Can. J. Math., Vol. XXV, No. 3, 1973, pp. 567–577

THE DUAL OF $H^p(R_+^{n+1})$ FOR $p < 1$

T. WALSH

Introduction. The dual of H^p of the unit disk for $0 < p < 1$ has been characterized by Duren, Romberg and Shields (see [**3**]). The present paper is concerned with the analogous result for $H^p(R_+^{n+1})$ in the sense of Stein and Weiss (see [**11**]). In this connection it may be recalled that the dual of H^1 has been characterized by Fefferman (see [**4**]). Recall that a system

$$F = \{F_0, F_1, \ldots, F_n\}$$

of $(n + 1)$ harmonic functions in the half space

$$R_+^{n+1} = \{(x, y) : x \in R^n, y > 0\}$$

belongs to $H^p(R_+^{n+1})$ for $p \geqq (n - 1)/n$ if F is the gradient of a harmonic function and

$$||F||[H^p] = \sup\{||F(\cdot\,, y)||_p : y > 0\} < \infty.$$

Here $|F| = (\sum_{j=0}^{n} |F_j|^2)^{1/2}$. The Poisson kernel P is defined by

$$P(x, y) = c_n^{-1}y(y^2 + |x|^2)^{-(n+1)/2}, \quad c_n^{-1} = \pi^{-(n+1)/2}\Gamma((n + 1)/2).$$

Suppose that $\alpha > 0$, k is the least integer larger than α and $[\alpha]$ is the largest integer at most equal to α. Then the Lipschitz space Λ^α is defined to consist of all residue classes of measurable functions f modulo polynomials of degree $[\alpha]$ at most such that

$$||f||[\Lambda^\alpha] = \sup\{|h|^{-\alpha}||\Delta^k(h)f||_\infty : h \in R^n\} < \infty.$$

Remark 1. If the notation λ^α is used for the Lipschitz space $\Lambda_{\infty,\infty}{}^\alpha$ defined by Herz in [**6**] then the present Λ^α is linearly homeomorphic to the second dual $(\lambda^\alpha)''$ of λ^α in such a way that the usual embedding of λ^α into $(\lambda^\alpha)''$ followed by the inverse of this homeomorphism is the inclusion $\lambda^\alpha \subset \Lambda^\alpha$ (cf. [**9**]).

The following result will be proved.

PROPOSITION. *Suppose that $(n - 1)/n < p < 1$, m is an integer $> n(1/p - 1)$ and for $F \in H^p(R_+^{n+1})$, $\varphi \in \Lambda^{n(1/p-1)}$ define*

$$(1) \qquad \langle\varphi, F\rangle = (-1)^m 2^m[(m - 1)!]^{-1}\int_0^\infty \int_{R^n} F_0(x, y)\Phi^{(m)}(x, y)y^{m-1}dxdy$$

where $\Phi^{(m)}(x, y) = [(\partial/\partial y)^m P(\cdot\,, y)] * \varphi(x)$. This definition does not depend on

Received February 16, 1972.

https://doi.org/10.4153/CJM-1973-058-6 Published online by Cambridge University Press

m and the mapping $\varphi \rightarrow \langle\varphi, \cdot\rangle$ *is a topological isomorphism from* $\Lambda^{n(1/p-1)}$ *onto the dual* $H^p(R_+^{n+1})'$ *of* $H^p(R_+^{n+1})$.

Needless to say, the Proposition extends to H^p spaces consisting of more general systems of conjugate harmonic functions (see, e.g., [**2**]) with lower bounds for p which are possibly different from $(n-1)/n$.

Proof of the Proposition. The proof will require two lemmas the first of which is well-known and is stated separately for the sake of easy reference only. If $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index of non-negative integers put $|\beta| = \sum_{i=1}^n \beta_j$; if l is a non-negative integer and f is a sufficiently often differentiable function in R_+^{n+1} put $f^{(\beta,l)} = (\partial/\partial x)^\beta(\partial/\partial y)^l f$. As usual the letter C will be used to denote constants whose dependence on quantities other than the dimension n may be indicated by subscripts.

