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On the Bernoulli property for rational maps

RICARDO MAÑÉ
Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brasil

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Abstract. Every rational function f with degree $d \geq 2$ has a unique invariant probability μ_f that maximizes entropy. It has been conjectured that the system (f, μ_f) is equivalent to the one sided Bernoulli shift $\sigma: B^+(1/d, \ldots, 1/d) \rightleftarrows$. In this paper we prove that there exists $m > 0$ such that (f^m, μ_f) is equivalent to $\sigma: B^+(1/d^m, \ldots, 1/d^m) \rightleftarrows$.

0. *Introduction*

Let f be an analytic endomorphism of the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, of topological degree $d \geq 2$. Such an endomorphism is always given by a rational function $f(z) = P(z)/Q(z)$, where P and Q are polynomials with no common roots and max {degree (P), degree (Q)} = d. In **[5]** and independently in **[2]**, **[6]** it was proved that there exists an f-invariant probability μ_f on the Borel σ-algebra of $\bar{\mathbb{C}}$ exhibiting several interesting properties. First, *it is the unique f-invariant probability on the Borel σ-algebra such that*

$$\mu_f(f(A)) = d\mu_f(A) \tag{1}$$

for every Borel set A such that $f|A$ is injective. Second, *it is the unique f-invariant probability that maximizes entropy* i.e. $h_\mu(f) = \log d$ (recall that by **[7]** and **[3]** the topological entropy of f is $\log d$). Third, with respect to μ_f, *f is exact* i.e. if $\mathscr{A}$ denotes the Borel σ-algebra of $\bar{\mathbb{C}}$ then every set $A \in \bigcap_{n \geq 1} f^{-n}(\mathscr{A})$ satisfies $\mu_f(A) = 0$ or 1. Finally, *μ_f describes the asymptotic distribution of the roots of the equation* $f^n(z) = a$. More precisely, if $z_i^{(n)}(a)$, $1 \leq i \leq d^n$ are the roots of the equation $f^n(z) = a$, define a probability

$$\mu^{(n)}(a) = d^{-n} \sum_{i=1}^{d^n} \delta_{z_i^{(n)}(a)}.$$

Then $\mu^{(n)}(a) \to \mu_f$ for every $a \in \bar{\mathbb{C}}$ with two possible exceptions that can be characterized explicitly. From this property it is not difficult to deduce that the support of μ_f is exactly the Julia set $J(f)$ of f. When f is a polynomial, Brolin **[1]** proved that μ_f is the equilibrium distribution of the Julia set (in the sense of potential theory, see **[1]** for the precise statement).

In both **[2]** and **[5]** it was conjectured that the system (f, μ_f) is equivalent to the *one sided* shift $\sigma: B^+(1/d, \ldots, 1/d) \rightleftarrows$ (or, what is the same, the transformation $z \to z^d$ of the circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ endowed with Lebesgue measure). We shall give below some arguments to sustain this conjecture. The purpose of this paper is to prove that *for some $m \geq 1$, the system (f^m, μ_f) is equivalent to the one sided Bernoulli shift $\sigma: B^+(1/d^m, \ldots, 1/d^m) \rightleftarrows$.* In particular the natural extension of f^m is an invertible Bernoulli shift. Since roots of invertible Bernoulli shifts are also

Bernoulli [8], we conclude that the natural extension of (f, μ_f) is Bernoulli. Moreover we shall prove that *the equivalence between* $(f^n|J(f), \mu_f)$ *and* σ: $B^+(1/d^n, \ldots, 1/d^n) \circlearrowleft$ *can be realized by a topological equivalence between full measure invariant residual sets.*

Before going into the formal statement of our result and discussing some questions raised by the equivalence conjecture, we shall give a brief idea of why this conjecture is so natural. To prove the conjecture it suffices to find a Borel set X, with $\mu_f(X) = 1$ and $f^{-1}(X) = X$, and a partition $X = \bigcup_{i=1}^d X_i$ in disjoint Borel sets satisfying:

(a) for all $1 \le i \le d$, $f|X_i$ is injective and $f(X_i) = X$;

(b) if $n \ge 1$ and $1 \le i_j \le d$, $j = 1, \ldots, n$:

$$\mu_f\left(\bigcap_{j=1}^{n} f^{-j}(X_{i_j})\right) = d^{-n};$$

(c) if $\mathscr{P}$ denotes the partition $\{X_1, \ldots, X_d\}$ then $\bigvee_{n=0}^{\infty} f^{-n}(\mathscr{P})$ coincides mod (0) with the Borel σ-algebra of $\bar{\mathbb{C}}$.

Now observe that (b) is an easy consequence of (a) and (1). Then we have to worry only about (a) and (c). As we shall see, (a) is quite easy to satisfy and (c) becomes, at least in this approach, the obstruction to solving the conjecture. Take an arc γ containing all the critical values of f (i.e. the images under f of its critical points) satisfying $\mu_f(\gamma) = 0$. Standard measure-theoretical arguments show the existence of such arcs. Then $\hat{X} = \bar{\mathbb{C}} - \gamma$ is a topological disk (i.e. a set homeomorphic to the unit disk) and doesn't contain critical values of f. Then, there exist branches $\varphi_i : \hat{X} \to \bar{\mathbb{C}}$ $i = 1, \ldots, d$, of $f^{-1}|\hat{X}$. Set $X_i = \varphi_i(\hat{X})$. It is easy to see that $f|X_i$ is injective and $f(\hat{X}_i) = \hat{X}$ for every $1 \le i \le d$. Moreover

$$\mu_f(\hat{X}) = 1 - \mu_f(\gamma) = 0.$$

But it is not necessarily true that $f^{-1}(\hat{X}) = \hat{X}$. To correct this, define

$$X = \left(\bigcup_{n\ge0} f^{-n}\left(\bigcup_{m\ge0} f^m(\hat{X}^c)\right)\right)^c.$$

This set has full measure. This follows from the fact X^c has zero measure, the f-invariance of μ_f, and the property (not listed above but proved in [2] or [5]) that f transforms zero measure sets into zero measure sets. It also satisfies $f^{-1}(X) = X$. Now set

$$X_i = \hat{X}_i \cap X.$$

Property (a) still holds for $X, X_1, \ldots, X_d$. *The problem is to decide whether* (c) *is satisfied.* It is reasonable to think that the answer may depend on the choice of γ. We shall show that, for high powers of f, this method works for a suitable choice of the curve γ.

An alternative approach to solve the conjecture could be the following. Suppose that defining $\hat{X}$, X, $X_1, \ldots, X_d$ as above, condition (c) is not satisfied. Write $\mathscr{A}_0 = \bigvee_{n\ge0} f^{-n}(\mathscr{P})$. Then $\mathscr{A} \ne \mathscr{A}_0$ (where $\mathscr{A}$ denotes the Borel σ-algebra). If we can find an f-invariant probability measure $\mu : \mathscr{A} \to [0, 1]$ such that $\mu|\mathscr{A}_0 = \mu_f|\mathscr{A}_0$ but $\mu \ne \mu_f$, the conjecture will be proved because $h_\mu(f, \mathscr{P}) = h_{\mu_f}(f, \mathscr{P})$ and this last term is $\log d$ (by (a) and (b)). Then we shall have another f-invariant probability with maximal entropy, contradicting the uniqueness of the invariant probability with this property.

THEOREM. *There exists* $N \geq 1$, *Borel sets* $\Lambda_1 \subset J(f)$, $\Lambda_2 \subset B^+(1/d^N, \ldots, 1/d^N)$ *and a homeomorphism* $h: \Lambda_1 \to \Lambda_2$ *such that*:

(a) $f^{-N}(\Lambda_1) = \Lambda_1$, $\sigma^{-1}(\Lambda_2) = \Lambda_2$;

(b) $\mu_f(\Lambda_1) = 1$, $\mu(\Lambda_2) = 1$ (*where* μ *denotes the Bernoulli measure of* $B^+(1/d^m, \ldots, 1/d^m)$);

(c) Λ_1 *and* Λ_2 *are residual*;

(d) $\sigma h(x) = hf^N(x)$, *for all* $x \in \Lambda_1$.

