Ergod. Th. & Dynam. Sys., (2024), **44**, 749–768 © The Author(s), 2023. Published by Cambridge University Press.
doi:10.1017/etds.2023.34

Dynamics of iteration operators on self-maps of locally compact Hausdorff spaces

CHAITANYA GOPALAKRISHNA†, MURUGAN VEERAPAZHAM‡ and WEINIAN ZHANG§

† *Statistics and Mathematics Unit, Indian Statistical Institute, R.V. College Post, Bengaluru-560059, India*
(e-mail: cberbalaje@gmail.com)
‡ *Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Mangalore-575 025, India*
(e-mail: murugan@nitk.edu.in)
§ *School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China*
(e-mail: matzwn@126.com)

(*Received* 15 *March* 2022 *and accepted in revised form* 26 *March* 2023)

Abstract. In this paper, we prove the continuity of iteration operators $\mathcal{J}_n$ on the space of all continuous self-maps of a locally compact Hausdorff space X and generally discuss dynamical behaviors of them. We characterize their fixed points and periodic points for $X = \mathbb{R}$ and the unit circle S^1. Then we indicate that all orbits of $\mathcal{J}_n$ are bounded; however, we prove that for $X = \mathbb{R}$ and S^1, every fixed point of $\mathcal{J}_n$ which is non-constant and equals the identity on its range is not Lyapunov stable. The boundedness and the instability exhibit the complexity of the system, but we show that the complicated behavior is not Devaney chaotic. We give a sufficient condition to classify the systems generated by iteration operators up to topological conjugacy.

Key words: Iteration operator, periodic points, chaos, Babbage equation, topological conjugacy
2020 Mathematics Subject Classification: 37C25 (Primary); 37B02, 39B12, 47H30 (Secondary)

1. *Introduction*

Iteration of a map is a composition of the map and itself. More precisely, the *nth order iterate* of a map $f : X \to X$ on a non-empty set X, denoted by f^n, is defined inductively by $f^n = f \circ f^{n-1}$ and $f^0 = \mathrm{id}$, the identity map. As known in [5, 20], iteration is one of the most important operations in contemporary mathematics because the dynamical system theory is based on iteration, numerical computation needs iteration, and all computer loop programs are iteration. Thus, iteration has attracted interests of research for a long time (see e.g. [2, 5, 6, 9, 11, 13, 14, 20, 27]).

Consider iteration of function f in a space $\mathcal{X}$ of functions and let $\mathcal{J}_n$ denote the correspondence

$$\mathcal{J}_n : f \in \mathcal{X} \mapsto f^n, \tag{1.1}$$

called the *nth order iteration operator*. This operator was proved to be continuous on $\mathcal{X} := \mathcal{C}([a, b])$, the space of continuous self-maps of a compact interval, in [28], and later on $\mathcal{C}(X)$ with a general compact metric space X in [23]. Furthermore, the authors [23] considered the semi-dynamical system $(\mathcal{C}(X), \mathcal{J}_n)$ generated by $\mathcal{J}_n$, which is the iteration semigroup $\{\mathcal{J}_n^k : k = 0, 1, \ldots\}$, and discussed its periodic points. They computed periodic points for $X = [a, b]$, gave boundedness of orbits and instability of the fixed points which are identity on their range in $\mathcal{C}([a, b])$, and proved that $\mathcal{J}_n$ is not topologically transitive in $\mathcal{C}(X)$ with a compact metric space X. They showed that $\mathcal{J}_n$ is not chaotic in Devaney's sense although boundedness with instability exhibits a complicated behavior.

In addition to the case that X is a compact interval, it is also interesting to discuss dynamics of iteration operators for $X = S^1$ (the unit circle in $\mathbb{C}$), a compact space without boundary, $X = \mathbb{R}$ without compactness, and more generally a locally compact Hausdorff space X, which can be a space of p-adic numbers (metrizable) or a manifold (not necessarily metrizable). So, we will generally consider X to be a locally compact Hausdorff space (not necessarily metrizable) and let $\mathcal{C}(X) := C^0(X, X)$ consist of all continuous self-maps of X, which is not necessarily a metric space but a topological space in the compact-open topology as shown in §2. We want to know what dynamical behaviors of an iteration operator are common for a general locally compact Hausdorff space X and what properties of an iteration operator in a specified X are not shared in another.

In this paper, we discuss dynamical behaviors of the semi-dynamical system $(\mathcal{C}(X), \mathcal{J}_n)$ for X to be a locally compact Hausdorff space. After making preliminaries on locally compact Hausdorff spaces X in §2, we prove continuity of the iteration operator $\mathcal{J}_n$ on $\mathcal{C}(X)$, which is the same as given in [23] for compact metric spaces X but for which we use a different tool 'iterated evaluation map' in the proof, and investigate periodic points of $\mathcal{J}_n$ in $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(S^1)$ in §3. We characterize all fixed points and periodic points of the system by discussing the Babbage equation, part of which looks similar to those of [23] but their applicability is enhanced to a larger extent. We show that in $\mathcal{C}(S^1)$, each $\mathcal{J}_n$ may have periodic points of period $k \geq 2$, whereas in $\mathcal{C}(\mathbb{R})$, each $\mathcal{J}_n$ only has fixed points. In §4, we discuss the stability for $\mathcal{J}_n$. First, we generally note that all orbits of $\mathcal{J}_n$ are bounded. Then we prove that in $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(S^1)$, every fixed point of $\mathcal{J}_n$ which is non-constant and equals the identity on its range is not Lyapunov stable. The boundedness and the instability exhibit the complexity of the system, but we observe that the complicated behavior is not Devaney chaotic. In §5, we give a sufficient condition to classify the systems generated by iteration operators up to topological conjugacy and prove that $(\mathcal{C}(X), \mathcal{J}_2)$ is not conjugate to $(\mathcal{C}(Y), \mathcal{J}_m)$ for every locally compact Hausdorff space Y and odd positive integer

m whenever $\mathcal{C}(X)$ contains an involutory map different from id. Finally, we give some remarks and leave some questions for future discussion in §6.

2. *Preliminaries*

In this section, we give some preliminaries for locally compact Hausdorff spaces, which we will work on in this paper. Unless mentioned particularly, let X, Y, Z denote topological spaces and $C(X, Y)$ the set of all continuous maps of X into Y. Define on $C(X, Y)$ the *compact-open topology* (see [21, p. 285]), that is, the topology generated by the collection of all finite intersections of sets in

$$\{\mathcal{F}(K, U) : K \text{ is a compact subset of } X \text{ and } U \text{ is open in } Y\},$$

where $\mathcal{F}(K, U) := \{f \in C(X, Y) : f(K) \subseteq U\}$. When Y is a metric space equipped with metric d, we have another topology on $C(X, Y)$, *viz.* the *topology of compact convergence* generated by

$$\{\mathcal{B}_K(f, \epsilon) : f \in C(X, Y), K \text{ is a compact subset of } X, \text{and } \epsilon > 0\},$$

where $\mathcal{B}_K(f, \epsilon) := \{g \in C(X, Y) : \sup\{d(f(x), g(x)) : x \in K\} < \epsilon\}$.

LEMMA 1

(i) [21, Theorem 46.7, p. 285] *If X is a compact space and Y a metric space with metric d, then on $C(X, Y)$, the uniform topology induced by the metric $\rho(f, g) :=$ $\sup\{d(f(x), g(x)) : x \in X\}$ and the topology of compact convergence coincide.*

(ii) [21, Theorem 46.8, p. 285] *If Y is a metric space with metric d, then on $C(X, Y)$, the compact-open topology and the topology of compact convergence coincide.*

LEMMA 2. [21, Theorem 46.10, p. 286] *If X is locally compact Hausdorff and $C(X, Y)$ have the compact-open topology, then the evaluation map $\mathcal{E} : X \times C(X, Y) \to Y$ defined by $\mathcal{E}(x, f) = f(x)$ is continuous.*

LEMMA 3. [21, Theorem 46.11, p. 287] *Let $C(X, Y)$ have the compact-open topology. If $h : X \times Z \to Y$ is continuous, then so is its induced map $F : Z \to C(X, Y)$ defined by $F(z)(x) = h(x, z)$.*

As defined in [1], a topological space X is said to be *hemicompact* if there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of compact sets of X such that if K is any compact subset of X, then $K \subseteq \bigcup_{i=1}^{k} K_{m_i}$ for some finitely many $K_{m_1}, K_{m_2}, \ldots, K_{m_k}$.

LEMMA 4

(i) [1, Theorem 7] *If X is hemicompact and Y is metrizable, then $C(X, Y)$ in the compact-open topology is metrizable with metric D given by*

$$D(f, g) := \sum_{j=1}^{\infty} \mu_j(f, g), \tag{2.2}$$

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

where

$$\mu_j(f,g) := \min\left\{\frac{1}{2^j}, \rho_j(f,g)\right\} \quad \textit{for all } j \in \mathbb{N}, \tag{2.3}$$

$$\rho_j(f,g) := \sup\{d(f(x), g(x)) : x \in K_j\} \quad \textit{for all } j \in \mathbb{N}, \tag{2.4}$$

with d inducing the topology of Y.

(ii) [17, Problem 1, p. 68] *If $C(X,Y)$ in the compact-open topology is metrizable, then X is hemicompact and Y is metrizable.*

3. *Continuity and periodic points*

Let X be a locally compact Hausdorff space. In this section, we prove $(\mathcal{C}(X), \mathcal{J}_n)$ to be a discrete semi-dynamical system indeed by showing the continuity of $\mathcal{J}_n$ on $\mathcal{C}(X)$.

