\, d\mu(x) \le C Mv(y).$$

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It remains to consider the term I_2. Obviously,

$$I_2 \lesssim \lambda^{-4/3} \int_{\mathcal{X}} \left| T\Big(\sum_j D_{t_j} a_j\Big)(x)\right|^{4/3} w(x)\chi_{\mathcal{X}\setminus E_\lambda}(x)\, d\mu(x).$$

Using (1.4) and (1.5) together with some basic estimates yields that, for any j, function v, and $y \in V_j$,

$$\int_{\mathcal{X}} h_{t_j}(x, y)|v(x)|\, d\mu(x) \lesssim \inf_{z\in V_j} Mv(z), \tag{4.6}$$

which gives us that

$$\begin{aligned}\left|\int_{\mathcal{X}} D_{t_j} a_j(x) v(x)\, d\mu(x)\right| &\le \int_{V_j} |a_j(y)| \left[\int_{\mathcal{X}} h_{t_j}(x, y)|v(x)|\, d\mu(x)\right] d\mu(y)\\ &\lesssim \int_{V_j} |a_j(y)|\, d\mu(y) \inf_{z\in V_j} Mv(z)\\ &\lesssim \lambda\mu(V_j) \inf_{z\in V_j} Mv(z)\\ &\lesssim \lambda \int_{\mathcal{X}} Mv(x)\chi_{V_j}(x)\, d\mu(x).\end{aligned}$$

Recall, by Lemma 3.3, that $(M^3(w\chi_{\mathcal{X}\setminus E_\lambda}))^{-3} \in A_\infty$. Let $\widetilde{T}$ be the *adjoint operator* of T. From the above estimate, Hölder's inequality, Theorem 1.2 with $k = 1$ and (4.2), we deduce that for $h \in L^4(\mathcal{X}, (w\chi_{\mathcal{X}\setminus E_\lambda})^{-3})$ with $\|h\|_{L^4(\mathcal{X},(w\chi_{\mathcal{X}\setminus E_\lambda})^{-3})} \le 1$,

$$\begin{aligned}&\left|\int_{\mathcal{X}} T\Big(\sum_j D_{t_j} a_j\Big)(x) h(x)\, d\mu(x)\right|\\ &\quad= \left|\int_{\mathcal{X}} \widetilde{T}h(x) \sum_j D_{t_j} a_j(x)\, d\mu(x)\right|\\ &\quad\lesssim \lambda \int_{\mathcal{X}} M(\widetilde{T}h)(x) \sum_j \chi_{V_j}(x)\, d\mu(x)\\ &\quad\lesssim \lambda \left\{\int_{\mathcal{X}} (M(\widetilde{T}h)(x))^4 (M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x))^{-3}\, d\mu(x)\right\}^{1/4}\\ &\qquad\times \left\{\int_{\mathcal{X}} \sum_j \chi_{V_j}(x) M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\, d\mu(x)\right\}^{3/4}\end{aligned}$$

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$$
\begin{aligned}
&\lesssim \lambda \left\{ \int_{\mathcal{X}} (M^2 h(x))^4 (M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(x))^{-3} \, d\mu(x) \right\}^{1/4} \\
&\quad \times \left\{ \int_{\mathcal{X}} \sum_j \chi_{V_j}(x) M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(x) \, d\mu(x) \right\}^{3/4} \\
&\lesssim \lambda \left\{ \int_{\mathcal{X}} \sum_j \chi_{V_j}(x) M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(x) \, d\mu(x) \right\}^{3/4}.
\end{aligned}
$$

This along with the inequality (4.5) leads to that

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \int_{\mathcal{X}} \sum_j \chi_{c_{10}B_j}(x) M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(x) \, d\mu(x) \\
&\lesssim \sum_j \inf_{y\in c_{10}B_j} M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(y) \mu(B_j) \\
&\lesssim \lambda^{-1} \sum_j \inf_{y\in c_{10}B_j} M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(y) \int_{B_j} |f(x)| \, d\mu(x) \\
&\lesssim \lambda^{-1} \int_{\mathcal{X}} |f(x)| \, M^3 (w\chi_{\mathcal{X}\setminus E_\lambda})(x) \, d\mu(x),
\end{aligned}
$$

which then completes the proof of Theorem 1.6. □

5. Proofs of Theorems 1.7 and 1.8

We begin with a generalization of Hölder's inequality.

LEMMA 5.1. *Let k be a nonnegative integer. Then there exists a constant $C > 0$ depending only on k such that for any measurable set V with $\mu(V) < \infty$, functions v_1 and v_2,*

$$
\|v_1 v_2\|_{L(\log L)^k, V} \le C \|v_1\|_{\exp L, V} \|v_2\|_{L(\log L)^{k+1}, V};
$$

here and in what follows $\|v_1\|_{\exp L, V}$ denotes the norm defined by

$$
\|v_1\|_{\exp L, V} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(V)} \int_V \exp\left(\frac{|v_1(x)|}{\lambda} \right) d\mu(x) \le 2 \right\}.
$$

For the proof of Lemma 5.1, see [17, p. 58].