The map h is constructed, as usual in this kind of result, using symbolic dynamics. In fact, given an analytic endomorphism $g: \bar{\mathbb{C}} \rightleftarrows$ of degree $l \geq 2$ and a topological disk U with $\mu_g(U) = 1$ not containing critical values of g we can define a measurable map $h: \Lambda \to B^+(1/l, \ldots, 1/l)$ where $\Lambda \subset J(g)$ is a residual set satisfying $g(\Lambda) \subset \Lambda$, such that $\sigma h(x) = hg(x)$ for all $x \in \Lambda$ and $h^*(\mu_g)$ is the Bernoulli measure of $B^+(1/l, \ldots, 1/l)$. This map is constructed as follows. Let $\varphi_i: U \to \bar{\mathbb{C}}$, $1 \leq i \leq l$, be the branches of $f^{-1}|U$. Then define $U_i = \varphi_i(U)$, $\Lambda = J(g) \cap (\bigcap_{n \geq 0} g^{-n}(\bigcup_i U_i))$ and $h: \Lambda \to B^+(1/l, \ldots, 1/l)$ by the condition:

$$g^n(x) \in U_{h(x)(n)}$$

for all $n \geq 0$. It is easy to check that h has the previous properties. The proof of the theorem consists of finding a power $N \geq 1$ and a topological disk $U \supset J(f^N)$ such that the map h constructed above satisfies the properties in the theorem. The boundary of the set U will be a finite union of continuous curves.

Our theorem naturally raises the question of whether a measure preserving map $T: B^+(1/d^N, \ldots, 1/d^N) \rightleftarrows$ such that $T^N = \sigma$ is equivalent to the Bernoulli shift of $B^+(1/d, \ldots, 1/d)$. Another question is whether an exact measure preserving map T of a probability space $(X, \mathcal{A}, \mu)$ which for some integer $d > 1$ satisfies $\mu(T(A)) = d(A)$ for every $A \in \mathcal{A}$ where $T|A$ is injective and such that X can be partitioned into sets $X_1, \ldots, X_d$ such that $T|X_i$ is a bijection between X_i and X for every i, is equivalent to the Bernoulli shift of $B^+(1/d, \ldots, 1/d)$. Obviously an affirmative answer to any of these questions implies an affirmative answer of the conjecture. But as interesting as knowing that $(f|J(f), \mu_f)$ is equivalent to $\sigma: B^+(1/d, \ldots, 1/d) \rightleftarrows$ is to understand geometrically the map that realizes the equivalence. More precisely, from the viewpoint of the understanding of the dynamics of f, the proper statement of the conjecture is *whether a full measure topological disk* U (preferably having a nice geometry) *can be found such that the map* $h: \Lambda \to B^+(1/d, \ldots, 1/d)$ *associated to* U *as above, is a measure preserving topological equivalence between full measure residual subsets.*

I. *Proof of the theorem*

In this section we shall deduce the theorem from the following lemma, which will be proved in the following four sections. Recall that if $g: \bar{\mathbb{C}} \rightleftarrows$ is an analytic map and $U \subset \bar{\mathbb{C}}$ is an open set, we say that a function $\varphi: U \to \bar{\mathbb{C}}$ is a branch of $g^{-1}|U$ if φ is continuous and $g(\varphi(x)) = x$ for all $x \in U$ (and then φ is analytic).

FUNDAMENTAL LEMMA. *There exists $N>1$, $\alpha>0$ and a topological disk $\tilde{U}$ with $\mu_f(\tilde{U})>0$ such that for all $n\geq 1$ there exist branches $\varphi_i^{(n)}:\tilde{U}\to\bar{\mathbb{C}}$ of $f^{-nN}|\tilde{U}$, $i=1,\ldots,k_n$ satisfying*:

(i) $\lim_{n\to+\infty}\max\{\operatorname{diam}(\varphi_i^{(n)}(\tilde{U}))|1\leq i\leq k_n\}=0$;

(ii) $\mu_f(\bigcup_i\varphi_i^{(n)}(\tilde{U}))\geq\alpha\mu_f(\tilde{U})$;

(iii) $\varphi_i^{(n)}(\tilde{U})\subset\tilde{U}$;

(iv) $f^N(\bigcup_i\varphi_i^{(n+1)}(\tilde{U}))\subset\bigcup_i\varphi_i^{(n)}(\tilde{U})$;

for all $n\geq 1$, $1\leq i\leq k_n$.

The first step in applying the fundamental lemma to prove the theorem, is to show that $\mu_f(\tilde{U})=1$. This will follow from the following lemma:

LEMMA I.1. *Let T be a mixing transformation of the probability space $(X,\mathscr{A},\mu)$ and suppose that $A_n\in\mathscr{A}$, $n\geq 1$, is a sequence of sets satisfying*:

(a) $\mu(A_0)>0$;

(b) $A_n\subset T^{-1}(A_{n-1})$ *for all* $n\geq 1$;

(c) *there exists* $\alpha>0$ *such that for all* $n\geq 1$:

$$\mu(A_n)\geq\alpha\mu(A_0).$$

Then $\mu(\bigcup_{j\geq l}A_j)=1$ *for all* $l\geq 0$.

Proof. Define $\alpha_n=\mu(A_n)/\mu(A_0)$. By (b) $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha>0$. Define $\alpha_\infty=\lim_{n\to\infty}\alpha_n$. If $n\geq m\geq 0$, (b) implies

$$T^{-(n-m)}(A_m)\supset A_n$$

and

$$\mu(T^{-(n-m)}(A_m)-A_n)=\mu(A_m)-\mu(A_n)=(\alpha_m-\alpha_n)\mu(A_0).$$

Then for all $S\in\mathscr{A}$ and $n\geq m\geq 0$:

$$\begin{aligned}\mu\Big(\Big(\bigcup_{j\geq m}A_j\Big)\cap S\Big)&\geq\mu(A_n\cap S)\\&=\mu(T^{-(n-m)}(A_m)\cap S)-\mu((T^{-(n-m)}(A_m)-A_n)\cap S)\\&\geq\mu(T^{-(n-m)}(A_m)\cap S)-(\alpha_m-\alpha_n)\mu(A_0).\end{aligned}$$

Since T is mixing, taking the limit as $n\to+\infty$, we obtain

$$\begin{aligned}&\lim_{n\to+\infty}(\mu(T^{-(n-m)}(A_m)\cap S)-(\alpha_m-\alpha_n)\mu(A_0))\\&\quad=\lim_{n\to+\infty}\mu(T^{-n}(T^{-m}(A_m))\cap S)-(\alpha_m-\alpha_\infty)\mu(A_0)\\&\quad=\mu(T^{-m}(A_m))\mu(S)-(\alpha_m-\alpha_\infty)\mu(A_0)\\&\quad=\mu(A_m)\mu(S)-(\alpha_m-\alpha_\infty)\mu(A_0)\\&\quad=(\alpha_m\mu(S)-(\alpha_m-\alpha_\infty))\mu(A_0)>0,\end{aligned}$$

if m is so large that $\alpha_m\mu(S)-(\alpha_m-\alpha_\infty)>0$. Hence

$$\mu\Big(\Big(\bigcup_{j\geq m}A_j\Big)\cap S\Big)>0$$

for every $m\geq 0$, $S\in\mathscr{A}$. Then $\mu(\bigcup_{j\geq m}A_j)=1$ for every $m\geq 0$. □

Thus the set $\tilde{U}$ in the fundamental lemma has full measure. This follows by applying I.1 to $f^N = T$, $A_0 = \tilde{U}$, $A_n = \bigcup_{j=1}^{k_n} \varphi_i^{(n)}(\tilde{U})$. Then

$$\mu_f\left(\bigcup_{n\geq m}\bigcup_{i=1}^{k_n}\varphi_i^{(n)}(\tilde{U})\right)=1 \qquad \text{for all } m,$$

and (iii) implies $\mu_f(\tilde{U}) = 1$.