THEOREM 1. *$\mathcal{J}_n$ is continuous on $\mathcal{C}(X)$ for each $n \in \mathbb{N}$.*

Proof. First, we prove by induction that the *iterated evaluation map* $\mathcal{E}_n : X \times \mathcal{C}(X) \to X$ defined by

$$\mathcal{E}_n(x, f) = f^n(x) \tag{3.5}$$

is continuous on $X \times \mathcal{C}(X)$ for each $n \in \mathbb{N}$. The case $n = 1$ follows by Lemma 2, where $Y = X$ and $\mathcal{E}_1 = \mathcal{E}$. Suppose that $\mathcal{E}_n$ is continuous for certain $n \geq 2$. To prove $\mathcal{E}_{n+1}$ is continuous on $X \times \mathcal{C}(X)$, consider the map $\mathcal{H}_n : X \times \mathcal{C}(X) \to X \times \mathcal{C}(X)$ defined by $\mathcal{H}_n = (\mathcal{E}_n, p)$, where $p : X \times \mathcal{C}(X) \to \mathcal{C}(X)$ is the projection map defined by $p(x, f) = f$. Since $\mathcal{E}_n$ and p are continuous, so is $\mathcal{H}_n$. Now,

$$(\mathcal{E} \circ \mathcal{H}_n)(x, f) = \mathcal{E}(\mathcal{E}_n(x, f), p(x, f)) = \mathcal{E}(f^n(x), f) = f^{n+1}(x) = \mathcal{E}_{n+1}(x, f)$$

for each $(x, f) \in X \times \mathcal{C}(X)$, implying that $\mathcal{E}_{n+1} = \mathcal{E} \circ \mathcal{H}_n$. Therefore, $\mathcal{E}_{n+1}$ is continuous, being the composition of continuous maps $\mathcal{E}$ and $\mathcal{H}_n$. Hence, the continuity of $\mathcal{E}_n$ is proved for all $n \in \mathbb{N}$.

In fact, by putting $Y = X$, $Z = C(X)$, and $h = \mathcal{E}_n$ in Lemma 3, we get $F : C(X) \to C(X)$ and $F(f)(x) = h(x, f) = \mathcal{E}_n(x, f) = \mathcal{J}_n(f)(x)$ for all $f \in \mathcal{C}(X)$ and $x \in X$, implying that $F = \mathcal{J}_n$. Thus, by Lemma 3, we see that $\mathcal{J}_n$ is continuous on $\mathcal{C}(X)$ for each $n \in \mathbb{N}$. □

Theorem 1 looks the same, but is more general than [23, Theorem 2.1]. Unlike [23], Theorem 1 does not assume X to be metrizable, which causes more difficulties in its proof. In the special case that X is a compact metric space with a metric d, the uniform topology induced by the metric ρ gives the compact-open topology on $\mathcal{C}(X)$ because of Lemma 1.

As indicated in §1, $\mathcal{J}_n$ defines a discrete semi-dynamical system on the space $\mathcal{C}(X)$ of continuous functions with its iteration semigroup $\{\mathcal{J}_n^k : k \geq 0\}$. Remark that $\mathcal{J}_n$ usually does not define a dynamical system because a homeomorphism f of a compact interval may have infinitely many iterative roots as indicated in [14].

3.1. *Periodic points in general case.*

One of the most fundamental problems on dynamical systems or semi-dynamical systems concerns fixed points and periodic points.

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

To emphasize the dependence on the space X, we use $\mathrm{Fix}(f; X)$ and $\mathrm{Per}(f; X)$ to denote the set of all fixed points and the set of all periodic points of f in X, respectively. Since $\mathcal{J}_n^k f = f^{n^k}$ by equation (1.1), we see that $f \in \mathcal{C}(X)$ is a fixed point of $\mathcal{J}_n$ if and only if f satisfies the functional equation

$$f^n = f, \tag{3.6}$$

and $f \in \mathcal{C}(X)$ is a k-periodic point of $\mathcal{J}_n$ if and only if f satisfies

$$f^{n^k} = f, \ f^{n^{k-1}} \neq f \quad \text{and} \quad f^{n^i} \neq f \quad \text{for all } i = 2, \ldots, k-2 \quad \text{with } i \nmid (k-1). \tag{3.7}$$

So, the fixed points and periodic points of $\mathcal{J}_n$ are related to solutions of the Babbage equation [14]

$$\phi^m = \mathrm{id}, \tag{3.8}$$

where $m > 0$ is a certain integer. The functional equations in equations (3.6) and (3.7) cannot be simply treated as the Babbage equation (3.8) because the range of f may not be the whole X. In what follows, we need to consider restriction of maps. For any $f \in \mathcal{C}(X)$ and $A \subseteq X$, let $\bar{A}$ denote the closure of A, $R(f)$ the range of f, and $f|_A$ the restriction of f to A. As in [21], a space X is said to be *first countable* if each point has a countable local basis, that is, for each $x \in X$, there exists a countable collection $\mathcal{B}_x$ of neighborhoods of x such that each open set U containing x contains an element B of $\mathcal{B}_x$.

LEMMA 5. *Let $n \in \mathbb{N}$ and suppose that X is first countable. Then $f \in \mathcal{C}(X)$ is a solution of the equation*

$$\phi^n = \phi \tag{3.9}$$

on X if and only if $R(f)$ is closed and there exists $g \in \mathcal{C}(X)$ such that $R(g) = R(f)$, $g|_{R(g)}$ satisfies the Babbage equation

$$\phi^{n-1} = \mathrm{id}$$

on $R(g)$, and $f|_{R(g)} = g|_{R(f)}$.

Proof. If $f \in \mathcal{C}(X)$ is a solution of equation (3.9), then $f^{n-1}(f(x)) = f(x)$ for all $x \in X$, that is, $f^{n-1}(y) = y$ for all $y \in R(f)$. Also, since X is first countable, for each $y \in \overline{R(f)}$, there exists a sequence (y_n) in $R(f)$ such that $y_k \to y$, implying by continuity of f that $f(y_k) \to f(y)$ as $k \to \infty$. However, $y_k = f^{n-1}(y_k) \to f^{n-1}(y)$ as $k \to \infty$. Therefore, as X is Hausdorff, we have $f(y) = y$. Thus, $y \in R(f)$, and so $\overline{R(f)} = R(f)$. Let $g := f$. Then f is a continuous extension of the solution $g|_{R(g)}$ on $R(g)$ to X such that $R(f)$ is closed, $R(g) = R(f)$, and $f|_{R(g)} = g|_{R(f)}$. The proof of the converse is trivial, since for every $g \in \mathcal{C}(X)$, whose restriction $g|_{R(g)}$ satisfies $\phi^{n-1} = \mathrm{id}$ on $R(f)$, we have $f^n(x) = f^{n-1}(f(x)) = g^{n-1}(f(x)) = f(x)$ for all $x \in X$. □

Usually, a solution ϕ of equation (3.8) is referred to as an *mth order unit iterative root* if m is the smallest positive integer such that equation (3.8) is satisfied. Clearly, each unit iterative root is invertible. As in [14, p. 290], for $m = 2$, every solution of equation (3.8),

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

whose inverse is itself, is called an *involutory function*. In general, if f is an mth order unit iterative root and $f^k = \mathrm{id}$, then m divides k clearly. On a general set E, the mth order unit iterative roots are formulated below.

LEMMA 6. [14, Theorem 15.1] *Let* $\{m_0, \ldots, m_r\}$, *where* $1 = m_0 < m_1 < \cdots < m_r = m$, *be the complete set of divisors of* m *and let* $E = \bigcup_{i=0}^{r} \bigcup_{j=1}^{m_i} U_j^i$ *be a decomposition of E into disjoint sets such that the sets* $U_1^i, U_{m_1}^i, \ldots, U_{m_i}^i$ *have the same cardinality for each* $1 \le i \le r$. *For* $1 \le i \le r$ *and* $1 \le j \le m_i - 1$, *let* f_{ij} *be an arbitrary one-to-one map of* U_j^i *onto* U_{j+1}^i. *Then the formula*

$$f(x) := \begin{cases} x & \text{for } x \in U_1^0, \\ f_{ij}(x) & \text{for } x \in U_j^i,\ j = 1, 2, \ldots, m_i - 1,\ i \ge 1, \\ f_{i1}^{-1}(\cdots(f_{i,m_i-1}^{-1}(x))\cdots) & \text{for } x \in U_{m_i}^i,\ i \ge 1 \end{cases}$$

defines the general solution of $\phi^m = \mathrm{id}$ *on E.*

Having Lemma 6, we are ready to define

$$\mathcal{U}_E^m := \{f \in \mathcal{C}(X) : f|_E \text{ is an } m\text{th order unit iterative root on } E \text{ and } R(f) = E\}$$

for any subset E of X and $m \in \mathbb{N}$. Since $\mathrm{Fix}(\mathcal{J}_1; X) = \mathcal{C}(X)$, that is, the problem of fixed points of $\mathcal{J}_1$ is trivial, we focus on $\mathcal{J}_n$ with $n \ge 2$. We give the following results on fixed points and periodic points, whose proof is similar to that of [23, Theorem 2.4].

THEOREM 2. *Let* $n, k \ge 2$ *be integers and suppose that X is first countable. Then: (i)* $f \in \mathcal{C}(X)$ *is a fixed point of* $\mathcal{J}_n$ *if and only if* $f \in \mathcal{U}_E^m$ *for a closed subset E of X and an integer* $m \ge 1$ *dividing* $n - 1$ *exactly; (ii)* $f \in \mathcal{C}(X)$ *is a k-periodic point of* $\mathcal{J}_n$ *if and only if* $f \in \mathcal{U}_E^m$ *for a closed subset E of X and an integer* $m > 1$ *satisfying that*

$$m \mid (n^k - 1) \quad \text{and} \quad m \nmid (n^j - 1) \quad \text{for } 1 \le j \le k - 1. \tag{3.10}$$

Remark that we were able to infer that E is compact in [23] because E is indeed $R(f)$ and domain X was assumed to be compact, but in Theorem 2, we can only conclude that E is closed because of using Lemma 5.

Example 1. Consider $f(x) = |x|$. Then $f \in \mathcal{C}(\mathbb{R})$. Clearly, $f \in \mathcal{U}_{[0,\infty)}^1$. By result (i) of Theorem 2, f is a fixed point of $\mathcal{J}_n$ for each $n \ge 2$.

Example 2. The *doubling map* $f \in \mathcal{C}(S^1)$ defined by $f(e^{it}) = e^{2it}$ is not a k-periodic point of $\mathcal{J}_n$ for $n, k \ge 2$. In fact, if $f^{n^k} = f$ for some $n, k \ge 2$, then $e^{it(2^{n^k}-2)} = 0$ for all $t \in [0, 2\pi)$, which is a contradiction.

Example 3. The *rotation map* $R_{2\pi/(n^k-1)} : S^1 \to S^1$ defined by $R_{2\pi/(n^k-1)}(e^{it}) = e^{i(t+(2\pi/(n^k-1)))}$ is a k-periodic point of $\mathcal{J}_n$ in $\mathcal{C}(S^1)$ for $n \ge 2$ and $k \ge 1$.