LEMMA 5.2. *Under the hypotheses of Theorem 1.7, there exists a constant $C > 0$ such that for any bounded function f with bounded support and $x \in \mathcal{X}$,*

$$
M^{\sharp}_{\widetilde{D}}(T_b f)(x) \le C \|b\|_{\mathrm{BMO}\,(\mathcal{X})} (M^2 (Tf)(x) + M^3 f(x)). \tag{5.1}
$$

https://doi.org/10.1017/S1446788708000657 Published online by Cambridge University Press

PROOF. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}\,(\mathcal{X})} = 1$. For each fixed $x \in \mathcal{X}$ and each fixed ball B containing x, let r_B be the radius of B. Recall, by [5], that T_b is bounded on $L^p(\mathcal{X})$ with $p \in (1, \infty)$. For a bounded function f with bounded support, we decompose f as

$$f(y) = f(y)\chi_{c_9 B}(y) + f(y)\chi_{\mathcal{X}\setminus c_9 B}(y) = f_1(y) + f_2(y).$$

Let $m_B(b)$ be the *mean value of b on B*, namely, $m_B(b) = (1/\mu(B)) \int_B b(y)\, d\mu(y)$. Write

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_B |T_b f(y) - \widetilde{D}_{t_B}(T_b f)(y)|\, d\mu(y) \\
&\quad \le \frac{1}{\mu(B)} \int_B |b(y) - m_B(b)||Tf(y)|\, d\mu(y) \\
&\qquad + \frac{1}{\mu(B)} \int_B |T((b - m_B(b)) f_1)(y)|\, d\mu(y) \\
&\qquad + \frac{1}{\mu(B)} \int_B |\widetilde{D}_{t_B}((b - m_B(b)) Tf)(y)|\, d\mu(y) \\
&\qquad + \frac{1}{\mu(B)} \int_B |\widetilde{D}_{t_B} T((b - m_B(b)) f_1)(y)|\, d\mu(y) \\
&\qquad + \frac{1}{\mu(B)} \int_B |T((b - m_B(b)) f_2)(y) - \widetilde{D}_{t_B} T((b - m_B(b)) f_2)(y)|\, d\mu(y) \\
&\quad = \sum_{j=1}^{5} U_j.
\end{aligned}$$

It follows from Lemma 5.1 that

$$U_1 \lesssim \|b - m_B(b)\|_{\exp L, B} \|Tf\|_{L\log L, B} \lesssim M^2(Tf)(x),$$

where we have employed the *John–Nirenberg inequality*, which says that for some positive constants c_{11} and $c_{12} > 1$,

$$\frac{1}{\mu(B)} \int_B \exp\left(\frac{|b(x) - m_B(b)|}{c_{11}\|b\|_{\mathrm{BMO}\,(\mathcal{X})}}\right) d\mu(x) \le c_{12}$$

(see [2]) and so

$$\|b - m_B(b)\|_{\exp L, B} \lesssim \|b\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

On the other hand, by Lemmas 3.1 and 5.1,

$$U_2 \lesssim \|(b - m_B(b)) f_1\|_{L\log L, c_9 B} \lesssim \|f\|_{L(\log L)^2, c_9 B} \lesssim M^3 f(x).$$

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As in the proof of (3.1), it follows from (1.4), (1.5) and Lemma 5.1 that

$$
\begin{aligned}
U_3 &\le \frac{1}{\mu(B)} \int_{2c_9 B} \left\{ \int_B h_{t_B}(y,z)\, d\mu(y) \right\} |b(z) - m_B(b)||Tf(z)|\, d\mu(z) \\
&\quad + \frac{1}{\mu(B)} \sum_{j=1}^{\infty} \int_{2^{j+1}c_9 B \setminus 2^j c_9 B} \left\{ \int_B h_{t_B}(y,z)\, d\mu(y) \right\} \\
&\qquad \times |b(z) - m_B(b)||Tf(z)|\, d\mu(z) \\
&\lesssim \frac{1}{\mu(B)} \int_{2c_9 B} |b(z) - m_B(b)||Tf(z)|\, d\mu(z) \\
&\quad + \frac{1}{\mu(B)} \sum_{j=1}^{\infty} s(2^{m(j-1)}) |m_{2^{j+1}c_9 B}(b) - m_B(b)| \int_{2^{j+1}c_9 B} |Tf(z)|\, d\mu(z) \\
&\quad + \frac{1}{\mu(B)} \sum_{j=1}^{\infty} s(2^{m(j-1)}) \int_{2^{j+1}c_9 B} |b(z) - m_{2^{j+1}c_9 B}(b)||Tf(z)|\, d\mu(z) \\
&\lesssim \|Tf\|_{L\log L, 2c_9 B} + \sum_{j=1}^{\infty} j 2^{jn} s(2^{m(j-1)}) \frac{1}{\mu(B)} \|Tf\|_{L^1, 2^{j+1}c_9 B} \\
&\quad + \sum_{j=1}^{\infty} 2^{jn} s(2^{m(j-1)}) \|Tf\|_{L\log L, 2^{j+1}c_9 B} \\
&\lesssim M_{L\log L}(Tf)(x) \\
&\lesssim M^2(Tf)(x).
\end{aligned}
$$

Similarly, by (1.4), (1.5), Lemmas 3.1 and 5.1, we see that

$$
\begin{aligned}
U_4 &\lesssim \frac{1}{\mu(B)} \|T((b - m_B(b)) f_1)\|_{L^1, 2c_9 B} \\
&\quad + \frac{1}{\mu(B)} \sum_{j=1}^{\infty} 2^{jn} s(2^{m(j-1)}) \|T((b - m_B(b)) f_1)\|_{L^1, 2^{j+1}c_9 B \setminus 2^j c_9 B} \\
&\lesssim \|(b - m_B(b)) f_1\|_{L\log L, c_9 B} \\
&\lesssim M^3 f(x).
\end{aligned}
$$

For the term U_5, the condition (ii) via a standard argument gives us that, for any $y \in B$,

$$
\begin{aligned}
&|T((b - m_B(b)) f_2)(y) - \widetilde{D}_{t_B} T((b - m_B(b)) f_2)(y)| \\
&\quad \lesssim r_B^{\alpha} \int_{d(y,z) \ge c_2 r_B} \frac{|b(z) - m_B(b)|}{\mu(B(y, d(y,z)))} \frac{|f(z)|}{(d(y,z))^{\alpha}}\, d\mu(z) \\
&\quad \lesssim \sum_{i=1}^{\infty} 2^{-i\alpha} \frac{1}{\mu(B(y, 2^i c_2 r_B))} \int_{d(y,z) \le 2^i c_2 r_B} |b(z) - m_B(b)||f(z)|\, d\mu(z)
\end{aligned}
$$