To prove the theorem set $g = f^N$ and take a topological disk $U \subset \tilde{U}$ such that $\mu_f(\tilde{U} - U) = 0$ and U doesn't contain critical values of g. Such a topological disk is easily obtained by deleting some arcs of measure zero joining the critical values of g in $\tilde{U}$ with $\partial\tilde{U}$. Since U doesn't contain critical values of g, there exist branches $\varphi_i^{(1)}: U \to \bar{\mathbb{C}}$, $i = 1, \ldots, d^N$ of $g^{-1}|U$. Set $V_i = \varphi_i^{(1)}(U) \cap J(f)$, $V = \bigcup_i V_i$ and define

$$\tilde{\Lambda} = J(f) - \bigcup_{n\geq 0} g^{-n}\left(\bigcup_{m\geq 0} g^m(V^c)\right).$$

It is easy to check that $\mu_f(\tilde{\Lambda}) = 1$ and $g^{-1}(\tilde{\Lambda}) = \tilde{\Lambda}$. Define

$$\Lambda_0 = \bigcap_{m=1}^{\infty}\bigcup_{n\geq m}\bigcup_{i=1}^{k_n}\varphi_i^{(n)}(U \cap J(f)).$$

Then $\mu_f(\Lambda_0) = 1$. But it is not true that $g^{-1}(\Lambda_0) = \Lambda_0$. To obtain this property too we introduce the set

$$\hat{\Lambda} = J(f) - \bigcup_{n\geq 0} g^{-n}\left(\bigcup_{m\geq 0} g^m(J(f) - \Lambda_0)\right).$$

This set satisfies $\mu_f(\hat{\Lambda}) = 1$ and $g^{-1}(\hat{\Lambda}) = \hat{\Lambda}$. Hence, if we set

$$\Lambda_1 = \hat{\Lambda} \cap \tilde{\Lambda}$$

we get $\mu_f(\Lambda_1) = 1$ and $g^{-1}(\Lambda_1) = \Lambda_1$. Now we can define a map $h: \tilde{\Lambda} \to B^+(1/d^N, \ldots, 1/d^N)$ by the condition

$$g^n(x) \in V_{h(x)(n)}$$

for all $n \geq 0$. It is clearly a measurable map and

$$\sigma h(x) = hg(x) \tag{1}$$

for all $x \in \tilde{\Lambda}$. Now we shall show that $\mu_f(h^{-1}(A)) = \mu(A)$ for every Borel set $A \subset B^+(1/d^N, \ldots, 1/d^N)$. It is sufficient to prove this relation when A has the form $A = \bigcap_{j=0}^n \sigma^{-j}(S_{i_j})$, where $S_i = \{\theta \in B^+(1/d^N, \ldots, 1/d^N): \theta(0) = i\}$, because these sets generate the Borel σ-algebra. We leave to the reader the task of verifying by induction on n that

$$h^{-1}(A) = \tilde{\Lambda} \cap \left(\bigcap_{j=0}^{n} g^{-j}(V_{i_j})\right) \tag{2}$$

and that $g^n|(\bigcap_{j=0}^n g^{-j}(V_{i_j}))$ is a bijection onto $J(f) \cap U$. Then

$$\mu_f(h^{-1}(A)) = \mu_f\left(\bigcap_{j=0}^{n} g^{-j}(V_{i_j})\right)$$

because $\mu_f(\tilde{\Lambda}) = 1$, and

$$\mu_f\left(\bigcap_{j=0}^{n} g^{-j}(V_{i_j})\right) = 1/d^{nN}$$

because of property (1) in the introduction. On the other hand $\mu(A) = \prod_{j=0}^n \mu(S_{i_j}) = 1/d^{nN}$, completing the proof that $\mu_f(h^{-1}(A)) = \mu(A)$. Now let us prove that $\hat{\Lambda}$ and

$\tilde{\Lambda}$ are residual subsets of $J(f)$. This will prove that so is $\Lambda_1 = \hat{\Lambda} \cap \tilde{\Lambda}$. The sets $\bigcup_{n\geq m} \bigcup_i \varphi_i^{(n)}(U \cap J(f))$ are open in $J(f)$, and also dense because they have full measure. Then Λ_0, that is their intersection, is residual in $J(f)$. Therefore $J(f) - \Lambda_0$ is meagre and then so is the set $\bigcup_{n\geq 0} g^{-n}(\bigcup_{m\geq 0} g^m(J(f) - \Lambda_0))$. Hence $\hat{\Lambda}$ is residual in $J(f)$. Moreover V is open in $J(f)$. But it has full measure because

$$\mu_f(V) = \mu_f(g^{-1}(U) \cap J(f)) = \mu_f(U) = 1.$$

Then it is open and dense. Therefore V^c is closed and its interior is empty. It follows that $\bigcup_{n\geq 0} g^{-n}(\bigcup_{m\geq 0} g^m(V^c))$ is meagre and $\tilde{\Lambda}$ is residual in $J(f)$. Hence Λ_1 is residual in $J(f)$.

Our next step is to prove that h is injective and a homeomorphism onto $h(\Lambda_1)$. For this we shall need the following lemma:

LEMMA I.2. *For every* $x \in \Lambda_1$

$$\lim_{n\to+\infty} \operatorname{diam}\left(\bigcap_{j=0}^{n} g^{-j}(\varphi_{h(x)(j)}(U)\right) = 0.$$

Proof. We claim that if $n \geq 1$ and $x \in \varphi_{i_n}^{(n)}(U)$ for some $1 \leq i_n \leq k_n$:

$$\bigcap_{j=0}^{n-1} g^{-j}(\varphi^{(1)}_{h(x)(j)}(U)) \subset \varphi^{(n)}_{i_n}(U).$$

We shall prove this by induction. Assume it is true for $n = m$. Suppose that $x \in \varphi^{(m+1)}_{i_{m+1}}(U)$ for some $1 \leq i \leq k_{m+1}$. By property (iv) of the fundamental lemma $g(x) \in \varphi^{(m)}_{i_m}(\tilde{U})$ for some $1 \leq i_m \leq k_m$. But we know that $x \in \Lambda_1$, that means that $g^j(x) \in U$ for all $j \geq 0$. Then $g(x) \in \varphi^{(m)}_{i_m}(U)$ because if $g(x) \in \varphi^{(m)}_{i_m}(\tilde{U}) - \varphi^{(m)}_{i_m}(U)$ then $g^m(g(x)) \in \tilde{U} - U$, hence $g^{m+1}(x) \notin U$. Now we apply the induction hypothesis to the point $g(x)$ and we obtain:

$$\bigcap_{j=0}^{m-1} g^{-j}(\varphi^{(1)}_{h(g(x))(j)}(U)) \subset \varphi^{(m)}_{i_m}(U).$$

Then

$$\begin{aligned} x &\in \bigcap_{j=0}^{m} g^{-j}(\varphi^{(1)}_{h(x)(j)}(U)) \\ &= \varphi^{(1)}_{h(x)(0)}(U) \cap g^{-1}\left(\bigcap_{j=0}^{m-1} g^{-j}(\varphi^{(1)}_{h(g(x))(j)}(U)\right) \\ &\subset \varphi^{(1)}_{h(x)(0)}(U) \cap g^{-1}(\varphi^{(m)}_{i_m}(U)). \end{aligned} \tag{3}$$

Now observe that $\varphi^{(1)}_{h(x)(0)}\varphi^{(m)}_{i_m}|U$ is a branch of $g^{-(m+1)}|U$ and

$$\varphi^{(1)}_{h(x)(0)}\varphi^{(m)}_{i_m}(U) = \varphi^{(1)}_{h(x)(0)}(U) \cap g^{-1}(\varphi^{(m)}_{i_m}(U)). \tag{4}$$

By (3)

$$x \in \varphi^{(1)}_{h(x)(0)}(U) \cap g^{-1}(\varphi^{(m)}_{i_m}(U)).$$

Then by (4) $\varphi^{(1)}_{h(x)(0)}\varphi^{(m)}_{i_m}|U$ is a branch of $g^{-(m+1)}|U$ whose image covers x. So is $\varphi^{(m+1)}_{i_{m+1}}|U$. Then $\varphi^{(1)}_{h(x)(0)}\varphi^{(m)}_{i_m}|U = \varphi^{(m+1)}_{i_{m+1}}$. Hence, using (3) and (4):

$$\begin{aligned} \bigcap_{j=0}^{m} g^{-j}(\varphi^{(1)}_{h(x)(j)}(U)) &\subset \varphi^{(1)}_{h(x)(0)}(U) \cap g^{-1}(\varphi^{(m)}_{i_m}(U)) \\ &= \varphi^{(m+1)}_{i_{m+1}}(U), \end{aligned}$$

as we wished to prove. To prove the lemma we use the fact that $x \in \bigcap_{m\geq 0} \bigcup_{j\geq m} g^{-j}(U)$ means that $x \in \varphi_{i_n}^{(n)}(U)$ for some $1 \leq i_n \leq k_n$, for infinitely many values of n. Then the claim implies

$$\lim_{n\to+\infty} \sup_{x\in\Lambda_1} \operatorname{diam}\left(\bigcap_{j=0}^{n} g^{-j}(\varphi_{h(x)(j)}^{(1)}(U))\right) \leq \lim_{n\to+\infty} \sup\,(\max_i \operatorname{diam}(\varphi_i^{(n)}(U))).$$

The last limit is zero by property (i) of the fundamental lemma. □

To prove that h has the desired properties we introduce the set:

$$\Sigma = \bigcap_{j\geq 0} g^{-j}(\bigcup_i V_i)$$

and the map $h_0: \Sigma \to B^+(1/d^N, \ldots, 1/d^N)$ defined by the condition:

$$g^j(x) \in V_{h_0(x)(j)} = \varphi_{(j)}^{(1)}(U) \cap J(f).$$

Clearly $\Sigma \supset \Lambda_1$ and $h_0|\Lambda_1 = h$.