3.2. *Periodic points in* $\mathcal{C}(\mathbb{R})$. For more detailed results, in this section, we focus on the case $X = \mathbb{R}$, the real line in the usual topology. For $a, b \in \mathbb{R}$ with $a < b$, let $|a, b|$ denote either an open interval (a, b), a semi-closed interval $[a, b)$ or $(a, b]$, or a closed interval

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

$[a, b]$, where one or both of the endpoints may be infinite. We have the following result, which can also be deduced from Lemma 6.

LEMMA 7. [18, 24] *If $f \in \mathcal{C}(|a, b|)$ is a solution of equation (3.8), then either $f = \mathrm{id}$ or f is a decreasing involutory function on $|a, b|$. More precisely, any solution of equation (3.8) on $|a, b|$ for general m is also a second-order unit iterative root. If m is odd, then the solution is uniquely the identity* id*; if m is even, then the solution is either id or a decreasing involutory function on $|a, b|$.*

In the following theorem, the proof of which is similar to that of [23, Theorem 3.2] on a compact interval, we refer to monotonically increasing (or decreasing) functions satisfying equation (3.9) as *monotonically increasing* (or *decreasing*) *fixed point* of $\mathcal{J}_n$.

THEOREM 3. *Every monotonic fixed point f of $\mathcal{J}_2$ in $\mathcal{C}(\mathbb{R})$ is of the first form in equation (3.11). Every monotonic fixed point f of $\mathcal{J}_3$ in $\mathcal{C}(\mathbb{R})$ is of one of the following two forms:*

$$f(x) = \begin{cases} a & \text{if } x \in (-\infty, a|, \\ x & \text{if } x \in |a, b|, \\ b & \text{if } x \in |b, \infty), \end{cases} \qquad f(x) = \begin{cases} b & \text{if } x \in (-\infty, a|, \\ g(x) & \text{if } x \in |a, b|, \\ a & \text{if } x \in |b, \infty), \end{cases} \tag{3.11}$$

where $|a, b|$ is a closed interval in $\mathbb{R}$ satisfying that $a \le b$ and $g \in \mathcal{C}(|a, b|)$ is a decreasing involutory map.

The above theorem is only applicable to monotonic fixed points. There exist fixed points of $\mathcal{J}_2$ or $\mathcal{J}_3$ which are not monotonic. For example, as considered in Example 1, the function $f(x) = |x|$ on $\mathbb{R}$ is not a monotonic map on $\mathbb{R}$ but a fixed point of $\mathcal{J}_2$.

The 'monotone' in Theorem 3 need not mean 'strict monotone'. In fact, if $a \in \mathbb{R}$ in the first form of equation (3.11), then we have $f(x) = a$ for $x \in (-\infty, a]$, implying that f is not strictly monotone. For strictly monotonic ones, we have the following result, the proof of which is similar to that of [23, Corollary 3.3] on a compact interval.

COROLLARY 1. *If f is a strictly monotonic fixed point of $\mathcal{J}_3$ in $\mathcal{C}(\mathbb{R})$, then either $f = \mathrm{id}$ or f is a strictly decreasing involutory function on $\mathbb{R}$.*

Theorem 3 gives results only for $\mathcal{J}_2$ and $\mathcal{J}_3$, but not for the generic $\mathcal{J}_n$. In what follows, we show that those monotonic fixed points of $\mathcal{J}_2$ and $\mathcal{J}_3$ are important representatives for the generic $\mathcal{J}_n$. Let $\mathcal{C}_{\mathrm{id}}(\mathbb{R})$ and $\mathcal{C}_{\mathrm{inv}}(\mathbb{R})$ consist of all continuous self-maps of $\mathbb{R}$ which are the identity and decreasing involutions on their range, respectively. By Theorem 3, monotonic fixed points of $\mathcal{J}_3$ are in both classes $\mathcal{C}_{\mathrm{id}}(\mathbb{R})$ and $\mathcal{C}_{\mathrm{inv}}(\mathbb{R})$ but monotonic fixed points of $\mathcal{J}_2$ are all in the same class $\mathcal{C}_{\mathrm{id}}(\mathbb{R})$. The following theorem, the proof of which is similar to that of [23, Theorem 3.4] on a compact interval, describes all fixed points of $\mathcal{J}_n$ for any $n \ge 2$.

THEOREM 4. *The following statements are true for system $(\mathcal{C}(\mathbb{R}), \mathcal{J}_n)$:*

(i) $\mathrm{Fix}(\mathcal{J}_m; \mathcal{C}(\mathbb{R})) = \mathrm{Fix}(\mathcal{J}_n; \mathcal{C}(\mathbb{R}))$ *if integers $m, n \ge 2$ satisfy $m \equiv n \pmod 2$;*

(ii) $\mathrm{Fix}(\mathcal{J}_2; \mathcal{C}(\mathbb{R})) \subsetneq \mathrm{Fix}(\mathcal{J}_3; \mathcal{C}(\mathbb{R}))$. *More concretely,*
$\mathrm{Fix}(\mathcal{J}_2; \mathcal{C}(\mathbb{R})) = \mathcal{C}_{\mathrm{id}}(\mathbb{R})$ *and* $\mathrm{Fix}(\mathcal{J}_3; \mathcal{C}(\mathbb{R})) = \mathcal{C}_{\mathrm{id}}(\mathbb{R}) \cup \mathcal{C}_{\mathrm{inv}}(\mathbb{R})$.

Theorem 3 together with result (i) of Theorem 4 shows that Theorem 3 is indeed true for every integer $n \geq 2$. More precisely, monotonic fixed points of $\mathcal{J}_n$ coincide with those of $\mathcal{J}_2$ and $\mathcal{J}_3$ accordingly as n is even and odd, respectively. So, Theorem 3 actually gives results for the representatives.

Although we can find many fixed points of $\mathcal{J}_n$, there are no non-trivial periodic points as seen from the following result, the proof of which is similar to that of [23, Theorem 3.5] on a compact interval.

THEOREM 5. *For each $n \in \mathbb{N}$, $\mathcal{J}_n$ does not have periodic points of period $k \geq 2$ in $\mathcal{C}(\mathbb{R})$.*

As observed earlier, every element of $\mathcal{C}(\mathbb{R})$ is a fixed point for $\mathcal{J}_1$. So, $\mathcal{J}_1$ has a dense set of periodic points in $\mathcal{C}(\mathbb{R})$. However, since $\mathcal{C}(\mathbb{R})$ is metrizable with metric D by result (i) of Lemma 4 and $\mathcal{J}_n$ is not the identity operator on $\mathcal{C}(\mathbb{R})$, it follows that $\mathrm{Fix}(\mathcal{J}_n; \mathcal{C}(\mathbb{R}))$ is not dense in $\mathcal{C}(\mathbb{R})$ for $n \geq 2$. This implies by Theorem 5 that $\mathrm{Per}(\mathcal{J}_n; \mathcal{C}(\mathbb{R}))$ is not dense in $\mathcal{C}(\mathbb{R})$ for $n \geq 2$.

3.3. *Periodic points in $\mathcal{C}(S^1)$.* As illustrated in Example 3, $\mathcal{J}_n$ may have periodic points of period $k \geq 2$ in $\mathcal{C}(S^1)$ but, in contrast, $\mathcal{J}_n$ only has fixed points, that is, trivial periodic points of period 1, in $\mathcal{C}(\mathbb{R})$ as shown in Theorem 5. In this section, we investigate all periodic points of $\mathcal{J}_n$ in $\mathcal{C}(S^1)$.

Given $z_0, z_1, \ldots, z_{m-1} \in S^1$ with $m \geq 2$, we write $z_0 \prec z_1 \prec \cdots \prec z_{m-1}$ if there exist $t_1, t_2, \ldots, t_{m-1} \in \mathbb{R}$ such that $0 < t_1 < t_2 < \cdots < t_{m-1} < 1$ and $z_j = z_0 e^{2\pi i t_j}$ for $1 \leq j \leq m-1$. In this case, we have

$$z_{j(\mathrm{mod\ m})} \prec z_{j+1(\mathrm{mod\ m})} \prec \cdots \prec z_{j+m-1(\mathrm{mod\ m})} \quad \text{for all } j \in \mathbb{N},$$

so that '$\prec$' is indeed a *cyclic order* on S^1.

For any two distinct points $z_1, z_2 \in S^1$, define the *arcs* (z_1, z_2), $[z_1, z_2)$, and $(z_1, z_2]$ by $(z_1, z_2) := \{z \in S^1 : z_1 \prec z \prec z_2\}$, $[z_1, z_2) := (z_1, z_2) \cup \{z_1\}$, and $(z_1, z_2] := (z_1, z_2) \cup \{z_2\}$. Then we have $(z_1, z_2) = \{e^{2\pi i t} \in S^1 : t \in (t_1, t_2)\}$, where t_1, t_2 are unique real such that $z_1 = e^{2\pi i t_1}$, $z_2 = e^{2\pi i t_2}$, and $0 \leq t_1 < t_2 < t_1 + 1 < 2$.

It is known (cf. [4, 25]) that for every homeomorphism $F : S^1 \to S^1$, there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ satisfying one of the Abel equations:

$$\begin{aligned} f(t+1) &= f(t) + 1 \quad \text{if } f \text{ is strictly increasing;} \\ f(t+1) &= f(t) - 1 \quad \text{if } f \text{ is strictly decreasing} \end{aligned}$$

such that $F(e^{2\pi i t}) = e^{2\pi i f(t)}$ for all $t \in \mathbb{R}$. Every such f is called a *lift* of F. We say that F *preserves* (or *reverses*) orientation accordingly as f is strictly increasing (or decreasing) on $\mathbb{R}$.

The following two lemmas together describe the general solution of equation (3.8) on S^1, each of which can also be deduced from Lemma 6.