LEMMA 1. *If* $(n-1)/n \leqq p \leqq \infty$, *then*

$$y^{|\beta|+l}||F^{(\beta,l)}(\cdot\,,y)||_p \leqq C_{|\beta|+l}||F||[H^p]. \tag{2}$$

Furthermore if $(n-1)/n < p < \infty$, *then*

$$\lim_{y\to\infty} y^{|\beta|+l}||F^{(\beta,l)}(\cdot\,,y)||_p = 0. \tag{3}$$

Proof. Suppose ψ is a radial C^∞ function in R^{n+1} supported in the ball of radius 1 about the origin, $\int \psi(z)dz = 1$ and as usual define $\psi_r(z) = r^{-n-1}\psi(r^{-1}z)$ for $r > 0$ and $z \in R^{n+1}$. Differentiation of the relation $F = \psi_{y/2}*F$ valid for any harmonic function F yields

$$y^{|\beta|+l}|F^{(\beta,l)}(x,y)| \leqq C_{|\beta|+l}\max\{|F(u,t)| : |u-x|^2 + |t-y|^2 \leqq (y/2)^2\}. \tag{4}$$

As in [**11**] let s denote the least harmonic majorant of $|F|^{(n-1)/n}$ in R_+^{n+1}. Then by Harnack's inequality the maximum on the right-hand side of (4) is bounded by $Cs(x,y)^{n/(n-1)}$. Hence $y^{|\beta|+l}|F^{(\beta,l)}(x,y)| \leqq C_{|\beta|+l}s(x,y)^{n/(n-1)}$. Now (2) and (3) follow from the results of [**11**] for the harmonic majorant s.

LEMMA 2. *For* $F \in H^p$, $(n-1)/n \leqq p \leqq 1$ *and* $\nu > n(1/p-1)$ *define*

$$T_\nu F(x,y) = \int y^{\nu+1}(y + |x-u|)^{-n-\nu-1}F_*(u,y)du$$

where

$$F_*(x,y) = \max\{|F(u,t)| : |u-x| + |t-y| \leqq y/2\}.$$

Then for $\delta = \nu/(1/p-1) - n$,

$$||T_\nu F(\cdot\,,y)||_p \leqq C_0 C_1^{1/p+\nu}\delta^{-(1/p-1)}||F||[H^p]. \tag{5}$$

Proof. The case $p = 1$ being obvious it will be assumed that $p < 1$. Let s_p denote the least harmonic majorant of $|F|^p$ in R_+^{n+1}. Hence by the results of [**11**]

$s_p(x, y) = P(\cdot, y) * \mu(x)$ where μ is a finite (positive) Borel measure on R^n and

(6) $$\mu(R^n)^{1/p} = ||F||[H^p].$$

(If $p > (n-1)/n$ μ is necessarily absolutely continuous with respect to Lebesgue measure.)

As in the proof of Lemma 1, $F_*(u, y)^p \leqq Cs_p(u, y)$. Hence

(7) $$T_\nu F(x, y) \leqq C \sup_u [y^\nu (y + |x-u|)^{-\nu} s_p(u, y)^{1/p-1}] \times \int y(y + |x-u|)^{-n-1} s_p(u, y) du.$$

Note that

$$[y/(y + |x-u|)]^{\nu/(1/p-1)} s_p(u, y) \leqq [y/(y + |x-u|)]^{\nu/(1/p-1)} \int y/(y + |u-v|)^{n+1} d\mu(v).$$

Split the domain of integration R^n in the last integral into the set $\{v : |u-v| \geqq 2|x-u|\}$ and its complement to obtain

$$[y/(y + |x-u|)]^{\nu/(1/p-1)} s_p(u, y) \leqq (3/2)^{\nu/(1/p-1)} y^{-n} \int [y/(y + |x-v|)]^{\nu/(1/p-1)} d\mu(v) + y^{-n}(y/r)^{\nu/(1/p-1)} \int_{|x-v| \leqq 3r} d\mu(v)$$

where $r = y + |x-u|$.