LEMMA I.3. (a) *If $x \in \Lambda_1$, $y \in \Sigma$ and $y \neq x$, then $h_0(x) \neq h_0(y)$.*

(b) *If $\Sigma_0 \subset \Sigma$ and $x \in \Lambda_1$ is in the interior of Σ_0, then $h(x)$ is in the interior of $h(\Sigma_0)$ in $h(\Sigma)$.*

(c) *$h(A)$ is dense in $B^+(1/d^N, \ldots, 1/d^N)$ if $A \subset \Sigma$ is dense.*

(d) *For all $x \in \Lambda_1$:*

$$\lim_{n\to+\infty} \operatorname{diam} \bigcap_{j=0}^{n} g^{-j}(V_{h(x)(j)}) = 0.$$

(e) *$h(\Lambda_1)$ is a residual subset of $B^+(1/d^N, \ldots, 1/d^N)$.*

Proof. (a) If $x \in \Lambda_1$ then $x \in \bigcap_{m\geq 0} \bigcap_{n\geq m} g^{-n}(U)$. Therefore by I.2

$$y \notin \bigcap_{j=0}^{n} g^{-j}(\varphi_{h_0(x)(j)}^{(1)}(U))$$

if n is large. This means $g^j(y) \notin \varphi_{h_0(x)(j)}^{(1)}(U)$ for some $0 \leq j \leq n$. Therefore $h_0(y)(j) \neq h_0(x)(j)$, or $h(x) \neq h(y)$.

(d) follows from intersecting the set in I.2 with $J(f)$.

(c) We have to prove that for any $l \geq 0$ and $\theta \in B^+(1/d^N, \ldots, 1/d^N)$ there exists $x \in A$ with $h(x)(j) = \theta(j)$ if $0 \leq j \leq l$. The set

$$\bigcap_{j=0}^{l} g^{-j}(V_{\theta(j)}) \cap \Sigma$$

is open and non-empty in Σ. Then it contains a point $x \in A$ because A is dense in Σ. Hence $g^j(x) \in V_{\theta(j)}$ if $0 \leq j \leq l$. Then $h(x)(j) = \theta(j)$ for $0 \leq j \leq l$.

(b) If $x \in \operatorname{Int}(\Sigma_0)$, by (d):

$$\bigcap_{j=0}^{n} g^{-1}(V_{h(x)(j)}) \cap \Sigma \subset \Sigma_0$$

for n large enough. Consider the open set in $h(\Sigma)$ given by:

$$h(\Sigma) \cap \{\theta: \theta(j) = h(x), 0 \leq j \leq n\}.$$

If θ belongs to this set it means that $\theta=h(y)$, $y\in\Sigma$ and $h(y)(j)=h(x)(j)$ for $0\le j\le n$. Then $g^{-j}(y)\in V_{h(x)(j)}$ for $0\le j\le n$. This means that:

$$y\in\bigcap_{j=0}^{n} g^{-j}(V_{h(x)(j)})\cap\Sigma.$$

Therefore:

$$h(\Sigma_0)\supset h\left(\bigcap_{j=0}^{n} g^{-j}(V_{h(x)(j)})\cap\Sigma\right)\supset h(\Sigma)\cap\{\theta\colon \theta(j)=h(x), 0\le j\le n\}$$

(e) To show that $h(\Lambda_1)$ is a residual set introduce the sets Σ_n of points $\theta\in B^+(1/d^N,\ldots,1/d^N)$ such that for some $m\ge n$:

$$\overline{\bigcap_{j=0}^{m} g^{-j}(V_{\theta(j)})}\subset\bigcap_{j=0}^{n} g^{-j}(V_{\theta(j)}).$$

Now we shall prove that $\bigcup_{n\ge1}\Sigma_n\supset h(\Lambda_1)$. We have to prove that $h(x)\in\Sigma_n$ for every $x\in\Sigma$ and $n\ge1$. If $x\in\Lambda_1$, $x\in\bigcap_{j=0}^{n-1} g^{-j}(V_{h(x)(j)})$. This set is open. Hence, by (d):

$$\overline{\bigcap_{j=0}^{m} g^{-j}(V_{h(x)(j)})}\subset\bigcap_{j=0}^{n-1} g^{-j}(V_{h(x)(j)})$$

if $m>n$ is large enough. Then $h(x)\in\Sigma_n$ as we wished to show. Moreover each Σ_n is open. Since it contains the set $h(\Lambda_1)$, which is dense by (c), it is also dense. Therefore $\Sigma_\infty=\bigcap_{n\ge1}\Sigma_n$ is a residual subset. Then, to prove that $h(\Lambda_1)$ is a residual subset of $B^+(1/d^N,\ldots,1/d^N)$ it suffices to prove that is residual in Σ_∞. Since Λ_1 is residual in Σ we can write $\Lambda_1=\bigcap_{n\ge0}A_n$, where each A_n is open and dense in Σ. By (a)

$$h(\Lambda_1)=\bigcap_{n\ge0} h(A_n).$$

But $h(\Lambda_1)\subset\Sigma_\infty$. Then

$$h(\Lambda_1)=\bigcap_{n\ge0} h(A_n)\cap\Sigma_\infty.$$

Moreover $\Lambda_1\subset A_n=\operatorname{Int}(A_n)$ for all n. By (b), $h(\Lambda_1)\subset\operatorname{Int}(A_n)$. Again, since $h(\Lambda_1)\subset\Sigma_\infty$, we obtain:

$$h(\Lambda_1)\subset\bigcap_{n\ge1}\operatorname{Int}(A_n)\cap\Sigma_\infty.$$

Then:

$$\bigcap_{n\ge0} h(A_n)\cap\Sigma_\infty=h(\Lambda_1)\subset\bigcap_{n\ge0}\operatorname{Int} h(A_n)\cap\Sigma_\infty\subset\bigcap_{n\ge0} h(A_n)\cap\Sigma_\infty.$$

This implies

$$h(\Lambda_1)=\bigcap_{n\ge0}\operatorname{Int} h(A_n)\cap\Sigma_\infty.$$

Then $h(\Lambda_1)$ is the intersection of the open subsets of Σ_∞ given by $\operatorname{Int} h(A_n)\cap\Sigma_\infty$, $n\ge1$. Since it is also dense (by (c)) it follows that is residual. $\square$

To complete the proof of the theorem, it remains only to show that $h^{-1}\colon h(\Lambda_1)\to\Lambda_1$ is continuous. Suppose that $\lim_{k\to+\infty}h(y_k)=h(x)$. We want to show that $\lim_{k\to+\infty}y_k=x$. But $h(y_k)\to h(y)$ means that $h(y_k)(j)=h_k(y)(j)$ for $0\le j\le n_k$, where $n_k\to+\infty$.

Then:

$$y_k \in \bigcap_{j=0}^{n_k} g^{-j}(V_{h(y_k)(j)}) = \bigcap_{j=0}^{n_k} g^{-j}(V_{h(x)(j)})$$

$$x \in \bigcap_{j=0}^{n_k} g^{-1}(V_{h(x)(j)}).$$

Then

$$d(y_k, x) \le \operatorname{diam} \bigcap_{j=0}^{n_k} g^{-j}(V_{h(x)(j)})$$

By I.3(d), it follows that $\lim_{k\to+\infty} y_k = x$. □

II. *Proof of the fundamental lemma*

Suppose that $a_1, \ldots, a_r$ are the periodic critical points of f. Without loss of generality we can suppose that they are fixed points of f. Take disjoint closed disks $D_1, \ldots, D_r$ centred at $a_1, \ldots, a_r$, not intersecting the Julia set of f, and satisfying $f(D_i) \subset D_i$. Set $D = \bigcup_i D_i$. Observe that $\mu_f(D^c) = 1$. To prove the fundamental lemma we shall need two preliminary lemmas.