LEMMA 8. [12] *Let $\phi \in \mathcal{C}(S^1)$ be a solution of equation (3.8) and have a fixed point in S^1. Then: (i) ϕ is the identity map if ϕ is orientation-preserving; or (ii) ϕ is an involution if ϕ is orientation-reversing.*

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

LEMMA 9. [12] *All mth-order iterative roots of identity in $\mathcal{C}(S^1)$ having no fixed points in S^1 are given by*

$$f(z) = \begin{cases} \phi_0(z) & \text{if } z \in [z_0, z_{m-1}), \\ (\phi_1 \circ \phi_2 \circ \cdots \circ \phi_{m-1})^{-1}(z) & \text{if } z \in [z_{m-1}, z_0) \end{cases} \tag{3.12}$$

with $\phi_j := \phi_0|_{[z_{j(m-k)-1}, z_{j(m-k)})}$ for $1 \le j \le m-1$, where k is an integer in $\{1, 2, \ldots, m-1\}$ relatively prime to m and $z_0, z_1, \ldots, z_{m-1}$ are some points in S^1 such that $z_0 \prec z_1 \prec \cdots \prec z_{m-1}$, and $\phi_0 : [z_0, z_{m-1}) \to [z_k, z_{k-1})$ is any arbitrary homeomorphism such that $\phi_0([z_{j-1}, z_j)) = [z_{j-1+k}, z_{j+k})$ for $1 \le j \le m-1$ with $z_j := z_{j (\mathrm{mod\ m})}$ for $j \ge m$.

The following two theorems characterize fixed points and periodic points of $\mathcal{J}_n$ in $\mathcal{C}(S^1)$.

THEOREM 6. *Let $n \in \mathbb{N}$. Then $f \in \mathcal{C}(S^1)$ is a fixed point of $\mathcal{J}_n$ if and only if one of the following conditions is satisfied: (i) $f|_{R(f)}$ is the identity map; (ii) $f|_{R(f)}$ is an orientation-reversing involution; or (iii) f is of the form equation (3.12) for a divisor m of $n-1$.*

Proof. Let $n \in \mathbb{N}$. To find all fixed points of $\mathcal{J}_n$ in $\mathcal{C}(S^1)$, in view of Lemma 5, it suffices to find all $f \in \mathcal{C}(S^1)$ satisfying the equation $\phi^{n-1} = \mathrm{id}$ on $R(f)$. So let $f \in \mathcal{C}(S^1)$ be such that $f^{n-1} = \mathrm{id}$ on $R(f)$. Since S^1 is connected and compact, $R(f)$ is either a singleton set, an arc, or the whole of S^1.

Suppose $R(f) = S^1$. If f has a fixed point, then by Lemma 8, f is either the identity map or an involution according as f is orientation-preserving or orientation-reversing. If f has no fixed points, then by Lemma 9, f is of the form in equation (3.12) for some divisor m of $n-1$.

Now suppose that $R(f) \subsetneq S^1$. Then $R(f)$ is either a singleton set or an arc in S^1. If $R(f)$ is a singleton set, then f is a constant map on S^1. If $R(f)$ is an arc, say $[z_1, z_2]$ for some $z_1, z_2 \in S^1$, then consider a homeomorphism $h_f : R(f) \to [t_1, t_2]$, where t_1, t_2 are unique reals satisfying the conditions $z_1 = e^{2\pi i t_1}$, $z_2 = e^{2\pi i t_2}$ and $0 \le t_1 < t_2 < t_1 + 1 < 2$. Define a map $\mathcal{H}_f : \mathcal{C}(R(f)) \to \mathcal{C}([t_1, t_2])$ by

$$\mathcal{H}_f(g) := h_f \circ g \circ h_f^{-1} \quad \text{for all } g \in \mathcal{C}(R(f)).$$

Then $\mathcal{H}_f$ is a bijective, bi-continuous map such that $\mathcal{H}_f \circ \mathcal{J}_n = \mathcal{J}_n \circ \mathcal{H}_f$ (that is, $(\mathcal{C}(R(f)), \mathcal{J}_n)$ is topologically conjugate to $(\mathcal{C}([t_1, t_2]), \mathcal{J}_n)$). Now since $f^{n-1} = \mathrm{id}$ on $R(f)$, we have $\mathcal{H}_f(f^{n-1}) = \mathcal{H}_f(\mathrm{id}) = \mathrm{id}$ on $[t_1, t_2]$, that is, $(h_f \circ f \circ h_f^{-1})^{n-1} = \mathrm{id}$ on $[t_1, t_2]$. Therefore, by Lemma 7, $h_f \circ f \circ h_f^{-1}$ is either the identity map or a decreasing involutory map on $[t_1, t_2]$. This implies that $f|_{R(f)}$ is either an identity map or an orientation-reversing involutory map.

Conversely, if $f \in \mathcal{C}(S^1)$ satisfies either of the conditions (i) or (ii), then $f^n = f$ on S^1 implying that f is a fixed point of $\mathcal{J}_n$. If f satisfies condition (iii), then by Lemma 9, we have $f^m = \mathrm{id}$ on S^1, and therefore $f^{n-1} = \mathrm{id}$ on S^1 as m divides $n-1$. Therefore, f is a fixed point of $\mathcal{J}_n$. □

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

THEOREM 7. *Let $n, k \geq 2$. Then $f \in \mathcal{C}(S^1)$ is a k-periodic point of $\mathcal{J}_n$ if and only if f is of the form in equation (3.12) for some $m > 1$ such that $m \mid (n^k - 1)$ and $m \nmid (n^j - 1)$ for $1 \leq j \leq k - 1$.*

Proof. Let $f \in \mathcal{C}(S^1)$ be a k-periodic point of $\mathcal{J}_n$. Then f satisfies

$$f^{n^k} = f \quad \text{and} \quad f^{n^j} \neq f \quad \text{for } 1 \leq j \leq k-1 \tag{3.13}$$

on S^1 and also by result (ii) of Theorem 2, $f \in \mathcal{U}_E^m$ for some compact subset E of K and $m > 1$ satisfying equation (3.10). In fact, here $E = R(f)$. We assert that $E = S^1$. Note that E, being the image of connected and compact set S^1 under f, is either a singleton set, an arc, or S^1. If E is a singleton, then f is a constant map on S^1, and therefore $f^n = f$, which is a contradiction to equation (3.13). If E is an arc, then $f|_E$ is either the identity map or an orientation-reversing involution. In any case, we arrive at a contradiction to equation (3.13). Therefore, $E = S^1$ so that $f \in \mathcal{U}_{S^1}^m$. This implies by Lemma 9 that f is of the form in equation (3.12).

Conversely, if f is of the form in equation (3.12) for some $m > 1$ such that $m \mid (n^k - 1)$ and $m \nmid (n^j - 1)$ for $1 \leq j \leq k-1$, then clearly $f \in \mathcal{U}_{S^1}^m$ with $m > 1$ satisfying equation (3.10) so that by result (ii) of Theorem 2, f is a k-periodic point of $\mathcal{J}_n$. □

4. *Stability in $\mathcal{J}_n$*

In this section, we study the (Lyapunov) stability of fixed points of the iteration operator $\mathcal{J}_n$. In view of Lemma 4, to investigate boundedness of orbits, in the following, we consider X to be a hemicompact metrizable space. Clearly, all the orbits of $\mathcal{J}_n$ are bounded because $D(f, g) = \sum_{j=1}^{\infty} \mu_j(f, g) \leq \sum_{j=1}^{\infty} 1/2^j = 1$ for all $f, g \in \mathcal{C}(X)$. In particular, when X is a compact metric space with metric d, we can let $X = \bigcup_{j=1}^{\infty} K_j$ such that $K_1 = X$ and $K_j = \emptyset$ for all $j \geq 2$, implying that the metric D of $\mathcal{C}(X)$ is $D(f, g) = \mu(f, g) = \min\{\frac{1}{2}, \rho(f, g)\}$, which is also equivalent to the uniform metric ρ on it. Further, the boundedness of orbits of $\mathcal{J}_n$ in $\mathcal{C}(X)$ can also be proved as follows. Since X is compact, there exist $y \in X$ and $M > 0$ such that $d(x, y) \leq M/2$, implying that $d(f^{n^k}(x), f^{n^l}(x)) \leq d(f^{n^k}(x), y) + d(y, f^{n^l}(x)) \leq M$ for all $x \in X$ and $k, l \in \mathbb{N}$. So, we can work with metric ρ of $\mathcal{C}(X)$ instead of D whenever X is compact.

As defined in [19], the orbit $(f^n(x))_{n \in \mathbb{N} \cup \{0\}}$ of a discrete semi-dynamical system (X, f), where X is a metric space equipped with the metric d, is said to be *(Lyapunov) stable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(f^k(x), f^k(y)) < \epsilon$ for all $k \in \mathbb{N}$ whenever $y \in X$ satisfies $d(x, y) < \delta$. We say that a point $x \in X$ is *(Lyapunov) stable* for f if its orbit $(f^n(x))_{n \in \mathbb{N} \cup \{0\}}$ is (Lyapunov) stable. The following two theorems prove that most fixed points of $\mathcal{J}_n$ in $\mathcal{C}(X)$ are not stable for $X = \mathbb{R}$ and $X = S^1$, respectively.

THEOREM 8. *Let $n \in \mathbb{N}$. If $f \in \mathcal{C}_{\text{id}}(\mathbb{R})$ is non-constant, then f is not stable for $\mathcal{J}_n$.*

Proof. Let $f \in \mathcal{C}_{\text{id}}(\mathbb{R})$ be a non-constant map. Then there exist $a, b \in \mathbb{R}$ with $a < b$ such that $[a, b] \subseteq R(f)$ and $f|_{[a,b]} = \text{id}$. For each $\eta > 0$, let $g_\eta : \mathbb{R} \to \mathbb{R}$ be the map defined by

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

$$g_\eta(x) = \begin{cases} f(x) & \text{if } x \in (-\infty, a] \cup [b, \infty), \\ a + (x-a)(1-\eta) & \text{if } x \in \left[a, \dfrac{a+b}{2}\right], \\ x(1+\eta) - b\eta & \text{if } x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Then $g_\eta \in \mathcal{C}(\mathbb{R})$ for each $\eta > 0$. Consider the metric D on $\mathcal{C}(\mathbb{R})$ defined as in equations (2.2)–(2.4) with the partition $\mathbb{R} = \bigcup_{j=1}^{\infty} K_j$, where

$$K_j := \begin{cases} [a, b] & \text{if } j = 1, \\ \left[b + \dfrac{j-3}{2}, b + \dfrac{j-1}{2}\right] & \text{if } j = 3, 5, \ldots, \\ \left[a - \dfrac{j}{2}, a - \dfrac{j-2}{2}\right] & \text{if } j = 2, 4, \ldots. \end{cases}$$

Let $\epsilon = \min\{(b-a)/8, \frac{1}{2}\}$ and for any $\delta > 0$, choose $\eta_\delta > 0$ such that $\eta_\delta < \min\{\delta_0/(b-a), \frac{1}{2}\}$ for some $0 < \delta_0 < \min\{\delta, \frac{1}{2}\}$.