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$$\begin{aligned}
&\lesssim \sum_{i=1}^{\infty} 2^{-i\alpha} \frac{1}{\mu(B(y, 2^i c_2 r_B))} \\
&\quad \times \int_{d(y,z)\le 2^i c_2 r_B} |b(z) - m_{B(y,2^i c_2 r_B)}(b)||f(z)|\, d\mu(z) \\
&\quad + \sum_{i=1}^{\infty} 2^{-i\alpha} |m_B(b) - m_{B(y,2^i c_2 r_B)}(b)| \\
&\quad \times \frac{1}{\mu(B(y, 2^i c_2 r_B))} \int_{d(y,z)\le 2^i c_2 r_B} |f(z)|\, d\mu(z) \\
&\lesssim \inf_{z\in B} M^2 f(z).
\end{aligned}$$

Combining the estimates for U_j $(1 \le j \le 5)$ then establishes (5.1). □

Proof of Theorem 1.7. We assume that $\|b\|_{\mathrm{BMO}\,(\mathcal{X})} = 1$ and $M^{[3p]+1}w$ is finite almost everywhere. As in the proof of Theorem 1.5, it suffices to prove that for any $p \in (1, \infty)$ and any bounded function f with bounded support,

$$\begin{aligned}
&\int_{\mathcal{X}} (M(T_b f)(x))^{p'} (M^{[3p]+1} w(x))^{1-p'}\, d\mu(x) \\
&\qquad \lesssim \int_{\mathcal{X}} (M^3 f(x))^{p'} (M^{[3p]+1} w(x))^{1-p'}\, d\mu(x). \qquad (5.2)
\end{aligned}$$

To prove (5.2), applying the $L^{p'}(\mathcal{X})$-boundedness of T_b, we see that for any $\epsilon > 0$,

$$\int_{\mathcal{X}} (M(T_b f)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \lesssim \epsilon^{1-p'} \|f\|^{p'}_{L^{p'}(\mathcal{X})} < \infty.$$

Recall, by Lemma 3.3, that $(M^{[3p]+1}w)^{1-p'} \in A_\infty$. If $\mu(\mathcal{X}) = \infty$, Theorem 2.1, along with Lemma 5.2 and Theorem 1.2 with $k = 2$, states that

$$\begin{aligned}
&\int_{\mathcal{X}} (M(T_b f)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\lesssim \int_{\mathcal{X}} (M^{\sharp}_{\tilde{D}}(T_b f)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\lesssim \int_{\mathcal{X}} (M^2(Tf)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\quad + \int_{\mathcal{X}} (M^3 f(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\lesssim \int_{\mathcal{X}} (M^3 f(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x). \qquad (5.3)
\end{aligned}$$

For the case of $\mu(\mathcal{X}) < \infty$, observe that for any ball $B \subset \mathcal{X}$ and any positive integer j,

$$|m_B(b) - m_{2^j B}(b)| \le \frac{1}{\mu(B)} \int_B |b(x) - m_{2^j B}(b)|\, d\mu(x) \lesssim \frac{\mu(2^j B)}{\mu(B)} \lesssim \frac{\mu(\mathcal{X})}{\mu(B)},$$

and

$$\begin{aligned}
&|m_{\mathcal{X}}(b) - m_B(b)| \\
&\quad \le \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |b(x) - m_B(b)|\, d\mu(x) \\
&\quad \le \frac{1}{\mu(\mathcal{X})} \lim_{j\to\infty} \bigg(\int_{2^j B} |b(x) - m_{2^j B}(b)|\, d\mu(x) + \mu(2^j B)|m_B(b) - m_{2^j B}(b)| \bigg) \\
&\quad \lesssim \frac{\mu(\mathcal{X})}{\mu(B)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} \exp\bigg(\frac{|b(x) - m_{\mathcal{X}}(b)|}{c_{11}} \bigg)\, d\mu(x) \\
&\quad \le \lim_{j\to\infty} \frac{1}{\mu(\mathcal{X})} \int_{2^j B} \exp\bigg(\frac{|b(x) - m_{2^j B}(b)|}{c_{11}} \bigg)\, d\mu(x) \exp\bigg(\frac{|m_{2^j B}(b) - m_{\mathcal{X}}(b)|}{c_{11}} \bigg) \\
&\quad \le c_{12} \lim_{j\to\infty} \exp\bigg(\frac{|m_{2^j B}(b) - m_{\mathcal{X}}(b)|}{c_{11}} \bigg) \\
&\quad \lesssim c_{12} \lim_{j\to\infty} \exp\bigg(\frac{\mu(\mathcal{X})}{c_{11}\mu(2^j B)} \bigg) \\
&\quad \lesssim 1.
\end{aligned}$$

It then follows from Lemmas 5.1 and 3.1 that

$$\begin{aligned}
&\int_{\mathcal{X}} |T_b f(x)|\, d\mu(x) \\
&\quad \le \int_{\mathcal{X}} |b(x) - m_{\mathcal{X}}(b)||Tf(x)|\, d\mu(x) + \int_{\mathcal{X}} |T((b - m_{\mathcal{X}}(b))f)(x)|\, d\mu(x) \\
&\quad \lesssim \inf_{x\in\mathcal{X}} M^2(Tf)(x) + \|(b - m_{\mathcal{X}}(b))f\|_{L\log L, \mathcal{X}} \\
&\quad \lesssim \inf_{x\in\mathcal{X}} (M^2(Tf)(x) + M^3 f(x)).
\end{aligned}$$