Next observe that

$$\sup_{y \geqq r} (y/r)^{\nu/(1/p-1)} \int_{|x-v| \leqq 3r} d\mu(v) \leqq \sum_{l=0}^{\infty} 2^{-(l-1)\nu/(1/p-1)} \int_{|x-v| \leqq 3.2^l y} d\mu(v).$$

Thus if χ denotes the characteristic function of the ball about the origin in R^n of radius 3,

$$\gamma(x) = (3/2)^{\nu/(1/p-1)} (1 + |x|)^{-\nu/(1/p-1)} + \sum_{l=0}^{\infty} 2^{-(l-1)\nu/(1/p-1)} \chi^{(2^{-l}\cdot)}$$

and

$$\sigma(x, y) = \sup\{[y/(y + |x-u|)]^{\nu/(1/p-1)} s_p(u, y) : u \in R^n\}$$

then $\sigma(x, y) \leqq y^{-n}\gamma(y^{-1}\cdot) * \mu(x)$ hence by Young's inequality for convolutions and (6),

$$||\sigma(\cdot, y)||_1 \leqq ||\gamma||_1 \mu(R^n) \leqq C2^\delta \left(\delta^{-1} + \sum_{l=0}^{\infty} 2^{-l\delta}\right) ||F||[H^p]^p.$$

Thus

(8) $$||\sigma(\cdot, y)||_1 \leqq C2^\delta \delta^{-1} ||F||[H^p]^p.$$

https://doi.org/10.4153/CJM-1973-058-6 Published online by Cambridge University Press

Now (7), (8), $||s_p(\cdot\,, y)||_1 \leqq \mu(R^n)$ and (6) imply

$$||T_\nu F(\cdot\,, y)||_p \leqq C||\sigma(\cdot\,, y)||_1{}^{1/p-1}||s_p(\cdot\,, y)||_1 \leqq C_0 C_1{}^{1/p+\nu}\delta^{-(1/p-1)}||F||[H^p].$$

This completes the proof of Lemma 2.

The main result can now be proved similarly as the result for the unit disk. Additional difficulties seem to be mainly due to the fact that H^q is not contained in H^p for $p < q$. Let $Q(x, y) = (Q_1(x, y), \ldots, Q_n(x, y)) = c_n{}^{-1}x(y^2 + |x|^2)^{-(n+1)/2}$ denote the conjugate Poisson kernel and put $S = (P, Q)$ so that S forms a system of conjugate harmonic functions. Observe that

$$|S^{(\beta, l)}(x, y)| \leqq C_{|\beta|+l}(y + |x|)^{-n-|\beta|-l}. \text{ Hence for } |\beta| + l > n(1/p - 1)$$

and $y > 0$

$$||S^{(\beta, l)}(\cdot + (0, y))||[H^p] \leqq C_{p,|\beta|+l}y^{n(1/p-1)-|\beta|-l}. \tag{9}$$

Suppose now that $\lambda \in (H^p)'$ and define $\Phi^{[\beta, l]}$ by

$$\Phi^{[\beta, l]}(-x, y) = (-1)^{|\beta|}\lambda(S^{(\beta, l)}(\cdot + (x, y)))$$

so that

$$|\Phi^{[\beta, l]}(x, y)| \leqq C_{p,|\beta|+l}||\lambda||[(H^p)']y^{n(1/p-1)-|\beta|-l}. \tag{10}$$

From harmonicity and the compatibility conditions satisfied by the derivatives of S it is easy to see that there exists a function Φ in $R_+{}^{n+1}$ such that $\Phi^{(\beta, l)} = \Phi^{[\beta, l]}$ for $|\beta| + l = m$. This, of course, is also a consequence of general existence theorems for over-determined systems of differential equations with constant coefficients.