Claim. $D(f, g_{\eta_\delta}) < \delta$ and $D(f, g_{\eta_\delta}^{n^{k_0}}) \geq \epsilon$ for some $k_0 \in \mathbb{N}$.

Consider any $x \in \mathbb{R}$. If $x \in K_1$ with $x \leq (a+b)/2$, then

$$|f(x) - g_{\eta_\delta}(x)| = |x - (a + (x-a)(1-\eta_\delta))| = (x-a)\eta_\delta < (x-a)\frac{\delta_0}{b-a} < \delta_0.$$

If $x \in K_1$ with $x > (a+b)/2$, then

$$|f(x) - g_{\eta_\delta}(x)| = |x - (x(1+\eta_\delta) - b\eta_\delta)| = (b-x)\eta_\delta < (b-x)\frac{\delta_0}{b-a} < \delta_0.$$

If $x \in K_j$ with $j > 1$, then

$$|f(x) - g_{\eta_\delta}(x)| = |f(x) - f(x)| = 0.$$

Therefore, $\rho_1(f, g_{\eta_\delta}) \leq \delta_0$ and $\rho_j(f, g_{\eta_\delta}) = 0$ for all $j > 1$, implying that $\mu_1(f, g_{\eta_\delta}) \leq \delta_0$ and $\mu_j(f, g_{\eta_\delta}) = 0$ for all $j > 1$. Hence, $D(f, g_{\eta_\delta}) \leq \delta_0 < \delta$.

Now for any $x \in [a, (a+b)/2]$, we have

$$g_{\eta_\delta}(x) = x(1-\eta_\delta) + a\eta_\delta \geq a(1-\eta_\delta) + a\eta_\delta = a,$$
$$g_{\eta_\delta}(x) \leq \frac{a+b}{2}(1-\eta_\delta) + a\eta_\delta = \frac{a+b}{2} - \frac{b-a}{2}\eta_\delta < \frac{a+b}{2},$$

implying that $g_{\eta_\delta}(x) \in [a, (a+b)/2]$. Therefore, $g_{\eta_\delta}([a, (a+b)/2]) \subseteq [a, (a+b)/2]$, and hence by induction,

$$g_{\eta_\delta}^k(x) = a + (x-a)(1-\eta_\delta)^k \tag{4.14}$$

for every $x \in [a, (a+b)/2]$ and $k \in \mathbb{N}$. Let $y = (a+b)/2$. Since $\eta_\delta \in (0, 1)$, equation (4.14) implies that $g_{\eta_\delta}^k(y) \to a$ as $k \to \infty$. So there exists $N \in \mathbb{N}$ such that $|g_{\eta_\delta}^k(y) - a| < \epsilon$ for all $k \geq N$. Choose $k_0 \in \mathbb{N}$ so large that $n^{k_0} > N$. Then

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

$g_{\eta_\delta}^{n^{k_0}}(y) - a < (b-a)/8$, that is, $-g_{\eta_\delta}^{n^{k_0}}(y) > -(7a+b)/8$. Thus,

$$f^{n^{k_0}}(y) - g_{\eta_\delta}^{n^{k_0}}(y) = \frac{b-a}{2}(1-(1-\eta_\delta)^{n^{k_0}}) > 0,$$

and therefore

$$|f^{n^{k_0}}(y) - g_{\eta_\delta}^{n^{k_0}}(y)| = f^{n^{k_0}}(y) - g_{\eta_\delta}^{n^{k_0}}(y) > \frac{a+b}{2} - \frac{7a+b}{8} = \frac{3(b-a)}{8} > \frac{b-a}{8},$$

implying that

$$\rho_1(f, g_{\eta_\delta}^{n^{k_0}}) = \rho_1(f^{n^{k_0}}, g_{\eta_\delta}^{n^{k_0}}) \geq |f^{n^{k_0}}(y) - g_{\eta_\delta}^{n^{k_0}}(y)| > \frac{b-a}{8}.$$

This proves that

$$\mu_1(f, g_{\eta_\delta}^{n^{k_0}}) = \min\left\{\frac{1}{2}, \rho_1(f, g_{\eta_\delta}^{n^{k_0}})\right\} \geq \min\left\{\frac{1}{2}, \frac{b-a}{8}\right\} = \epsilon.$$

Therefore, the claim holds, and hence f is not stable for $\mathcal{J}_n$. □

THEOREM 9. *Let $n \in \mathbb{N}$ and $f \in \mathcal{C}_{\mathrm{id}}(S^1)$, consisting of all continuous self-maps of S^1 which are the identity on their range. Then f is stable for $\mathcal{J}_n$ if and only if f is a constant map on S^1.*

Proof. Let $n \in \mathbb{N}$ and $f \in \mathcal{C}_{\mathrm{id}}(S^1)$ be a constant map on S^1. For given $\epsilon > 0$, choose $\delta = \epsilon$. Then for every $k \in \mathbb{N}$ and $g \in \mathcal{C}(S^1)$ with $\rho(f,g) < \delta$, we have

$$|f(x) - g^k(x)| = |f(g^{k-1}(x)) - g(g^{k-1}(x))| \leq \rho(f,g) < \epsilon \quad \text{for all } x \in S^1,$$

implying $\rho(f, g^k) < \epsilon$, and hence in particular $\rho(f^{n^k}, g^{n^k}) < \epsilon$. Therefore, f is stable.

Conversely, suppose that $f \in \mathcal{C}_{\mathrm{id}}(S^1)$ is a non-constant map on S^1. Then $f|_{R(f)} = \mathrm{id}$ such that either $R(f) = S^1$ or $R(f) = [z_1, z_2]$ for some $z_1 = e^{it_1}$, $z_2 = e^{it_2} \in S^1$ with $0 \leq t_1 < t_2 < 2\pi$. For each $\eta > 0$, let $g_\eta : S^1 \to S^1$ be the map defined by

$$g_\eta(e^{it}) = \begin{cases} f(e^{it}) & \text{if } t \in [0, 2\pi) \setminus [t_1, t_2], \\ e^{i[t_1 + (t-t_1)(1-\eta)]} & \text{if } t \in \left[t_1, \dfrac{t_1+t_2}{2}\right], \\ e^{i[t(1+\eta) - t_2\eta]} & \text{if } t \in \left[\dfrac{t_1+t_2}{2}, t_2\right]. \end{cases}$$

Then $g_\eta \in \mathcal{C}(S^1)$ for each $\eta > 0$. Let $\epsilon = (t_2 - t_1)/2\sqrt{2}\pi$ and for any $\delta > 0$, choose $\eta_\delta > 0$ such that $\eta_\delta < \min\{2\delta_0/(t_2 - t_1), 1\}$ for some $0 < \delta_0 < \delta$.

Claim. $\rho(f, g_{\eta_\delta}) < \delta$ and $\rho(f, g_{\eta_\delta}^{n^{k_0}}) \geq \epsilon$ for some $k_0 \in \mathbb{N}$.

Consider any $t \in [0, 2\pi)$. If $t \in [0, 2\pi) \setminus [t_1, t_2]$, then

$$|f(e^{it}) - g_{\eta_\delta}(e^{it})| = |f(e^{it}) - f(e^{it})| = 0 < \delta_0.$$

If $t \in (t_1, (t_1 + t_2)/2)$, then

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

$$\begin{aligned}|f(e^{it}) - g_{\eta_\delta}(e^{it})| &= |e^{it} - e^{i[t_1+(t-t_1)(1-\eta_\delta)]}|\\ &= |1 - e^{i\eta_\delta(t_1-t)}|\\ &\leq 2\left|\sin\left(\frac{\eta_\delta(t_1-t)}{2}\right)\right|\\ &\leq |\eta_\delta(t_1-t)| = (t-t_1)\eta_\delta < \frac{t_2-t_1}{2}\frac{2\delta_0}{t_2-t_1} = \delta_0.\end{aligned}$$

If $t \in [(t_1+t_2)/2, t_2)$, then by a similar argument, we have $|f(e^{it}) - g_{\eta_\delta}(e^{it})| < \delta_0$. Therefore, $|f(e^{it}) - g_{\eta_\delta}(e^{it})| < \delta_0$ for all $t \in [0, 2\pi)$ and hence $\rho(f, g_{\eta_\delta}) \leq \delta_0 < \delta$.

Now for any $t \in [t_1, (t_1+t_2)/2]$, we have

$$\begin{aligned}t_1 = t_1(1-\eta_\delta) + t_1\eta_\delta &\leq t(1-\eta_\delta) + t_1\eta_\delta\\ &= t_1 + (t-t_1)(1-\eta_\delta)\\ &\leq \frac{t_1+t_2}{2}(1-\eta_\delta) + t_1\eta_\delta\\ &= \frac{t_1+t_2}{2} - \frac{t_2-t_1}{2}\eta_\delta < \frac{t_1+t_2}{2},\end{aligned}$$

implying $g_{\eta_\delta}(e^{it}) \in [z_1, w]$, where $w = e^{i(t_1+t_2)/2}$. Therefore, $g_{\eta_\delta}([z_1, w]) \subseteq [z_1, w]$. Hence, it can be shown by induction that $g^k_{\eta_\delta}(e^{it}) = e^{i[t_1+(t-t_1)(1-\eta_\delta)^k]}$ for every $t \in [t_1, (t_1+t_2)/2]$ and $k \in \mathbb{N}$. Also, $\sin(t/2) \geq t/\pi$ for all $t \in [0, \pi]$, and therefore,

$$|e^{it} - 1| = \sqrt{2}\sin\left(\frac{t}{2}\right) \geq \frac{\sqrt{2}t}{\pi} \quad \text{for all } t \in [0, \pi]. \tag{4.15}$$