If $g: \bar{\mathbb{C}} \circlearrowleft$ is an analytic map, we say that a topological disk $U \subset \bar{\mathbb{C}}$ is *g-adapted* if it doesn't contain critical values of g and its boundary is a finite union of C^1 arcs. The first step in the proof of the fundamental lemma, (lemma II.1 below), says that there exists $m \ge 1$ such that for any f^m-adapted topological disk $\tilde{U} \subset D^c$ there exists a branch $\varphi: \tilde{U} \to \bar{\mathbb{C}}$ of $f^{-m}|\tilde{U}$ such that $\varphi(\tilde{U})$ has a neighbourhood U_1 where for any $n \ge 1$ a reasonable number of branches of $f^{-nm}|U_1$ can be defined and they contract U_1 exponentially with respect to n. The topological disk $\tilde{U}$ is not going to be the set U of the fundamental lemma, neither is n going to be the N in that lemma. We shall still have to decrease U_0 (by deleting certain continuous arcs) and increase n.

LEMMA II.1. *There exists $m \ge 1$ such that if $\tilde{U} \subset D^c$ is an f^m-adapted topological disk, then there exists a branch $\hat{\varphi}: \tilde{U} \to \bar{\mathbb{C}}$ of $f^{-m}|\tilde{U}$, a topological disk $U_1 \supset \hat{\varphi}(\tilde{U})$, constants $\hat{\alpha} > 0$, $C > 1$ and $0 < \lambda < 1$, and for every $n \ge 1$, branches $\hat{\varphi}_i^{(n)}: U_1 \to \bar{\mathbb{C}}$ of $f^{-mn}|U_1$, with $1 \le i \le \hat{k}_n$, satisfying*:

(i) $f^m(\bigcup_i \hat{\varphi}^{(n+1)}(U_1)) \subset \bigcup_i \hat{\varphi}_i^{(n)}(U_1)$;

(ii) $\mu_f(\bigcup_i \hat{\varphi}_i^{(n)} \hat{\varphi}(\tilde{U})) \ge \tilde{\alpha}\mu_f(\hat{\varphi}(\tilde{U}))$;

(iii) $|(\hat{\varphi}_i^{(n)})'(z)| \le C\lambda^n$;

for all $n \ge 1$, $1 \le i \le \hat{k}_n$, $z \in U_1$.

The gap between what the fundamental lemma requires and II.1 produces is essentially that the property $\hat{\varphi}_i^{(n)}(U_0) \subset U_0$ is not granted by II.1. To fill this gap we shall need two lemmas that will tell us how to delete certain arcs of U_0 in order to, after increasing m, satisfy this property.

LEMMA II.2. *For all $m \ge 1$ and $\varepsilon > 0$ there exists an f^m-adapted topological disk $\tilde{U} \subset D^c$ and $0 < t < 1$ such that writing $B(\alpha, \tilde{U}^c) = \{z: d(z, \tilde{U}^c) \le \alpha\}$ then*:

$$\sum_{n=1}^{\infty} \mu_f(B(t^n, \tilde{U}^c)) \le \varepsilon.$$

To prove the fundamental lemma take n given by II.1 and for this m take $\tilde{U}$ and $0<t<1$ given by II.2. Choose $s>0$ so large that

$$\sum_{j=1}^{\infty} \mu_f(B(t^{j+s}, \tilde{U}^c)) \leq \frac{\tilde{\alpha}}{2d^m} \mu_f(\tilde{U}). \tag{1}$$

Now choose a positive integer m_1 such that

$$C\lambda_1^{m_1 n-1} \operatorname{diam} \hat{\varphi}(\tilde{U}) \leq t^{n+s}, \tag{2}$$

where C is the constant given by II.1 when applied to $\tilde{U}$. Set $N=m_1 m$. To prove the fundamental lemma we have to find the branches $\varphi_i^{(n)}: \tilde{U} \to \bar{\mathbb{C}}$, $1 \leq i \leq k_n$, satisfying the required properties. We shall define k_n and the branches by induction on n. Let $\hat{\varphi}$, $\hat{k}_n$, $\hat{\varphi}_i^{(n)}$, $\hat{\alpha}$, C and $0<\lambda<1$ be given by lemma II.1 when applied to the $\tilde{U}$ chosen above. Observe that $\hat{\varphi}_i^{(m_1 n-1)}\hat{\varphi}: \tilde{U} \to \bar{\mathbb{C}}$ is a branch of $f^{-Nn}|\tilde{U}$. Based on this, we define

$$S_1 = \{1 \leq i \leq \hat{k}_{m_1-1} : \hat{\varphi}_i^{(m_1-1)}\hat{\varphi}(\tilde{U}) \subset \tilde{U}\}.$$

Reorder the indices i in order to have

$$S_1 = \{1, \dots, k_1\}.$$

This defines k_1. As branches $\varphi_i^{(1)}$, $1 \leq i \leq k_1$ of $f^{-N}|\tilde{U}$ we take:

$$\varphi_i^{(1)} = \hat{\varphi}_i^{(m_1-1)}\hat{\varphi}.$$

Now define S_n as the set of indexes $1 \leq i \leq \hat{k}_{m_1 n}-1$ such that:

$$\hat{\varphi}_i^{(m_1 n-1)}\hat{\varphi}(\tilde{U}) \subset \tilde{U}$$

and

$$f^N \hat{\varphi}^{(m_1 n-1)}\hat{\varphi} = \varphi_j^{(n-1)}$$

for some $1 \leq j \leq k_{n-1}$. Now reorder the set S_n to have

$$S_n = \{1, \dots, k_n\}.$$

Take this as the definition of k_n and define the branches by:

$$\varphi_i^{(n)} = \hat{\varphi}_i^{(m_1 n-1)}\hat{\varphi}.$$

Clearly these definitions imply properties (i), (iii) and (iv) of the fundamental lemma. Let us prove that (ii) is also satisfied. Define the sets

$$A_n = \bigcup_i \varphi_i^{(n)}(\tilde{U})$$

$$B_n = \bigcup_i \{\varphi_i^{(m_1 n-1)}\hat{\varphi}(\tilde{U}) : \hat{\varphi}_i^{(m_1 n-1)}\hat{\varphi}(\tilde{U}) \cap \tilde{U}^c \neq \varnothing\}.$$

and observe that

$$B_n - A_n = C_n \cup f^{-1}(B_{n-1} - A_{n-1})$$
$$B_1 - A_1 = C_1.$$

Hence

$$\mu_f(B_n - A_n) \leq \mu_f(C_n) + \mu_f(B_{n-1} - A_{n-1}).$$

Therefore:

$$\mu_f(B_n - A_n) \leq \sum_{j=1}^{n} \mu_f(C_j) \leq \sum_{j=1}^{\infty} \mu_f(C_j) \tag{3}$$

But by property (iii) of II.1 and (2)

$$\operatorname{diam} \hat{\varphi}^{(m_1 n-1)}\hat{\varphi}(\tilde{U}) \leq C\lambda^{m_1 n-1} \operatorname{diam} \hat{\varphi}(\tilde{U}) \leq t^{n+s}$$

for all $n \geq 1$. Then

$$C_j \subset B(t^{j+s}, \tilde{U}^c),$$

and by (i)

$$\sum_{j=1}^{\infty} \mu_f(C_j) \leq \sum_{j=1}^{\infty} \mu_f(B(t^{j+s}, \tilde{U}^c)) \leq \frac{\tilde{\alpha}}{2d^m} \mu_f(\tilde{U}).$$

This inequality together with (3) implies:

$$\mu_f(A_n) \geq \mu_f(B_n) - \frac{\tilde{\alpha}}{2d^m} \mu_f(\tilde{U}).$$

Moreover, property (ii) of lemma II.1 implies:

$$\mu_f(B_n) \geq \tilde{\alpha}\mu_f(\hat{\varphi}(\tilde{U})) = \frac{\tilde{\alpha}}{d^m} \mu_f(\tilde{U}).$$

Hence

$$\mu_f(A_n) \geq \frac{\tilde{\alpha}}{d^m} \mu_f(\tilde{U}) - \frac{\tilde{\alpha}}{2d^m} \mu_f(\tilde{U}) = \frac{\tilde{\alpha}}{2d^m} \mu_f(\tilde{U}).$$

Therefore, taking $\alpha = \tilde{\alpha}/2d^m$, property (ii) of the fundamental lemma is satisfied. □