Next it will be shown that

$$||\Phi(\cdot\,, y)||[\Lambda^{n(1/p-1)}] \leqq C_p||\lambda||[(H^p)'].$$

It is easy to see that for this it suffices to establish

$$||\Delta^2(h)\Phi^{(\beta)}(\cdot\,, y)||_\infty \leqq C_p||\lambda||[(H^p)']|h|^{n(1/p-1)-(\beta)} \tag{11}$$

for $|\beta|$ equal to the largest integer N less than α. This in turn follows similarly as in [**12**, p. 425] by applying $\Delta^2(h)$ to

$$\Phi^{(\beta)}(\cdot\,, y) = -\int_0^{|h|} t\Phi^{(\beta,2)}(\cdot\,, y + t)dt$$
$$+ |h|\Phi^{(\beta,1)}(\cdot\,, y + |h|) + \Phi^{(\beta)}(\cdot\,, y + |h|)$$

and writing $\Delta^2(h)\Phi^{(\beta,1)}(\cdot\,, y + |h|)$ and $\Delta^2(h)\Phi^{(\beta)}(\cdot\,, y + |h|)$ as integrals of the partial derivatives of $\Phi^{(\beta, j)}$ with respect to x which can be estimated by means of (10).

It is elementary to verify that (11) implies

$$\left|\Phi^{(\beta)}(x, y) - \Phi^{(\beta)}(0, y) - \sum_{j=1}^n c_j(y)x_j\right| \leqq C_\beta||\lambda||[(H^p)']|x|(1 + |\log|x||)$$

https://doi.org/10.4153/CJM-1973-058-6 Published online by Cambridge University Press

for some linear form $\sum_{j=1}^n c_j(y)x_j$, or

$$|\Phi^{(\beta)}(x, y) - \Phi^{(\beta)}(0, y)| \leqq C_p||\lambda||[(H^p)']|x|^{n(1/p-1)-N}, \ (|\beta| = N)$$

according as $n(1/p - 1)$ is an integer or not (cf. [**6**] and [**14**]). Hence for some polynomial $\Phi_p(0; \ldots, y)$ in $x_1, \ldots, x_n$ of degree $n(1/p - 1)$ at most (the Taylor polynomial of $\Phi(\cdot, y)$ about 0 if $n(1/p - 1)$ is not an integer) and $|\gamma| \leqq N$

$$(12) \qquad |\Phi^{(\gamma)}(x, y) - \Phi_p^{(\gamma)}(0; x, y)| \leqq C_p|\lambda||[(H^p)']|x|^{n(1/p-1)-|\gamma|}(1 + |\log|x||).$$

As a consequence of (11) and (12) the family $\{\Phi^{(\gamma)}(\cdot, y) - \Phi_p^{(\gamma)}(0; \cdot, y) : |\gamma| \leqq N, y > 0\}$ is equicontinuous and uniformly bounded in any compact set. Hence by the Arzelà-Ascoli theorem there exists a sequence $\{y_k\}$ and a function $\Phi(\cdot, 0)$ such that $\lim_{k\to\infty} y_k = 0$ and

$$\Phi^{(\gamma)}(\cdot, y_k) - \Phi_p^{(\gamma)}(\cdot; 0, y_k) - \Phi^{(\gamma)}(\cdot, 0) \to 0$$

locally uniformly as $k \to \infty$ hence (11) and (12) continue to hold for $y = 0$ with $\Phi_p(0; \cdot, 0) = 0$.

By (10) (see [**11**; **13**])

$$\begin{aligned}\Phi^{(m)}(x, y + y_k) &= P(\cdot, y) * \Phi^{(m)}(\cdot, y_k)(x) \\ &= P^{(m)}(\cdot, y) * \Phi(\cdot, y_k)(x),\end{aligned}$$

hence

$$(13) \qquad \Phi^{(m)}(x, y) = \lim_{k\to\infty} \Phi^{(m)}(x, y + y_k) = P^{(m)}(\cdot, y) * \Phi(\cdot, 0)(x).$$

(This would also follow from the fact that $\Lambda^{n(1/p-1)}$ is the dual of a space containing all functions $P^{(m)}(\cdot - x, y)$, for $x \in R^n$, $y > 0$, and the Banach-Alaoglu theorem; see Remark 1.)