Now for each $k \in \mathbb{N}$, we have

$$\begin{aligned}|f(w) - g^k_{\eta_\delta}(w)| &= |e^{i(t_1+t_2)/2} - e^{i[t_1+((t_1+t_2)/2-t_1)(1-\eta_\delta)^k]}|\\ &= |1 - e^{-i[(t_2-t_1)/2(1-(1-\eta_\delta)^k)]}|\\ &= |e^{i[(t_2-t_1)/2(1-(1-\eta_\delta)^k)]} - 1|\end{aligned} \tag{4.16}$$

and

$$0 \leq \frac{t_2-t_1}{2}[1-(1-\eta_\delta)^k] < \frac{t_2-t_1}{2} < \frac{t_2}{2} \leq \frac{2\pi}{2} = \pi,$$

implying by equation (4.15) that

$$\begin{aligned}|e^{i[(t_2-t_1)/2(1-(1-\eta_\delta)^k)]} - 1| &\geq \frac{\sqrt{2}}{\pi}\cdot\frac{t_2-t_1}{2}[1-(1-\eta_\delta)^k]\\ &= \frac{t_2-t_1}{\sqrt{2}\pi}[1-(1-\eta_\delta)^k],\end{aligned}$$

for each $k \in \mathbb{N}$. Then equation (4.16) implies that

$$|f(w) - g^k_{\eta_\delta}(w)| \geq \frac{t_2-t_1}{\sqrt{2}\pi}[1-(1-\eta_\delta)^k], \tag{4.17}$$

for each $k \in \mathbb{N}$. Since $1-(1-\eta_\delta)^k \to 1$ as $k \to \infty$, there exists $N \in \mathbb{N}$ such that $(1-\eta_\delta)^k < \frac{1}{2}$ for all $k \geq N$. Choose k_0 sufficiently large such that $n^{k_0} > N$. Then

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

$1-(1-\eta_\delta)^{n^{k_0}} > \frac{1}{2}$, and therefore from equation (4.17), we have

$$|f(w)-g_{\eta_\delta}^{n^{k_0}}(w)| \geq \frac{t_2-t_1}{\sqrt{2}\pi}\cdot\frac{1}{2} = \frac{t_2-t_1}{2\sqrt{2}\pi} = \epsilon,$$

which implies that

$$\rho(f, g_{\eta_\delta}^{n^{k_0}}) = \rho(f^{n^{k_0}}, g_{\eta_\delta}^{n^{k_0}}) \geq |f^{n^{k_0}}(w) - g_{\eta_\delta}^{n^{k_0}}(w)| \geq \epsilon.$$

This proves the claim, from which it follows that f is not stable, which is a contradiction. Hence, f is a constant map on S^1. □

As defined in [10], a discrete semi-dynamical system (X, f), where X is a metric space equipped with the metric d, is said to be *topologically transitive* if for every pair of open sets U, V in X, there exist $x \in U$ and $n \in \mathbb{N}$ such that $f^n(x) \in V$. Here, f is said to be *sensitively dependent on initial conditions* if there exists $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$, there exist $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$. As defined in [8], f is said to be *chaotic* in Devaney's sense if: (i) the set of periodic points of f is dense in X; (ii) f is topologically transitive; and (iii) f exhibits sensitive dependence on initial conditions. Since $\mathrm{Per}(\mathcal{J}_n; \mathcal{C}(\mathbb{R}))$ is not dense in $\mathcal{C}(\mathbb{R})$, as seen in §3, $\mathcal{J}_n$ is not chaotic $\mathcal{C}(\mathbb{R})$ for $n \geq 2$. Also, since S^1 is compact, by [23, Theorem 5.1], $\mathcal{J}_n$ is not chaotic on $\mathcal{C}(S^1)$ for $n \geq 2$. However, $\mathcal{J}_1$ is also not chaotic on $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(S^1)$ since it is not sensitively dependent on initial conditions. Thus, although we see that all orbits of $\mathcal{J}_n$ in $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(S^1)$ are bounded but most of the fixed points are unstable, which actually exhibits a complicated behavior of $\mathcal{J}_n$, the complicated behavior is *not chaotic*. The following examples show how complicated an orbit of $\mathcal{J}_n$ can be.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Let $n = 2$, $\epsilon = \frac{1}{8}$, $\delta = 0.13$, $\delta_0 = 0.07$, $\eta_\delta = 0.04$, and g_2 be the map g_{η_δ} as defined in Theorem 8 with $a = 0$ and $b = 1$. Define the maps $g_1, g_3 : \mathbb{R} \to \mathbb{R}$ by

$$g_1(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 0.99x & \text{if } 0 \leq x \leq 1, \\ 0.99 & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad g_3(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1.1x & \text{if } 0 \leq x \leq 0.5, \\ x + 0.05 & \text{if } 0.5 \leq x \leq 0.55, \\ -2x + 1.7 & \text{if } 0.55 \leq x \leq 0.6, \\ 1.25x - 0.25 & \text{if } 0.6 \leq x \leq 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Then $f \in \mathrm{Fix}(\mathcal{J}_2; \mathcal{C}(\mathbb{R}))$ and $g_1, g_2, g_3 \in \mathcal{C}(\mathbb{R})$. Consider the metric D on $\mathcal{C}(\mathbb{R})$ defined as in equations (2.2)–(2.4) with the partition $\mathbb{R} = \bigcup_{j=1}^{\infty} K_j$, where

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

$$K_j := \begin{cases} [0,1] & \text{if } j = 1, \\ \left[-\dfrac{j-1}{2}, -\dfrac{j-1}{2}+1\right] & \text{if } j = 3, 5, \ldots, \\ \left[\dfrac{j}{2}, \dfrac{j}{2}+1\right] & \text{if } j = 2, 4, \ldots. \end{cases}$$

An easy computation shows that $D(g_j, f) < \delta$ for all $j = 1, 2, 3$. However, $g_1^{2^6}(0.4) = 0.2102$, $g_2^{2^4}(0.4) = 0.2082$, and $g_3^{2^3}(0.4) = 0.5796$, implying that $D(g_1^{2^6} f) > \epsilon$, $D(g_2^{2^4} f) > \epsilon$, and $D(g_3^{2^3}, f) > \epsilon$. This illustrates the claim in the proof of Theorem 8 for f for the above-mentioned specific choices of δ, δ_0, and η_δ with $\epsilon = \frac{1}{8}$. Indeed, f is not stable for $\mathcal{J}_2$ as seen from Theorem 8.

To illustrate the complexity of $\mathcal{J}_2$, we investigate the asymptotic behavior of the orbits of g_1, g_2, and g_3. We have

$$g_1^k(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 0.99^k x & \text{if } 0 \le x \le 1, \\ 0.99^k & \text{if } x \ge 1, \end{cases}$$

for all $k \in \mathbb{N}$. Therefore, the sequence of maps $(g_1^k)_{k\in\mathbb{N}}$ and hence the orbit $(g_1^{2^k})_{k\in\mathbb{N}\cup\{0\}}$ of g_1 converges uniformly to the zero map on $\mathbb{R}$. As seen in the proof of Theorem 8, $g_2^k(x) \to 0$ as $k \to \infty$, for each $x \in [0, 0.5]$. Also, for each $x \in [0.5, 1)$, there exists $k_x \in \mathbb{N}$ such that $g_2^{k_x}(x) \in [0, 0.5]$. Further,

$$g_2^k(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x \ge 1, \end{cases}$$

for all $k \in \mathbb{N}$. Thus, the sequence of maps $(g_2^k)_{k\in\mathbb{N}}$ and hence the orbit $(g_2^{2^k})_{k\in\mathbb{N}\cup\{0\}}$ of g_2 converges pointwise to the discontinuous map $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

The point $x = 0.5$ is a 3-periodic point of g_3 and therefore, the orbit $(g_3^{2^k})_{k\in\mathbb{N}\cup\{0\}}$ of g_3 does not converge. In fact, by [**16**, Theorem 1], g_3 is chaotic in the sense of Li and Yorke. However, $g_3([0, 1]) \subseteq [0, 1]$, implying that g_3 is not topologically transitive and therefore is not chaotic in the sense of Devaney.

Thus, although all the orbits of $\mathcal{J}_2$ in $\mathcal{C}(\mathbb{R})$ are bounded, it is possible that an orbit may not converge or, even if it converges, the limit function may not be continuous. Furthermore, the choice of $\delta = 0.13$ is not special. Indeed, we can construct maps in $\mathcal{C}(\mathbb{R})$ that are similar to those of g_1, g_2, and g_3 in each δ-neighborhood of f.

Example 5. Let f be the identity map on S^1. Let $n = 2$, $\epsilon = \frac{1}{8}$, $\delta = 0.15$, $\delta_0 = 0.07$, $\eta_\delta = 0.05$, and g_2 be the map g_{η_δ} as defined in Theorem 9 with $t_1 = 0$ and $t_2 = \pi/2$. Define the maps $g_1, g_3 : S^1 \to S^1$ by $g_1(e^{it}) = e^{0.9it}$ for all $t \in [0, 2\pi)$, and

$$g_3(e^{it}) = e^{i(t+2\pi/7)} \quad \text{for all } t \in [0, 2\pi).$$

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

Then $f \in \mathrm{Fix}(\mathcal{J}_2; \mathcal{C}(S^1))$ and $g_1, g_2, g_3 \in \mathcal{C}(S^1)$. A computation as in Theorem 9 shows that $\rho(g_2, f) < \delta$. However,

$$\begin{aligned}|g_2^{2^4}(e^{i\pi/4}) - e^{i\pi/4}| &= |1 - e^{i\pi/4(1-0.95^{2^4})}| \\ &\geq \frac{1}{2\sqrt{2}}(1 - 0.95^{2^4}) \quad \text{(using equation (4.15))} \\ &\geq \frac{1}{4\sqrt{2}} > \epsilon,\end{aligned}$$

implying that $\rho(g_2^{2^4}, f) > \epsilon$. Indeed, f is not stable for $\mathcal{J}_2$ as seen from Theorem 9.

To illustrate the complexity of iteration operator $\mathcal{J}_2$, we investigate the asymptotic behavior of the orbits of g_1, g_2, and g_3. We have $g_1^k(e^{it}) = e^{0.9^k it}$ for all $k \in \mathbb{N}$ and for all $t \in [0, 2\pi)$. Therefore, the sequence of maps $(g_1^k)_{k\in\mathbb{N}}$ and hence the orbit $(g_1^{2^k})_{k\in\mathbb{N}\cup\{0\}}$ of g_1 converge uniformly to the constant map $f_1 : S^1 \to S^1$ defined by $f_2(e^{it}) = 1$ for all $t \in [0, 2\pi)$. As noted in the proof of Theorem 9, $g_2^k(e^{it}) \to 1$ as $k \to \infty$, for each $t \in [0, \pi/4]$. Also, for each $t \in [\pi/4, \pi/2)$, there exists $k_t \in \mathbb{N}$ such that $\arg(g_2^{k_t}(e^{it})) \in [0, \pi/4]$. Moreover, $g_2^k(e^{it}) = e^{it}$ for all $k \in \mathbb{N}$ and for all $t \in [\pi/2, 2\pi)$. Thus, the sequence of maps $(g_2^k)_{k\in\mathbb{N}}$ and hence the orbit $(g_2^{2^k})_{k\in\mathbb{N}\cup\{0\}}$ of g_2 converge pointwise to the discontinuous map $f_2 : S^1 \to S^1$ defined by

$$f_2(e^{it}) = \begin{cases} 1 & \text{if } 0 \leq t < \dfrac{\pi}{2}, \\ e^{it} & \text{if } \dfrac{\pi}{2} \leq t < 2\pi. \end{cases}$$

The map g_3 is a 3-periodic point of $\mathcal{J}_2$ and therefore the orbit $(g_3^{2^k})_{k\in\mathbb{N}\cup\{0\}}$ of g_3 converges to the periodic orbit $\{g_3, g_3^2, g_3^{2^2}\}$.