Again by Theorem 2.1, Lemma 5.2 and Theorem 1.2 with $k = 2$,

$$\begin{aligned}
&\int_{\mathcal{X}} (M(T_b f)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\quad \lesssim \int_{\mathcal{X}} (M^{\sharp}_{\widetilde{D}}(T_b f)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\qquad + \|T_b f\|^{p'}_{L^1(\mathcal{X})} \int_{\mathcal{X}} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x)
\end{aligned}$$

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$$
\begin{aligned}
&\lesssim \int_{\mathcal{X}} (M^2(Tf)(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\quad + \int_{\mathcal{X}} (M^3 f(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x) \\
&\lesssim \int_{\mathcal{X}} (M^3 f(x))^{p'} (M^{[3p]+1}(w+\epsilon)(x))^{1-p'}\, d\mu(x). \qquad (5.4)
\end{aligned}
$$

The inequality (5.2) follows by letting $\epsilon \to 0$ in (5.3) and (5.4), which completes the proof of Theorem 1.7. □

PROOF OF THEOREM 1.8. We use some ideas coming from [14]. Again we assume that $\|b\|_{\mathrm{BMO}\,(\mathcal{X})} = 1$. Let $\tau_{\mathcal{X}} = \|f\|_{L^1(\mathcal{X})}(\mu(\mathcal{X}))^{-1}$. As in the proof of Theorem 1.6, it suffices to consider the case that $\lambda > \tau_{\mathcal{X}}$. For each fixed $\lambda > \tau_{\mathcal{X}}$ and each bounded function f with bounded support, with the notation B_j, V_j, g, E_λ, a and a_j as in the proof of Theorem 1.6, we see that the proof of Theorem 1.8 can be reduced to proving that

$$
w(\{x \in \mathcal{X} \setminus E_\lambda : |T_b g(x)| > \lambda/2\}) \lesssim \lambda^{-1} \int_{\mathcal{X}} |f(x)|\, M^4 w(x)\, d\mu(x) \qquad (5.5)
$$

and

$$
\begin{aligned}
&w(\{x \in \mathcal{X} \setminus E_\lambda : |T_b a(x)| > \lambda/2\}) \\
&\quad \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log\left(2 + \frac{|f(x)|}{\lambda}\right) M^4 w(x)\, d\mu(x). \qquad (5.6)
\end{aligned}
$$

To prove (5.5), we apply Theorem 1.7 with $p = 5/4$ and obtain

$$
\begin{aligned}
&w(\{x \in \mathcal{X} \setminus E_\lambda : |T_b g(x)| > \lambda/2\}) \\
&\quad \lesssim \lambda^{-5/4} \int_{\mathcal{X}} |T_b g(x)|^{5/4}\, w(x) \chi_{\mathcal{X}\setminus E_\lambda}(x)\, d\mu(x) \\
&\quad \lesssim \lambda^{-5/4} \int_{\mathcal{X}} |g(x)|^{5/4}\, M^4(w \chi_{\mathcal{X}\setminus E_\lambda})(x)\, d\mu(x) \\
&\quad \lesssim \lambda^{-1} \int_{\mathcal{X}\setminus \bigcup_j V_j} |f(x)|\, M^4 w(x)\, d\mu(x) \\
&\qquad + \lambda^{-1} \sum_j \int_{V_j} |g(x)|\, M^4(w \chi_{\mathcal{X}\setminus E_\lambda})(x)\, d\mu(x),
\end{aligned}
$$

which via the argument used in the proof of (4.3) then yields (5.5).

We turn our attention to (5.6). Since f is bounded with bounded support, we can write

$$
T_b a(x) = \sum_j (b(x) - m_{B_j}(b)) T a_j(x) + T\left(\sum_j (m_{B_j}(b) - b) a_j\right)(x).
$$

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It follows from Theorem 1.6 and Lemma 5.1 that

$$
\begin{aligned}
w\bigg(\bigg\{x \in \mathcal{X} \setminus E_\lambda : \bigg|T\bigg(\sum_j (m_{B_j}(b) - b)a_j\bigg)(x)\bigg| > \lambda/4\bigg\}\bigg)
&\lesssim \lambda^{-1} \sum_j \int_{\mathcal{X}} |b(x) - m_{B_j}(b)||a_j(x)|\, M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\, d\mu(x)\\
&\lesssim \lambda^{-1} \sum_j \inf_{x \in c_{10}B_j} M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\mu(V_j)\\
&\quad \times \|b - m_{c_{10}B_j}(b)\|_{\exp L, V_j} \|a_j\|_{L\log L, V_j}\\
&\quad + \lambda^{-1} \sum_j \inf_{x \in c_{10}B_j} M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\|a_j\|_{L^1(\mathcal{X})}.
\end{aligned}
$$

Take $c_{11} > 1$ such that $(2c_{10}^n)^{1/c_{11}} < 2$. Note that, for any $\tau > 0$,

$$
\frac{1}{\mu(c_{10}B_j)} \int_{c_{10}B_j} \exp\bigg(\frac{|b(y) - m_{c_{10}B_j}(b)|}{\tau}\bigg)\, d\mu(y) \le 2
$$

implies

$$
\frac{1}{\mu(V_j)} \int_{V_j} \exp\bigg(\frac{|b(y) - m_{c_{10}B_j}(b)|}{\tau}\bigg)\, d\mu(y) \le 2c_{10}^n.
$$

It then follows from Hölder's inequality that

$$
\frac{1}{\mu(V_j)} \int_{V_j} \exp\bigg(\frac{|b(y) - m_{c_{10}B_j}(b)|}{c_{11}\tau}\bigg)\, d\mu(y) \le (2c_{10}^n)^{1/c_{11}} \le 2.
$$