III. *Proof of lemma* II.1

There exists $r > 0$ such that for any power f^m of f, the order of every critical point z_0 of f^m that is not a periodic critical point, is $\leq r$. Therefore for all $m \geq 1$, every critical point $z_0 \notin D$ has order $\leq r$. Take m satisfying:

$$2dmr < d^{m/2}, \tag{1}$$

$$2dm < (1 - 4d^{-m/2})d^m, \tag{2}$$

$$1/2 < 1 - 4d^{-m/2}. \tag{3}$$

Now let $\tilde{U}$ be an f^m-adapted topological disk contained in D^c. Since $\tilde{U}$ doesn't contain critical values of f^m, there exist d^m branches ψ_i: $U_0 \to \bar{\mathbb{C}}$, $i = 1, \ldots, d^m$ of $f^{-m}|\tilde{U}$. From the definition of f^m-adapted topological disk and the fact that $\tilde{U}$ is contained in D^c it follows that a point of $\bar{\mathbb{C}}$ can belong to at most r sets $\psi_i(\tilde{U})$ (in fact to r sets of type $\partial\psi_i(\tilde{U})$ since the sets $\psi_i(\tilde{U})$, $1 \leq i \leq d^m$ are disjoint). Then we can take topological disks $W_i \supset \psi_i(\tilde{U})$ such that a point can belong to at most r sets W_i. We claim there exists $1 \leq i_0 \leq d^n$, a sequence

$$0 \leq \varepsilon_n \leq 2d^{-mn/2} \qquad n = 1, 2, \ldots, \tag{4}$$

and branches $\varphi_i^{(n)}$: $W_{i_0} \to \bar{\mathbb{C}}$, $1 \leq i \leq \hat{k}_n$ of $f^{-mn}|W_{i_0}$ satisfying:

$$\hat{k}_n \geq \left(1 - \sum_{j=1}^{n} \varepsilon_j\right) d^{mn}, \tag{5}$$

$$f^m\left(\bigcup_i \varphi_i^{(n+1)}(W_{i_0})\right) \subset \bigcup_i \varphi_i^{(n)}(W_{i_0}), \tag{6}$$

$$a(\varphi_i^{(n)}(W_{i_0})) \leq d^{-mn/2}, \tag{7}$$

for all $n \geq 1$ where $a(\cdot)$ denotes the Lebesgue measure normalized to have $a(\bar{\mathbb{C}}) = 1$.

To determine i_0 observe that the number of critical values of f^m is bounded by $2dm$. Since every critical value of f^m belongs to at most r sets W_i, it follows that at most $2dmr$ of these sets can contain critical values of f^m. By (1) there exists i_0 such that W_{i_0} doesn't contain critical values of f^m. Then we can take branches $\varphi_i^{(1)}: W_{i_0} \to \bar{\mathbb{C}}$, $j = 1, \ldots, d^m$ of $f^{-m}|W_{i_0}$. Since the sets $\varphi_i^{(1)}(W_{i_0})$ are disjoint:

$$1 = a(\bar{\mathbb{C}}) \geq \sum_{i=1}^{d^n} a(\varphi_i^{(1)}(W_{i_0})).$$

If $S = \{1 \leq i \leq d^n: a(\varphi_i^{(1)}(W_{i_0})) \leq d^{-m/2}\}$ we obtain:

$$1 \geq \sum_{i \notin S} a(\varphi_i^{(1)}(W_{i_0})) \geq (d^m - \#S)d^{-m/2}.$$

This implies:

$$\#S \geq (1 - d^{-m/2})d^m. \tag{8}$$

Arrange the indices to have $S = \{1, \ldots, \hat{k}_1\}$ and define $\varepsilon_1 = d^{-m/2}$. By (8), $\hat{k}_1 \geq (1 - \varepsilon_1)d^m$. Then $\hat{k}_1$ and ε_1 satisfy (4) and (5). Moreover $a(\varphi_i^{(1)}(W_{i_0})) \leq d^{-m/2}$ for $1 \leq i \leq \hat{k}_1$. Hence (7) also is satisfied. Now suppose we have constructed ε_j, k_j, $\varphi_i^{(j)}$ for $j = 1, 2, \ldots, n$ satisfying (4), (5), (6) and (7). Since the number of critical values of f^m is bounded by $2dm$ and the sets $\varphi_i^{(n)}(W_{i_0})$, $1 \leq i \leq \hat{k}_n$, are disjoint, there exist $k_n - 2dm$ of these sets not containing critical values of f^m. Arrange the index i in such a way that these sets are $\varphi_i^{(n)}(W_{i_0})$, $1 \leq i \leq \hat{k}_n - 2dm$. Observe that by (5), (4), (3) and (2):

$$\begin{aligned} k_n - 2dm &\geq \left(1 - \sum_{j=1}^{n} \varepsilon_j\right) d^{mn} - 2dm \\ &\geq \left(1 - \sum_{j=1}^{n} 2d^{-mj/2}\right) d^{mn} - 2dm \\ &\geq (1 - 2d^{-m/2}(1 - d^{-m/2})^{-1})d^{mn} - 2dm \\ &\geq (1 - 4d^{-m/2})d^{mn} - 2dm > 0. \end{aligned}$$

Define $S_0 = \{1 \leq i \leq \hat{k}_n : \varphi_i^{(n)}(W_{i_0})$ doesn't contain critical values of $f^m\}$. Then for every $j \in S_0$ there exist branches $\varphi_{j,i}: \varphi_j^{(n)}(W_{i_0}) \to \bar{\mathbb{C}}$ of $f^{-m}|\varphi_j^{(n)}(W_{i_0})$ with $i = 1, \ldots, d^m$. The compositions $\varphi_{i,j}\varphi_j^{(n)}: W_{i_0} \to \bar{\mathbb{C}}$ give us $d^m(\#S_0)$ branches of $f^{-(n+1)m}|W_{i_0}$, that we shall denote $\varphi_i^{(n+1)}$, $i = 1, \ldots, t_n$, where $t_n = d^m(\#S_0)$. Observe that:

$$f^m\left(\bigcup_{i=1}^{t_n} \varphi_i^{(n+1)}(W_{i_0})\right) \subset \bigcup_{i=1}^{\hat{k}_n} \varphi_i^{(n)}(W_{i_0}). \tag{9}$$

Define $S = \{1 \leq i \leq t_n: a(\varphi_i^{(n+1)}(W_{i_0})) \leq D^{-m(n+1)/2}\}$. Then:

$$\begin{aligned} 1 = a(\bar{\mathbb{C}}) &\geq \sum_{i=1}^{t_n} a(\varphi_i^{(n+1)}(W_{i_0})) \geq \sum_{i \notin S} a(\varphi_i^{(n+1)}(W_{i_0})) \\ &\geq (t_n - \#S)d^{-m(n+1)/2}. \end{aligned}$$

Then:

$$\begin{aligned}\#S &\geq t_n - d^{m(n+1)/2} \geq d^m(\#S_0) - d^{m(n+1)/2}\\ &\geq (\hat{k}_n - 2dm)d^m - d^{m(n+1)/2}\\ &\geq \left(1 - \sum_{j=1}^{n} \varepsilon_j\right)d^{m(n+1)} - 2d^{m+1}m - d^{m(n+1)/2}\\ &\geq \left(1 - \sum_{j=1}^{n} \varepsilon_j - (2md^{1-mn} + d^{-m(n+1)/2})\right)d^{m(n+1)}. \end{aligned} \tag{10}$$

Define

$$\varepsilon_{n+1} = 2md^{1-mn} + d^{-m(n+2)/2} \tag{11}$$

and $\hat{k}_{n+1} = \#S$. Arrange the indexes to have $S = \{1, \ldots, \hat{k}_{n+1}\}$. Using the fact that $n > 1$ it is easy to check that (11) and (1) imply (4). Clearly (10) and (11) imply (5). Property (6) follows from (9) and (7) from the definition of S. This completes the proof of the claim. To complete the proof of II.1 we shall use Koebe's theorem ([3]) which says that given a topological disk D and a compact subset $K \subset D$ there exists $A > 0$ such that any injective analytic function $f: D \to \bar{\mathbb{C}}$ satisfies $|f'(z_1)| \geq A|f'(z_2)|$ for every z_1 and z_2 in K. Take $D = W_{i_0}$ and as K a topological disk U_1 satisfying

$$W_{i_0} \supset \bar{U}_1 \supset U_1 \supset \overline{\psi_{i_0}(U_0)}.$$

Then, for all $n \geq 1$, $z \in \varphi_i^{(n)}(U)$, $1 \leq i \leq k_n$:

$$\begin{aligned} d^{-mn/2} &\geq a(\varphi_i^{(n)}(U_1)) \geq a(U_1) \inf_{w \in U_1} |(\varphi_i^{(n)})'(w)|^2\\ &\geq a(U_1)A^2|(\varphi^{(n)})'(z)|^2. \end{aligned}$$