Thus, similar to $\mathcal{C}(\mathbb{R})$, although all the orbits of $\mathcal{J}_2$ in $\mathcal{C}(S^1)$ are bounded, it is possible that an orbit may not converge or, even if it converges, the limit function may not be continuous.

5. *Classification up to conjugacy*

Since $\mathbb{R}$ is not homeomorphic to S^1, it is natural to expect that the dynamics of $\mathcal{J}_n$ on $\mathcal{C}(\mathbb{R})$ is 'different' from that on $\mathcal{C}(S^1)$. This is also evident from our discussion as $\mathcal{J}_n$ has periodic points of all periods in $\mathcal{C}(S^1)$ whereas it has no non-trivial periodic points in $\mathcal{C}(\mathbb{R})$ for $n \geq 2$. This leads to the question: *Is the dynamics of $\mathcal{J}_n$ on $\mathcal{C}(X)$ identical to that on $\mathcal{C}(Y)$ whenever X is homeomorphic to Y?* We investigate this question in this section in locally compact spaces.

Let (X, f) and (Y, g) be two discrete semi-dynamical systems. As defined in [10], we say that (X, f) is *topologically conjugate* (or simply *conjugate*) to (Y, g) if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. In this case, h is called a *topological conjugacy*.

THEOREM 10. *Let X and Y be locally compact Hausdorff spaces. If X is homeomorphic to Y, then $(\mathcal{C}(X), \mathcal{J}_n)$ is conjugate to $(\mathcal{C}(Y), \mathcal{J}_n)$ for each $n \in \mathbb{N}$.*

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

Proof. Let h be a homeomorphism of X onto Y. Define a map $\mathcal{H} : \mathcal{C}(X) \to \mathcal{C}(Y)$ as

$$\mathcal{H}(f) := h \circ f \circ h^{-1} \quad \text{for all } f \in \mathcal{C}(X).$$

Then $\mathcal{H}$ is a well-defined bijective map. To prove the continuity of $\mathcal{H}$, first we claim that the map $\mathcal{F} : Y \times \mathcal{C}(X) \to Y$ defined by $\mathcal{F}(y, f) := h \circ f \circ h^{-1}(y)$ is continuous. In fact, consider the map $\mathcal{G} : Y \times \mathcal{C}(X) \to X \times \mathcal{C}(X)$ defined by $\mathcal{G} := (h^{-1} \circ p_1, p_2)$, where the p_i are the projection maps on $Y \times \mathcal{C}(X)$ given by $p_1(y, f) := y$ and $p_2(y, f) := f$. Since $h^{-1} \circ p_1$ and p_2 are continuous, so is $\mathcal{G}$. Also, by Lemma 2, the evaluation map $\mathcal{E} : X \times \mathcal{C}(X) \to X$, defined by $\mathcal{E}(x, f) := f(x)$, is continuous. Now,

$$\begin{aligned} h \circ \mathcal{E} \circ \mathcal{G}(y, f) &= h(\mathcal{E}(h^{-1} \circ p_1(y, f), p_2(y, f))) \\ &= h(\mathcal{E}(h^{-1}(y), f)) \\ &= h(f(h^{-1}(y))) = \mathcal{F}(y, f) \end{aligned}$$

for each $(y, f) \in Y \times \mathcal{C}(X)$, implying that $\mathcal{F} = h \circ \mathcal{E} \circ \mathcal{G}$. Therefore, being the composition of continuous maps h, $\mathcal{E}$, and $\mathcal{G}$, $\mathcal{F}$ is continuous, and the claim is proved.

The map $\mathcal{H}$ is actually the induced map of $\mathcal{F}$. Therefore, since $\mathcal{F}$ is continuous, by Lemma 3, we get that $\mathcal{H}$ is continuous on $\mathcal{C}(X)$. By a similar argument, it follows that $\mathcal{H}^{-1}$ is continuous on $\mathcal{C}(Y)$. Hence, $\mathcal{H}$ is a homeomorphism of $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$.

Finally, for each $f \in \mathcal{C}(X)$, we have

$$\begin{aligned} \mathcal{J}_n \circ \mathcal{H}(f) = \mathcal{J}_n(h \circ f \circ h^{-1}) &= (h \circ f \circ h^{-1})^n \\ &= h \circ f^n \circ h^{-1} = h \circ \mathcal{J}_n(f) \circ h^{-1} = \mathcal{H} \circ \mathcal{J}_n(f), \end{aligned}$$

implying that $\mathcal{J}_n \circ \mathcal{H} = \mathcal{H} \circ \mathcal{J}_n$. Therefore, $(\mathcal{C}(X), \mathcal{J}_n)$ is conjugate to $(\mathcal{C}(Y), \mathcal{J}_n)$. □

In contrast to the above theorem, it is more interesting to see whether $(\mathcal{C}(X), \mathcal{J}_n)$ is still conjugate to $(\mathcal{C}(Y), \mathcal{J}_n)$ for some $n \in \mathbb{N}$ and some spaces X and Y which are not homeomorphic to each other, unless we can prove the converse of the above theorem. Concerning the converse, we need to consider the weaker converse first and ask: Does the notion $\mathcal{C}(X)$ is homeomorphic to $\mathcal{C}(Y)$ imply that X is homeomorphic to Y? The answer is no in general because of the following counter-example.

Counter-example. $\mathcal{C}([a, b])$ is homeomorphic to $\mathcal{C}(\mathbb{R})$, although $[a, b]$ is not homeomorphic to $\mathbb{R}$.

In fact, as shown in [**22**, pp. 116–122], a *seminorm* on a real vector space X is a map $p : X \to \mathbb{R}$ satisfying the following conditions: (i) $p(x) \geq 0$ for all $x \in X$; (ii) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$ and $x \in X$; (iii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$. A real topological vector space X along with a family $\mathcal{P}$ of seminorms on it is said to be a *locally convex space*. As shown in [**26**, pp. 175–176], a topological space X is said to be *completely metrizable* if there exists a complete metric on it inducing its topology. As seen at the end of §3.2, $\mathcal{C}(\mathbb{R})$ in the the compact-open topology is metrizable with metric D defined as in equations (2.2)–(2.4), where d is the usual metric on $\mathbb{R}$. Also, by [**7**, Proposition 1.12, p. 145], D is complete. Hence, $\mathcal{C}(\mathbb{R})$ is completely metrizable.

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

Further, $\mathcal{C}(\mathbb{R})$ is locally convex because the family $\mathcal{P} = \{\rho_j : j \in \mathbb{N}\}$ of seminorms induces the compact-open topology on it. Moreover, by of [15, Theorem 5.2, p. 694], $\mathcal{C}(\mathbb{R})$ is separable. However, $C([a, b], \mathbb{R})$, the space of all continuous functions of $[a, b]$ into $\mathbb{R}$ in the uniform topology induced by the uniform metric ρ, is also a completely metrizable space. Further, by using the Weierstrass polynomial approximation theorem, it follows that $C([a, b], \mathbb{R})$ is also separable. Moreover, it is locally convex, being a normed linear space. Thus, both $C([a, b], \mathbb{R})$ and $\mathcal{C}(\mathbb{R})$ are infinite dimensional separable completely metrizable locally convex linear spaces. Hence, by the Anderson–Kadec theorem (cf. [3, Theorem 5.2, p. 189]), which states that every infinite dimensional separable Banach space is homeomorphic to the countable product $\mathbb{R}^\omega$ of $\mathbb{R}$ with itself in the product topology, it follows that they are homeomorphic as topological spaces. Moreover, by a known result (cf. [3, Theorem 6.2 p. 190]), which states that every closed convex set of non-empty interior in an infinite dimensional Banach space is homeomorphic to the whole space, we get that $\mathcal{C}([a, b])$ is homeomorphic to $C([a, b], \mathbb{R})$. Therefore, $\mathcal{C}([a, b])$ is homeomorphic to $\mathcal{C}(\mathbb{R})$.

Another concern is the question: *Even if $\mathcal{C}(X)$ is homeomorphic to $\mathcal{C}(Y)$, is $(\mathcal{C}(X), \mathcal{J}_m)$ conjugate to $(\mathcal{C}(Y), \mathcal{J}_n)$ for different m and n*? As observed from Theorem 4, the existence of decreasing involutory fixed points ensures that $\mathcal{J}_3$ is not conjugate to $\mathcal{J}_2$ on $\mathcal{C}(\mathbb{R})$. More generally, we have the following theorem.