Therefore,

$$
\|b - m_{c_{10}B_j}(b)\|_{\exp L, V_j} \le c_{11}\|b - m_{c_{10}B_j}(b)\|_{\exp L, c_{10}B_j} \lesssim 1.
$$

This via the fact that

$$
\begin{aligned}
\|a_j\|_{L\log L, V_j} &\le \lambda + \frac{1}{\mu(V_j)} \int_{V_j} |f(x)| \log\bigg(2 + \frac{|f(x)|}{\lambda}\bigg)\, d\mu(x)\\
&\lesssim \frac{1}{\mu(V_j)} \int_{V_j} |f(x)| \log\bigg(2 + \frac{|f(x)|}{\lambda}\bigg)\, d\mu(x)
\end{aligned}
$$

gives us that

$$
\begin{aligned}
w\bigg(\bigg\{x \in \mathcal{X} \setminus E_\lambda : \bigg|T\bigg(\sum_j (b - m_{B_j}(b))a_j\bigg)(x)\bigg| > \lambda/4\bigg\}\bigg)
&\lesssim \lambda^{-1} \sum_j \inf_{x \in c_{10}B_j} M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\mu(V_j)\|a_j\|_{L\log L, V_j}\\
&\lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log\bigg(2 + \frac{|f(x)|}{\lambda}\bigg) M^3(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\, d\mu(x).
\end{aligned}
$$

It remains to consider $\sum_j (b - m_{B_j}(b)) T a_j$. Write

$$\begin{aligned}
&\sum_j (b(x) - m_{B_j}(b)) T a_j(x) \\
&\quad = \sum_j (b(x) - m_{B_j}(b))(T a_j(x) - T D_{t_j} a_j(x)) \\
&\qquad + \sum_j (b(x) - m_{B_j}(b)) T D_{t_j} a_j(x) \\
&\quad = \sum_j (b(x) - m_{B_j}(b))(T a_j(x) - T D_{t_j} a_j(x)) \\
&\qquad + T_b\bigg(\sum_j D_{t_j} a_j\bigg)(x) + T\bigg(\sum_j (b - m_{B_j}(b)) D_{t_j} a_j\bigg)(x) \\
&\quad = W_1(x) + W_2(x) + W_3(x).
\end{aligned}$$

Let r_j be the radius of B_j and $t_j = r_j^m$. An argument involving the condition (iii) and Lemma 5.1 shows that for any j, $y \in V_j$ and weight v,

$$\begin{aligned}
&\int_{\mathcal{X} \setminus \vartheta B_j} |K(x, y) - K_{t_j}(x, y)| |b(x) - m_{B_j}(b)| \, v(x) \, d\mu(x) \\
&\quad \lesssim \sum_{i=1}^{\infty} \int_{c_3 2^{i-1} r_j \le d(x,y) < c_3 2^i r_j} \frac{|b(x) - m_{B_j}(b)|}{\mu(B(y, d(x, y)))} \frac{t_j^{\beta/m}}{[d(x, y)]^{\beta}} v(x) \, d\mu(x) \\
&\quad \lesssim \sum_{i=1}^{\infty} 2^{-\beta i} \frac{1}{\mu(c_3 2^i B)} \int_{d(x,y) < c_3 2^i r_j} |b(x) - m_{c_3 2^i B_j}(b)| \, v(x) \, d\mu(x) \\
&\qquad + \sum_{i=1}^{\infty} 2^{-\beta i} |m_{B_j}(b) - m_{c_3 2^i B_j}(b)| \frac{1}{\mu(c_3 2^i B_j)} \int_{d(x,y) < c_3 2^i r_j} v(x) \, d\mu(x) \\
&\quad \lesssim M^2 v(y),
\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}
&w(\{x \in \mathcal{X} \setminus E_\lambda : |W_1(x)| > \lambda/12\}) \\
&\quad \le \lambda^{-1} \sum_j \int_{\mathcal{X} \setminus \vartheta B_j} |T(a_j - D_{t_j} a_j)(x)| |b(x) - m_{B_j}(b)| \, w(x) \chi_{\mathcal{X} \setminus E_\lambda}(x) \, d\mu(x) \\
&\quad = \lambda^{-1} \sum_j \int_{\mathcal{X}} |a_j(y)| \int_{\mathcal{X} \setminus \vartheta B_j} |K(x, y) - K_{t_j}(x, y)| \\
&\qquad \times |b(x) - m_{B_j}(b)| \, w(x) \chi_{\mathcal{X} \setminus E_\lambda}(x) \, d\mu(x) \, d\mu(y) \\
&\quad \lesssim \lambda^{-1} \sum_j \int_{\mathcal{X}} |a_j(y)| \, M^2(w \chi_{\mathcal{X} \setminus E_\lambda})(y) \, d\mu(y) \\
&\quad \lesssim \lambda^{-1} \int_{\mathcal{X}} |f(y)| \, M^2 w(y) \, d\mu(y).
\end{aligned}$$

To deal with W_2, let $\widetilde{T}_b$ be the *adjoint operator* of T_b. If

$$h \in L^5(\mathcal{X}, (w\chi_{\mathcal{X}\setminus E_\lambda})^{-4})$$

with $\|h\|_{L^5(\mathcal{X},(w\chi_{\mathcal{X}\setminus E_\lambda})^{-4})} \le 1$, by (4.6), we obtain