Therefore:

$$|(\varphi_i^{(n)})'(z)| \leq A^{-1}a(U_1)^{-\frac{1}{2}}(d^{-m/4})^n \tag{12}$$

for all $n \geq 1$, $1 \leq i \leq k_n$, $z \in U_1$. Then (12) implies the existence of C and λ satisfying

$$|(\varphi_i^{(n)})'(z)| \leq C\lambda^n \tag{13}$$

for all $n \geq 1$, $1 \leq i \leq \hat{k}_n$, $z \in U_1$. Moreover, property (1) in the introduction implies:

$$\mu_f(\varphi_i^{(n)}(\varphi_{i_0}(\tilde{U}))) = d^{-mn}\mu_f(\varphi_{i_0}(\tilde{U}))$$

for all $n \geq 1$, $1 \leq i \leq \hat{k}_n$. Hence, by (5),

$$\begin{aligned} \mu_f\left(\bigcup_{i=1}^{\hat{k}_n} \varphi_i^{(n)}(\varphi_{i_0}(\tilde{U}))\right) &= \hat{k}_n d^{-mn}\mu_f(\varphi_{i_0}(\tilde{U}))\\ &\geq \left(1 - \sum_{j=1}^{n} c_j\right)\mu_f(\varphi_{i_0}(\tilde{U}))\\ &\geq \left(1 - \sum_{j=1}^{\infty} c_j\right)\mu_f(\varphi_{i_0}(\tilde{U}))\\ &= \left(1 - \sum_{j=1}^{\infty} \varepsilon_j\right)d^{-m}\mu_f(\tilde{U}). \end{aligned} \tag{14}$$

Define $\hat{\alpha} = (1-\sum_{j=1}^{\infty} \varepsilon_j)d^{-m}$. By, (2) and (3)

$$\hat{\alpha} = \left(1 - \sum_{j=1}^{\infty} \varepsilon_j\right) d^{-m} \geq (1 - 2d^{-m/2}(1-d^{-m/2})^{-1})d^m$$
$$> (1-4d^{-m/2})d^m > \tfrac{1}{2}d^m > 0. \tag{15}$$

Now to prove II.1 we just take $\hat{\varphi} = \varphi_{i_0}$, the topological disk U_1 and the constants $\hat{\alpha} > 0$, $C > 0$, $0 < \lambda < 1$. Then (15) implies (ii), (6) implies (i) and (13) implies (iii). By (15), $\hat{\alpha} > 0$. □

IV. *Proof of lemma* II.2

The proof of II.2 is based on the following property of probability measures on $\bar{\mathbb{C}}$.

LEMMA IV.1. *Let μ be a probability measure on the Borel σ-algebra of $\mathbb{C}$. Write $B(r) = \{z: |z| < r\}$. If for all $0 < t < 1$*

$$\sum_{n=1}^{\infty} \mu(B(t^n)) < +\infty,$$

then given any $0 < t < 1$ there exists $-1 \leq a \leq 1$ such that the segment $\gamma_a = \{u(1+ai): 0 \leq u \leq 1\}$ satisfies

$$\sum_{n=1}^{\infty} \mu(B(t^n, \gamma_a)) < +\infty$$

where $B(t^n, \gamma_a)$ denotes the set $\{z: d(z, \gamma_a) \leq t^n\}$.

In its turn, the proof of this lemma will require a preliminary result about finite measures on an interval.

LEMMA IV.2. *Let ν be a finite measure on the Borel σ-algebra of an interval $[a, b]$. Then for every $0 < t < 1$ and $0 < \alpha < 1$ the following property holds for almost every $x \in [a, b]$ with respect to Lebesgue measure: there exists $n(x)$ such that*

$$\nu((x - t^n, x + t^n)) \leq t^{\alpha n}$$

for all $n \geq n(x)$.

Proof. By identifying a with b we can translate the problem to a circle S (where it will be easier to handle). Suppose that $S = \{z: |z| = 1\}$. Write $S_n = \{x: \nu((x-t^n, x+t^n)) \leq t^{\alpha n}\}$. For every $n \geq 1$ let J_i, $1 \leq i \leq m_n$ be disjoint open intervals in S of length t^n, $m_n = [2\pi/t^n]$. For each $1 \leq i \leq m_n$ let $\hat{J}_i$ be an open interval with the same mid-point as J_i and diam $(\hat{J}_i) = 3$ diam J_i. Then:

$$\sum_{i=1}^{m_n} \nu(\hat{J}_i) \leq 3 \sum_{i=1}^{m_n} \nu(J_i),$$

because every J_i appears contained in at most three intervals of type $\hat{J}_i$. Then:

$$\sum_{i=1}^{m_n} \nu(\hat{J}_i) \leq 3\nu(S).$$

Defining $H_n = \{1 \leq i \leq m_n: \nu(\hat{J}_i) \leq t^{\alpha n}\}$ we obtain

$$3\nu(S) \geq \sum_{i \notin H_n} \nu(\hat{J}_i) \geq (m_n - \# H_n)t^{\alpha n}.$$

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Therefore

$$\# H_n \geq m_n - 3\nu(S)t^{-\alpha n} = [2\pi/t^n] - 3\nu(S)t^{-\alpha n}.$$

Writing λ for the Lebesgue measure on S:

$$\lambda\left(\bigcup_{i\in H_n} J_i\right) = t^n \# H_n \geq t^n[2\pi/t^n] - 3\nu(S)t^{(1-\alpha)n}$$
$$\geq 2\pi - t^n - 3\nu(S)t^{(1-\alpha)n}.$$

But

$$\bigcup_{i\in H_n} J_i \subset S_n$$

because if $x \in J_i$ and $i \in H_n$ we have $(x - t^n, x + t^n) \subset \hat{J}_i$, and then $\nu((x - t^n, x + t^n)) \leq \nu(\hat{J}_i) \leq t^{\alpha n}$. Then

$$\lambda(S_n^c) \leq \lambda\left(\left(\bigcup_{i\in H_n} J_i\right)^c\right) \leq t^n + 3\nu(S)t^{(1-\alpha)n}.$$

By the Borel-Cantelli lemma it follows that for λ-a.e. $x \in S$ there exists $n(x)$ such that x belongs to every set S_n for $n \geq n(x)$. This completes the proof of the lemma. □

Now let us prove IV.1. Consider the interval $L = \{2 + ti: |t| < 1\}$. If A is a subset of $\{2 + ti: t \in \mathbb{R}\}$, set $\hat{A} = \{sz: 0 < s < 1, z \in A\}$. Define a measure ν on L by $\nu(A) = \mu(\hat{A})$ if $A \subset L$ is a Borel set. Now we shall use the following elementary property whose verification we leave to the reader: There exist $\alpha_0 > 0$, $\beta_0 > 1$ and $C > 0$ such that if $0 < \alpha < \alpha_0$, $\beta_0 < \beta < 1/\alpha$ and $-\frac{1}{2} < a < \frac{1}{2}$, then

$$B(\alpha, \gamma_a) \subset B(\beta\alpha) \cup \hat{A}(a, \beta)$$

where:

$$A(a, \beta) = \{2 + si: 2a - C\beta^{-1} < s < 2a + C\beta^{-1}\}.$$

Apply this property to $\alpha = t^n$, $\beta = t^{-n/2}$. Then:

$$B(t^n, \gamma_a) \subset B(t^{n/2}) \cup \hat{A}(a, t^{-n/2}).$$

Hence

$$\mu(B(t^n, \gamma_a)) \leq \mu(B(t^{n/2})) + \mu(\hat{A}(a, t^{-n/2}))$$
$$= \mu(B(t^{n/2})) + \nu(\{2 + si: 2a - Ct^{n/2} < s < 2a + Ct^{n/2}\}).$$

By IV.2 we can choose $-\frac{1}{2} < a < \frac{1}{2}$ such that

$$\nu(\{2s + i: 2a - t^{\alpha n} < s < 2a + t^{\alpha n}\} \leq t^{\alpha^2 n}$$

for large n, say $n \geq n_1$. Then

$$\mu(B(t^n, \gamma_a)) \leq \mu(B(t^{n/2})) + t^{\alpha^2 n}$$

for $n \geq n_1$. Therefore

$$\sum_{n=n_1}^{\infty} \mu(B(t^n, \gamma_a)) \leq \sum_{n=n_1}^{\infty} \mu(B((\sqrt{t})^n)) + \sum_{n=n_1}^{\infty} (t^{\alpha^2})^n < +\infty.$$

In order to apply IV.1 to μ_f we shall first show that

$$\sum_{n=1}^{\infty} \mu_f(B(t^n, z)) < +\infty$$

for all $0<t<1$ and every critical value z of f (where $B(t^n, z)=\{w: |w-z|<t^n\}$). We shall prove the integrability of $\log|f'|$ with respect to ergodic measures of positive entropy.