THEOREM 11. *Let X be a locally compact Hausdorff space such that there exists an involutory map $f_0 \neq \mathrm{id}$ in $\mathcal{C}(X)$. Then, $(\mathcal{C}(X), \mathcal{J}_2)$ is not conjugate to $(\mathcal{C}(Y), \mathcal{J}_m)$ for each locally compact Hausdorff space Y and odd positive integer m.*

Proof. Suppose that $\mathcal{H}$ is a conjugacy of $(\mathcal{C}(Y), \mathcal{J}_m)$ and $(\mathcal{C}(X), \mathcal{J}_2)$. Then $\mathcal{H} : \mathcal{C}(Y) \to \mathcal{C}(X)$ is a homeomorphism such that

$$\mathcal{H}(g^m) = \mathcal{H}(g)^2 \quad \text{for all } g \in \mathcal{C}(Y).$$

Let $f_1 = f_0$ and $f_2 = \mathrm{id}$. Then there exist $g_1 \neq g_2 \in \mathcal{C}(Y)$ such that $f_1 = \mathcal{H}(g_1)$ and $f_2 = \mathcal{H}(g_2)$. We have $\mathrm{id} = f_1^2 = \mathcal{H}(g_1)^2 = \mathcal{H}(g_1^m)$ and $\mathrm{id} = f_2^2 = \mathcal{H}(g_2)^2 = \mathcal{H}(g_2^m)$, implying that $g_1^m = g_2^m$, because $\mathcal{H}$ is one-to-one. Since m is odd, f_1 is a fixed point for $\mathcal{J}_m$, implying that g_1 is a fixed point for $\mathcal{J}_2$, that is, $g_1^2 = g_1$ and therefore $g_1^m = g_1$. Similarly, since f_2 is a fixed point for $\mathcal{J}_m$, it follows that g_2 is a fixed point for $\mathcal{J}_2$, that is, $g_2^2 = g_2$, and therefore $g_2^m = g_2$. Hence, we have $g_1 = g_2$, which is a contradiction. So $(\mathcal{C}(Y), \mathcal{J}_m)$ is not conjugate to $(\mathcal{C}(X), \mathcal{J}_2)$ and the result follows. □

6. *Remarks and questions*

The iteration operator $\mathcal{J}_1$ has no non-trivial periodic points in both $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(S^1)$. Also, unlike $\mathcal{C}(\mathbb{R})$, where there are no non-trivial periodic points as seen in Theorem 5, $\mathcal{J}_n$ has periodic points of all periods in $\mathcal{C}(S^1)$ for $n \geq 2$ (see Example 3). Further, we have the following result.

PROPOSITION. *The Sharkovskii theorem, that is, the existence of 3-periodic points implies that of k-periodic points for all $k \geq 1$, is valid in both the systems $(\mathcal{C}(\mathbb{R}), \mathcal{J}_n)$ and $(\mathcal{C}(S^1), \mathcal{J}_n)$ for all $n \geq 1$.*

https://doi.org/10.1017/etds.2023.34 Published online by Cambridge University Press

In fact, in the cases of $(\mathcal{C}(\mathbb{R}), \mathcal{J}_n)$ with $n \geq 1$ and $(\mathcal{C}(S^1), \mathcal{J}_1)$, the result follows vacuously since there are no 3-periodic points. However, in the case of $(\mathcal{C}(S^1), \mathcal{J}_n)$ with $n \geq 2$, there are k-periodic points for all $k \geq 1$ as seen from Example 3. Observe that this proposition makes a contrast with the Sharkovskii theorem on the underlying spaces $\mathbb{R}$ and S^1 because the Sharkovskii theorem holds valid on $\mathbb{R}$ but not on S^1 (see, for example, the rotation map $R_{2\pi/3}$ on S^1 defined as in Example 3).

Additionally, from Theorem 10, we can conclude that it suffices to discuss the dynamics of $\mathcal{J}_n$ on $\mathcal{C}(X)$ for X to be a representative of the equivalence class $[X]$ under the homeomorphism equivalence relation. In view of this observation, it follows that dynamics of $\mathcal{J}_n$ on $\mathcal{C}(X)$ is identical to that on:

(i) $\mathcal{C}(\mathbb{R})$ (respectively $\mathcal{C}([a, b])$) whenever X is, for example, a curve in $\mathbb{C}$ homeomorphic to $\mathbb{R}$ (respectively $[a, b]$);

(ii) $\mathcal{C}(S^1)$ whenever X is, for example, a simple closed curve in $\mathbb{C}$, or the real projective line $\mathbb{RP}^1$.

Moreover, by the *Hahn–Mazurkiewicz theorem* [**26**], a Hausdorff topological space X is a continuous image of $[a, b]$ or S^1 if and only if it is a *Peano space* (that is, a compact, connected, locally connected, metric space). Therefore, the equivalence class $[[a, b]]$ of $[a, b]$ (respectively $[S^1]$ of S^1) also contains subspaces of some 'complicated' spaces in the Euclidean plane $\mathbb{R}^2$ like the space filling curves—Peano curve, Hilbert curve, Sierpiński curve (respectively like Sierpiński triangle).

Finally, we conclude the paper with some problems for future discussion. Since any non-constant map f in $\mathcal{C}_{\text{id}}(\mathbb{R})$ (respectively $\mathcal{C}_{\text{id}}(S^1)$) has a unique choice for $f|_{R(f)}$, namely id, we were indeed able to use the definition of stability to prove the instability of such maps in Theorem 8 (respectively Theorem 9). However, if f lies in $\mathcal{C}_{\text{inv}}(S^1)$ (consisting of all continuous self-maps of S^1 which are orientation-reversing involutions on their range), then $f|_{R(f)}$ has uncountably many choices and therefore the case-wise approach used in the proof of the above theorem is impractical. Thus, the problem of stability of fixed points of $\mathcal{J}_n$, which are in $\mathcal{C}_{\text{inv}}(S^1)$, is highly non-trivial. For the same reason, the problem of stability of fixed points of $\mathcal{J}_n$ in $\mathcal{C}_{\text{inv}}(\mathbb{R})$ is also highly non-trivial. Additionally, although $\mathcal{J}_n$ does not have a non-trivial periodic point in $\mathcal{C}(\mathbb{R})$ by Theorem 5, the problem of stability of periodic points of $\mathcal{J}_n$ in $\mathcal{C}(S^1)$ is also difficult. Further, we remind that we did not conclude the discussion in Theorem 11 for all pairs n and m, which is interesting to consider. We observed through the counter-example given in §5 that a weaker converse of Theorem 10 is not true in general. However, it is interesting to investigate if the actual converse of Theorem 10 is true, that is, does $(\mathcal{C}(X), \mathcal{J}_n)$ is conjugate to $(\mathcal{C}(Y), \mathcal{J}_n)$ imply that X is homeomorphic to Y?

Acknowledgements. Chaitanya Gopalakrishna is supported by the National Institute of Technology Karnataka Surathkal through Senior Research Fellowship, the Indian Statistical Institute Bangalore through the J. C. Bose Fellowship of Prof. B. V. Rajarama Bhat, and the National Board for Higher Mathematics, India through No. 0204/3/2021/R&D-II/7389. Murugan Veerapazham is supported by SERB, DST, Government of India, through ECR/2017/000765. Weinian Zhang is supported by NSFC Nos. 11831012, 12171336, and 11821001.

REFERENCES

[1] R. F. Arens. A topology for spaces of transformations. *Ann. of Math. (2)* **47**(3) (1946), 480–495.
[2] C. Babbage. An essay towards the calculus of functions. *Philos. Trans. Roy. Soc. London* **105** (1815), 389–423.
[3] C. Bessaga and A. Pełczyński. *Selected Topics in Infinite-Dimensional Topology (Mathematical Monographs, 58)*. PWN–Polish Scientific Publishers, Warsaw, 1975.
[4] L. S. Block and W. A. Coppel. *Dynamics in One Dimension (Lecture Notes in Mathematics, 1513)*. Springer, Berlin, 1992.
[5] A. M. Blokh. The set of all iterates is nowhere dense in $C([0, 1], [0, 1])$. *Trans. Amer. Math. Soc.* **333**(2) (1992), 787–798.
[6] U. T. Bödewadt. Zur iteration reeller functionen. *Math. Z.* **49** (1944), 497–516.
[7] J. B. Conway. *Functions of One Complex Variable (Graduate Texts in Mathematics, 11)*, 2nd edn. Springer, New York, 1978.
[8] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Westview Press, New York, 1991.
[9] M. K. Fort. The embedding of homeomorphisms in flows. *Proc. Amer. Math. Soc.* **6**(6) (1955), 960–967.
[10] R. A. Holmgren. *A First Course in Discrete Dynamical Systems*. Springer, New York, 1991.
[11] W. Jarczyk. On an equation of linear iteration. *Aequationes Math.* **51**(3) (1996), 303–310.
[12] W. Jarczyk. Babbage equation on the circle. *Publ. Math.* **63**(3) (2003), 389–400.
[13] M. Kuczma. On monotonic solutions of a functional equation I. *Ann. Polon. Math.* **9** (1960), 295–297.
[14] M. Kuczma. *Functional Equations in a Single Variable*. Polish Scientific Publishers, Warsaw, 1968.
[15] S. Kundu and P. Garg. The compact-open topology: a new perspective. *Topology Appl.* **156**(4) (2009), 686–696.
[16] T.-Y. Li and J. A. Yorke. Period three implies chaos. *Amer. Math. Monthly* **82**(10) (1975), 985–992.
[17] R. A. McCoy and I. Ntantu. *Topological Properties of Spaces of Continuous Functions (Lecture Notes in Mathematics, 1315)*. Springer, Berlin, 1988.
[18] N. McShane. On the periodicity of homomorphisms of the real line. *Amer. Math. Monthly* **68**(6) (1961), 562–563.
[19] J. Milnor. *Dynamics in One Complex Variable*, 3rd edn. Princeton University Press, Princeton, NJ, 2006.
[20] J. Milnor and W. Thurston. On Iterated Maps of the Interval. *Dynamical Systems (Lecture Notes in Mathematics, 1342)*. Ed. J. C. Alexander. Springer, New York, 1988, pp. 465–563.
[21] J. R. Munkres. *Topology*, 2nd edn. Prentice Hall, Inc., Upper Saddle River, NJ, 2000.
[22] L. Narici and E. Beckenstein. *Topological Vector Spaces (Pure and Applied Mathematics, 296)*, 2nd edn. CRC Press, Boca Raton, FL, 2011.
[23] M. Veerapazham, C. Gopalakrishna and W. Zhang. Dynamics of the iteration operator on the space of continuous self-maps. *Proc. Amer. Math. Soc.* **149**(1) (2021), 217–229.
[24] E. Vincze. Über die Charakterisierung der assoziativen Funktionen von mehreren Veränderlichen. *Publ. Math. Debrecen* **6** (1959), 241–253.
[25] C. T. C. Wall. *A Geometric Introduction to Topology*. Dover Publications, New York, 1993.
[26] S. Willard. *General Topology*. Dover Publications, Inc., Mineola, NY, 2004.
[27] W. Zhang. PM functions, their characteristic intervals and iterative roots. *Ann. Polon. Math.* **65**(2) (1997), 119–128.
[28] W. M. Zhang and W. Zhang. Continuity of iteration and approximation of iterative roots. *J. Comput. Appl. Math.* **235**(5) (2011), 1232–1244.