$$\begin{aligned}
&\left|\int_{\mathcal{X}} T_b\Big(\sum_j D_{t_j}a_j\Big)(x)h(x)\,d\mu(x)\right| \\
&\quad = \left|\int_{\mathcal{X}} \widetilde{T}_b h(x)\sum_j D_{t_j}a_j(x)\,d\mu(x)\right| \\
&\quad \lesssim \lambda\int_{\mathcal{X}} M(\widetilde{T}_b h)(x)\sum_j \chi_{V_j}(x)\,d\mu(x) \\
&\quad \lesssim \lambda\left\{\int_{\mathcal{X}} (M(\widetilde{T}_b h)(x))^5(M^4(w\chi_{\mathcal{X}\setminus E_\lambda})(x))^{-4}\,d\mu(x)\right\}^{1/5} \\
&\qquad \times\left\{\int_{\mathcal{X}} \sum_j \chi_{V_j}(x)M^4(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\,d\mu(x)\right\}^{4/5} \\
&\quad \lesssim \lambda\left\{\int_{\mathcal{X}} (M^3 h(x))^5(M^4(w\chi_{\mathcal{X}\setminus E_\lambda})(x))^{-4}\,d\mu(x)\right\}^{1/5} \\
&\qquad \times\left\{\int_{\mathcal{X}} \sum_j \chi_{V_j}(x)M^4(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\,d\mu(x)\right\}^{4/5} \\
&\quad \lesssim \lambda\left\{\int_{\mathcal{X}} \sum_j \chi_{V_j}(x)M^4(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\,d\mu(x)\right\}^{4/5},
\end{aligned}$$

where the second inequality follows from Hölder's inequality, the third inequality follows from the estimate (5.2), and the last inequality follows from (4.2). Therefore, as in the proof of Theorem 1.6,

$$\begin{aligned}
&w(\{x \in \mathcal{X}\setminus E_\lambda : |W_2(x)| > \lambda/12\}) \\
&\quad \lesssim \lambda^{-5/4}\int_{\mathcal{X}} \left|T_b\Big(\sum_j D_{t_j}a_j\Big)(x)\right|^{5/4} w(x)\chi_{\mathcal{X}\setminus E_\lambda}(x)\,d\mu(x) \\
&\quad \lesssim \sum_j \int_{c_{10}B_j} M^4(w\chi_{\mathcal{X}\setminus E_\lambda})(x)\,d\mu(x) \\
&\quad \lesssim \lambda^{-1}\int_{\mathcal{X}} |f(x)|\,M^4 w(x)\,d\mu(x).
\end{aligned}$$

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Finally, we consider W_3. For $h \in L^5(\mathcal{X}, (w\chi_{\mathcal{X}\setminus E_\lambda})^{-4})$, each fixed j and $z \in B_j$, an argument involving (1.4), (1.5) and Lemma 5.1, yields

$$\begin{aligned}
&\int_{\mathcal{X}} h_{t_j}(y,z)|m_{B_j}(b)-b(y)||\widetilde{T}h(y)|\,d\mu(y)\\
&\quad\le \int_{2\kappa B_j} h_{t_j}(y,z)|m_{B_j}(b)-b(y)||\widetilde{T}h(y)|\,d\mu(y)\\
&\qquad+\sum_{i=1}^{\infty}\int_{2^{i+1}\kappa B_j\setminus 2^i\kappa B_j} h_{t_j}(y,z)|m_{B_j}(b)-b(y)||\widetilde{T}h(y)|\,d\mu(y)\\
&\quad\le \int_{2\kappa B_j} h_{t_j}(y,z)|m_{B_j}(b)-b(y)||\widetilde{T}h(y)|\,d\mu(y)\\
&\qquad+\sum_{i=1}^{\infty} s(2^{m(i-1)})|m_{B_j}(b)-m_{2^{i+1}\kappa B_j}(b)|\\
&\qquad\times\int_{2^{i+1}\kappa B_j\setminus 2^i\kappa B_j}\frac{|\widetilde{T}h(y)|}{\mu(B(y,r_j))}\,d\mu(y)\\
&\qquad+\sum_{i=1}^{\infty} s(2^{m(i-1)})\int_{2^{i+1}\kappa B_j\setminus 2^i\kappa B_j}\frac{1}{\mu(B(y,r_j))}\\
&\qquad\times|m_{2^{i+1}\kappa B_j}(b)-b(y)||\widetilde{T}h(y)|\,d\mu(y)\\
&\quad\lesssim \inf_{\xi\in B_j} M^2(\widetilde{T}h)(\xi)\\
&\qquad+\sum_{i=1}^{\infty} is(2^{m(i-1)})2^{iN}[\mu(B_j)]^{-1}\mu(2^i\kappa B_j)\inf_{\xi\in B_j} M(\widetilde{T}h)(\xi)\\
&\qquad+\sum_{i=1}^{\infty} s(2^{m(i-1)})2^{iN}[\mu(B_j)]^{-1}\mu(2^i\kappa B_j)\inf_{\xi\in B_j} M^2(\widetilde{T}h)(\xi)\\
&\quad\lesssim \inf_{\xi\in B_j} M^2(\widetilde{T}h)(\xi).
\end{aligned}$$

Thus, if $\|h\|_{L^5(\mathcal{X},(w\chi_{\mathcal{X}\setminus E_\lambda})^{-4})}\le 1$, it follows that

$$\begin{aligned}
&\left|\int_{\mathcal{X}} T\Big(\sum_j (b-m_{B_j}(b))D_{t_j}a_j\Big)(x)h(x)\,d\mu(x)\right|\\
&\quad\le \sum_j\int_{\mathcal{X}}\left\{\int_{\mathcal{X}} h_{t_j}(y,z)|m_{B_j}(b)-b(y)||\widetilde{T}h(y)|\,d\mu(y)\right\}|a_j(z)|\,d\mu(z)\\
&\quad\le \sum_j \inf_{z\in B_j} M^2(\widetilde{T}h)(z)\int_{\mathcal{X}}|a_j(z)|\,d\mu(z)\\
&\quad\lesssim \lambda\int_{\mathcal{X}} M^2(\widetilde{T}h)(y)\sum_j \chi_{B_j}(y)\,d\mu(y).
\end{aligned}$$

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As in the argument for the term W_2, another application of Theorem 1.2 with $k = 2$ and the inequality (4.2) yields

$$w(\{x \in \mathcal{X} \setminus E_\lambda : |W_3(x)| > \lambda/12\}) \lesssim \lambda^{-1} \int_{\mathcal{X}} |f(x)|\, M^4 w(x)\, d\mu(x).$$

This completes the proof of Theorem 1.8. □

6. Proofs of Theorems 1.9 and 1.10

We first show Theorem 1.9.