LEMMA IV.3. *Let $g: \bar{\mathbb{C}} \circlearrowleft$ be an analytic endomorphism and μ a g-invariant ergodic probability defined on the Borel σ-algebra of $\bar{\mathbb{C}}$ with $h_\mu(g)>0$. Then*:

$$h_\mu(g) \le 2\int_{\bar{\mathbb{C}}} \log|g'|\, d\mu.$$

Proof. Observe that the function $z \to \log|g'(z)|$ is measurable and upper bounded. Then, even if its integral with respect to μ is $-\infty$, we can apply Birkhoff's theorem and get:

$$\lim_{n\to+\infty} \frac{1}{n}\log|(g^n)'(z)| = \int_{\bar{\mathbb{C}}} \log|g'|\, d\mu$$

for μ-a.e.z. Write λ for the integral on the right. Then λ is the Lyapunov exponent of g and its multiplicity is 2. By Ruelle's inequality:

$$h_\mu(g) \le 2\max\{\lambda, 0\}.$$

From $h_\mu(g)>0$ it follows that $\lambda>0$. Hence:

$$h_\mu(g) \le 2\lambda = 2\int_{\bar{\mathbb{C}}} \log|g'|\, d\mu. \qquad \square$$

LEMMA IV.4

$$\int_{\bar{\mathbb{C}}} \log d(z, w)\, d\mu_g(z) > -\infty$$

for every critical value w of g.

Proof. First observe that if w_0 is a critical point (i.e. $g'(w_0)=0$):

$$\int_{\bar{\mathbb{C}}} \log d(z, w_0)\, d\mu_g(z) > -\infty,$$

because in a sufficiently small disk D centred at w_0 there exists $C>0$ such that $\log d(z, w_0) \ge C\log|g'(z)|$ for $z \in D$. Then

$$\int_{\bar{\mathbb{C}}} \log d(z, w_0)\, d\mu_g(z) \ge \int_{D^c} \log d(z, w_0)\, d\mu_g(z) + C\int_D \log|g'(z)|\, d\mu_g(z).$$

But $z \to \log d(z, w_0)$ is continuous in D^c, hence the first integral is $>-\infty$, and so is the second by IV.3. Now set $w=g(w_0)$. Let D_1 be a disk centred at w, so small that it doesn't contain other critical values of g. Let γ be a segment joining w to ∂D_1 with $\mu_g(\gamma)=0$. Let $\varphi: D_1-\gamma \to \bar{\mathbb{C}}$ be a branch of $g^{-1}|(D_1-\gamma)$ such that $\lim_{z\to w}\varphi(z)=w_0$. By property (1) in the introduction $\mu_g(g(A))=d\mu_g(A)$, where d is the degree

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of g, for every Borel set $A\subset\varphi(D_1-\gamma)$. Then:

$$\int_{\varphi(D_1-\gamma)} d\log d(g(z),w)\,d\mu_g(z)=\int_{D_1-\gamma}\log(z,w)\,d\mu_g(z)$$
$$=\int_{D_1}\log(z,w)\,d\mu_g(z).$$

Take $0<k<1$ such that $d(g(z),w)\geq d(z,w)^k$ for all $z\in\varphi(D_1,\gamma)$. Then:

$$\log d(g(z),w)\geq k\log d(z,w).$$

Hence:

$$\int_{D_1}\log(z,w)\,d\mu_g(z)\geq dk\int_{\varphi(D_1-\gamma)}\log d(z,w)\,d\mu_g(z)>-\infty. \qquad \square$$

LEMMA IV.5. *If w is a critical value of g and $0<t<1$,*

$$\sum_{n=1}^{\infty}\mu_g(B(t^n,w))<+\infty.$$

Proof. Write $A_n=\{z:\ t^{n+1}<|w-z|<t^n\}$ and set

$$C=\int_{\bar{\mathbb{C}}}|\log d(z,w)|\,d\mu_g(z).$$

Then

$$C\geq\sum_{n=1}^{\infty}\int_{A_n}|\log d(z,w)|\,d\mu_g(z)\geq\sum_{n=1}^{\infty}(n+1)\log t\mu_g(A_n).$$

But $B(t^n,w)=\bigcup_{j=n}^{\infty}A_j$ and thus $\mu_g(B(t^n,w))=\sum_{j=n}^{\infty}\mu_g(A_j)$. Hence

$$\sum_{n=1}^{\infty}\mu_g(B(t^n,w))=\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}\mu_g(A_j)=\sum_{n=1}^{\infty}n\mu_g(A_n)\leq\frac{1}{\log t}C. \qquad \square$$

Now we are ready to prove II.2. Suppose first that D contains only one disk. Applying a C^∞ diffeomorphism of $\bar{\mathbb{C}}$ we can suppose that the D has the shape, and the critical values $w_1,\dots,w_k$ of f^m are arranged with respect to D, as in figure 1.

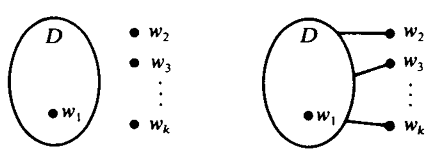

FIGURE 1

Using IV.5 and IV.1 we choose segments $\gamma_1,\dots,\gamma_k$ as in figure 1 such that

$$\sum_{n=1}^{\infty}\mu_f(B(t^n,\gamma_i))<+\infty \qquad i=1,\dots,k. \tag{1}$$

Then using (1) and the fact that D doesn't intersect the Julia set (and then has a neighbourhood with measure zero), $\tilde{U}=\bar{\mathbb{C}}-\bigcup_{i=1}^{k}\gamma_i$ satisfies II.2. If D is the union of two disks D_1, D_2, apply a diffeomorphism to place D_1, D_2 and the critical values as in figure 2. And choose $\gamma_1',\gamma_1,\dots,\gamma_k$ with property (1) (again using IV.1). Then

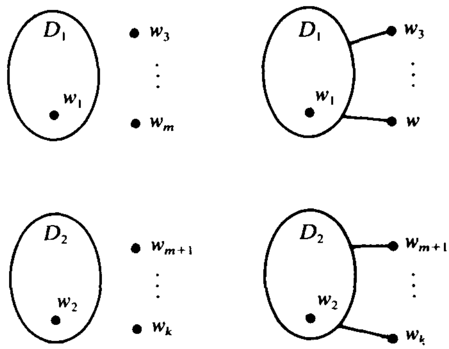

FIGURE 2

$\tilde{U} = \bar{\mathbb{C}} - \gamma_1' - \bigcup_{i=1}^{k} \gamma_i$ satisfies II.2. The case when D contains several disks is handled in a similar way. If $D = \varnothing$ take a segment γ situated with respect to $w_1, \ldots, w_k$ as in figure 3 (after applying a C^∞ diffeomorphism). Then choose $\gamma_1, \ldots, \gamma_k$ satisfying (1) as before and define $\tilde{U} = \bar{\mathbb{C}} - \bigcup_{i=1}^{k} \gamma_i$. □

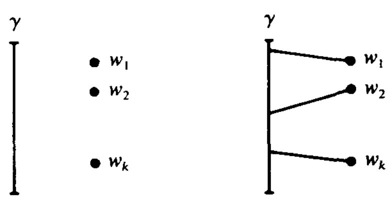

FIGURE 3

REFERENCES

[1] H. Brolin. Invariant sets under iteration of rational functions. *Arkiv für Matematik*, Band G, **6** (1966).
[2] A. Freire, A. Lopes & R. Mañé. An invariant measure for rational maps. *Bol. Soc. Bras. Mat.* **14** (1983) 45-62.
[3] M. Gromov. On the entropy of holomorphic maps. Preprint, IHES.
[4] E. Hille. *Analytic Functions Theory.*
[5] V. Ljubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergod. Th. & Dynam. Sys.* **3** (1983), 351-386.
[6] R. Mañé. On the uniqueness of the maximizing measure for rational maps. *Bol. Soc. Bras. Mat.* **14** (1983) 27-43.
[7] M. Misiurewicz & F. Przyticki. Topological entropy and degree of smooth mappings. *Bull. Acad. Pol. Sci.* **25** (1977), 573-574.
[8] D. Ornstein. Factors of Bernoulli shifts are Bernoulli. *Adv. in Math.* **5** (1970), 349-364.