PROOF OF THEOREM 1.9. By an estimate of Duong and McIntosh [4] (see also [9]), we know that under the hypotheses of Theorem 1.9,

$$T^* f(x) \lesssim MTf(x) + Mf(x). \tag{6.1}$$

Thus, Theorem 1.9 follows from Theorem 1.5 and Lemma 2.4 directly. □

PROOF OF THEOREM 1.10. By (2.2) with $k = 0$ and (6.1), it suffices to prove that if $u \in A_1$, then for any $\lambda > 0$ and bounded function f with bounded support,

$$u(\{x \in \mathcal{X} : M(Tf)(x) > \lambda\}) \lesssim \int_{\mathcal{X}} \frac{|f(x)|}{\lambda} \log\left(2 + \frac{|f(x)|}{\lambda}\right) u(x)\, d\mu(x). \tag{6.2}$$

An application of Theorem 1.3 with $k = 1$ together with Lemma 2.4 then gives us that

$$\begin{aligned}
&u(\{x \in \mathcal{X} : M(Tf)(x) > 1\}) \\
&\quad \lesssim \sup_{\tau > 0} \tau \log^{-1}(2 + \tau^{-1}) u(\{x \in \mathcal{X} : M^2 f(x) > \tau\}) \\
&\quad \lesssim \int_{\mathcal{X}} |f(x)| \log(2 + |f(x)|) u(x)\, d\mu(x),
\end{aligned}$$

which via homogeneity leads to (6.2). □

7. Holomorphic functional calculi of elliptic operators

This section is devoted to some applications of our theorems in Section 1 to holomorphic functional calculi of elliptic operators and Schrödinger operators. We first review some necessary background.

For fixed ω and ν with $0 \le \omega < \nu < \pi$, define the *closed sector* S_ω in the complex plane $\mathbb{C}$ by

$$S_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \le \omega\} \cup \{0\}$$

and denote its *interior* by S_ω^0. Let $H(S_\nu^0)$ be the *space of holomorphic functions on* S_ν^0, and $H^\infty(S_\nu^0)$ be the *subspace of* $H(S_\nu^0)$ defined by

$$H^\infty(S_\nu^0) = \left\{ f \in H(S_\nu^0) : \|f\|_\infty = \sup_{\zeta \in S_\nu^0} |f(\zeta)| < \infty \right\}.$$

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Let $\Psi(S_\nu^0)$ and $F(S_\nu^0)$ respectively be the *function spaces* defined by

$$\Psi(S_\nu^0) = \{\psi \in H(S_\nu^0) : \text{ there is an } \varepsilon > 0 \\ \text{such that } |\psi(\zeta)| \le C|\zeta|^\varepsilon (1 + |\zeta|^{2\varepsilon})^{-1}\}$$

and

$$F(S_\nu^0) = \{\psi \in H(S_\nu^0) : \text{ there is an } \varepsilon > 0 \text{ such that } |\psi(\zeta)| \le C(|\zeta|^{-\varepsilon} + |\zeta|^\varepsilon)\}.$$

It then follows that

$$\Psi(S_\nu^0) \subset H^\infty(S_\nu^0) \subset F(S_\nu^0).$$

Let $0 \le \omega < \pi$. A closed operator L on some Banach space $\mathbb{A}$ is said to be *of type* ω, if its spectrum $\sigma(L) \subset S_\omega$ and for each $\nu > \omega$, there exists a constant C_ν such that

$$\|(L - \zeta I)^{-1}\| \le C_\nu |\zeta|^{-1} \quad \forall \zeta \notin S_\nu.$$

By the Hille–Yosida theorem, an operator L of type ω with $\omega < \pi/2$ is the generator of a bounded holomorphic semigroup e^{-zL} on the sector S_ν^0 with $\nu = \pi/2 - \omega$.

Now let L be a *one-to-one operator of type* ω *with dense domain and dense range in* $\mathbb{A}$. The *functional calculi* of L can be defined as follows.

If $\psi \in \Psi(S_\nu^0)$, then

$$\psi(L) = \frac{1}{2\pi i} \int_\Gamma (L - \zeta I)^{-1} \psi(\zeta)\, d\zeta,$$

where Γ is the contour $\{\zeta = re^{\pm i\theta} : r \ge 0\}$ parameterized clockwise around S_ω and $\omega < \theta < \nu$. As pointed out in [4], this integral is absolutely convergent in $\mathcal{L}(\mathbb{A})$, and the definition is independent of the choice of $\theta \in (\omega, \nu)$.

Now we take $f \in F(S_\nu^0)$ satisfying $|f(\zeta)| \le C(|\zeta|^{-k} + |\zeta|^k)$ for certain $C > 0$ and $k > 0$, and any $\zeta \in S_\nu^0$, and choose

$$\psi(\zeta) = \left(\frac{\zeta}{(1+\zeta)^2}\right)^{k+1}.$$

Then ψ, $f\psi \in \Psi(S_\nu^0)$ and $\psi(L)$ is one-to-one. Therefore, $(f\psi)(L)$ is a bounded operator on $\mathbb{A}$, and $(\psi(L))^{-1}$ is a closed operator on $\mathbb{A}$. Now we define

$$f(L) = (\psi(L))^{-1} (f\psi)(L).$$

As in [4, 5, 9], we obtain the following Theorem 7.1 from Theorems 1.5 and 1.6, and Theorem 7.2 from Theorems 1.7 and 1.8. We omit the details.

THEOREM 7.1. *Let u be a weight on $\mathcal{X}$ and Ω be a measurable set of a space of homogeneous type $(\mathcal{X}, d, \mu)$. Let $0 \le \omega < \nu \le \pi$ and L be an operator of type $\omega < \pi/2$, so that $-L$ generates a holomorphic semigroup e^{-zL}, in the set $0 \le |\arg(z)| < \pi/2 - \omega$. Suppose that:*

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(a$_1$) *the holomorphic semigroup* e^{-zL}, $|\arg(z)| < \pi/2 - \omega$ *is represented by kernels* $a_z(x, y)$ *which satisfy that, for all* $\theta > \omega$, *an estimate*

$$|a_z(x, y)| \le C_\theta h_{|z|}(x, y),$$

for all $x, y \in \Omega$ *and* $|\arg(z)| < \pi/2 - \theta$, *where* h_t *is defined on* $\mathcal{X} \times \mathcal{X}$ *by Definition 1.1;*

(a$_2$) *the operator* L *has a bounded holomorphic functional calculus in* $L^2(\Omega)$, *that is, for any* $\nu > \omega$ *and* $f \in H^\infty(S_\nu^0)$, *the operator* $f(L)$ *satisfies*

$$\|f(L)\|_{L^2(\Omega)\to L^2(\Omega)} \le C_\nu \|f\|_\infty.$$

Then for any $p \in (1, \infty)$, $\nu > \omega$ *and* $f \in H^\infty(S_\nu^0)$,

$$\|f(L)\|_{L^p(\Omega, M^{[2p]+1}u)\to L^p(\Omega,u)} \le C_{p,\nu} \|f\|_\infty.$$

Furthermore, when $p = 1$ *we have that*

$$\|f(L)\|_{L^1(\Omega, M^3 u)\to L^{1,\infty}(\Omega,u)} \le C_\nu \|f\|_\infty.$$

THEOREM 7.2. *Let* u *be a weight on* $\mathcal{X}$, Ω *be a measurable set of a space of homogeneous type* $(\mathcal{X}, d, \mu)$ *and* b *be a function on* Ω *such that* $\widetilde{b}(x) = b(x)\chi_\Omega(x)$ *is in* BMO $(\mathcal{X})$. *Let* $0 \le \omega < \nu \le \pi$ *and* L *be an operator of type* $\omega < \pi/2$, *which is the same as in Theorem 7.1. For* $f \in H^\infty(S_\nu^0)$, *let* $f(L)_b$ *be defined as in (1.13) with* T *replaced by* $f(L)$. *Then for any* $p \in (1, \infty)$, $\nu > \omega$ *and* $f \in H^\infty(S_\nu^0)$,

$$\|f(L)_b\|_{L^p(\Omega, M^{[3p]+1}u)\to L^p(\Omega,u)} \le C_{p,\nu} \|\widetilde{b}\|_{\mathrm{BMO}\,(\mathcal{X})} \|f\|_\infty.$$

Furthermore, when $p = 1$ *we have that*

$$\|f(L)_b\|_{L\log L(\Omega, M^4 u)\to L^{1,\infty}(\Omega,u)} \le C_\nu \|\widetilde{b}\|_{\mathrm{BMO}\,(\mathcal{X})} \log(2 + \|\widetilde{b}\|_{\mathrm{BMO}\,(\mathcal{X})}) \|f\|_\infty,$$

where $\|f(L)_b\|_{L\log L(\Omega, M^4 u)\to L^{1,\infty}(\Omega,u)}$ *is the minimum constant* $C > 0$ *such that for all functions* h *satisfying* $\int_\Omega |h(x)| \log(2 + |h(x)|) M^4 u(x)\, d\mu(x) < \infty$,

$$\begin{aligned} &u(\{x \in \Omega : |f(L)_b(h)(x)| > \lambda\}) \\ &\quad \le C \int_\Omega \frac{|h(x)|}{\lambda} \log\left(2 + \frac{|h(x)|}{\lambda}\right) M^4 u(x)\, d\mu(x). \end{aligned}$$

At the end of this paper, we give two operators which satisfy the hypotheses of Theorems 7.1 and 7.2.

Let V be a nonnegative function on $\mathbb{R}^n$. The *Schrödinger operator with potential* V is defined by

$$L = -\Delta + V(x). \tag{7.1}$$

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The Trotter formula shows that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies an *upper bound of Gaussian type*, namely, there exist constant $C > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$0 < p_t(x, y) \leq \frac{C}{t^{n/2}} e^{-c|x-y|^2/t}.$$

As is well known, unless V satisfies additional conditions, the heat kernel may be a discontinuous function in the space variables.

Another example is the following elliptic operator. Let

$$Lf = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f \tag{7.2}$$

be an *elliptic divergence form operator of real, symmetric coefficients with Dirichlet boundary conditions* on a domain Ω of $\mathbb{R}^n$ which is defined by the variation method. This means that L is the positive self-adjoint operator associated with the form

$$Q(f, g) = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) \left(\frac{\partial}{\partial x_j} f(x) \right) \left(\frac{\partial}{\partial x_i} \bar{g}(x) \right) dx$$

on $Y \times Y$ by $\langle Lg, h \rangle = Q(g, h)$, where Y is the Sobolev space $H_0^1(\Omega)$. This operator also has Gaussian heat kernel bounds without any conditions on smoothness of the boundary of Ω.

As was pointed in [4, 5, 9], the operators L in both (7.1) and (7.2) satisfy the assumptions of Theorems 7.1 and 7.2. More general operators on open domains of $\mathbb{R}^n$ which possess Gaussian bounds can be found in [4, 5, 9].

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