Suppose now $F \in H^p$. Then for $y > 0$

$$(14) \qquad F^{(m)}(x, 2y) = \int S^{(m)}(x + u, y)F_0(-u, y)du.$$

Let $J = (j_1, \ldots, j_n)$ denote any integer lattice point in R^n. It will be shown that for any $y > 0$

$$(15) \qquad F^{(m)}(\cdot + (0, 2y)) = \lim_{l\to\infty} F_{l,y}^{(m)}$$

in H^p where

$$F_{l,y}^{(m)} = l^{-n} \sum_{|J|\leqq l^2} S^{(m)}(\cdot + (l^{-1}J, y))F_0(-l^{-1}J, y).$$

Clearly any $F_{l,y}^{(m)}$ belongs to H^p. Note that for T_mF as defined in Lemma 2 with $\nu = m$

$$\sup_{l>2/y} l^{-n} \sum_{|J|\leqq l^2} |S^{(m)}(x + l^{-1}J, y)F_0(-l^{-1}J, y)| \leqq T_mF(x, y)$$

hence by Lemma 2,

(16) $$||\sup_l |F_{l,y}{}^{(m)}(\cdot\,,0)|\,||_p \leqq C_{p,m}||F||[H^p].$$

Also by (14) and since $F_0(\cdot\,,y)$ is bounded and continuous

$$\lim_{l\to\infty} F_{l,y}{}^{(m)}(x,0) = F^{(m)}(x,2y)$$

for any $x \in R^n$. Hence by (16) and dominated convergence

$$\lim_{l\to\infty} F_{l,y}{}^{(m)}(\cdot\,,0) = F^{(m)}(\cdot\,,2y)$$

in L^p. Since the functions $F_{l,y}{}^{(m)}$ are in H^p for $l = 1, 2, \ldots$ (15) now follows from the general theory of H^p spaces. (15) implies

$$\lim_{l\to\infty} \lambda(F_{l,y}{}^{(m)}) = \lim_{l\to\infty} l^{-n} \sum_{|J|\leqq l^2} \Phi^{(m)}(l^{-1}J,y)F_0(l^{-1}J,y)$$

Hence by the continuity and boundedness of $\Phi^{(m)}(\cdot\,,y)$ and since $||F_*(\cdot\,,y)||_1 < \infty$ it follows that

(17) $$\lambda(F^{(m)}(\cdot + (0,2y))) = \int_{R^n} \Phi^{(m)}(x,y)F_0(x,y)dx.$$

Clearly

$$||\sup_{0\leqq t\leqq cy} |F^{(m)}(\cdot\,,y+t)|\,||_p \leqq C_{m,y,c}||F||[H^p]$$

for any $y > 0$, $c > 0$. Hence similarly as before it can be seen that for η, $R > 0$ (17) implies

(18) $$\lambda\left(\int_0^R F^{(m)}(\cdot + (0,2y+\eta))y^{m-1}dy\right) = \int_0^R \int \Phi^{(m)}(x,y)F_0(x,y+\eta)y^{m-1}dxdy.$$

Next note that for $(n-1)/n < p < \infty$

(19) $$\lim_{R\to\infty} (-1)^m[(m-1)!]^{-1}\int_0^R F^{(m)}(\cdot + (0,y))y^{m-1}dy = F$$

in H^p. For by integration by parts the expression after the limit sign equals

$$F + \sum_{l=1}^{m} (-1)^{m-l+1}[(m-l)!]^{-1}F^{(m-l)}(\cdot + (0,R))R^{m-l}$$

and as a result of Lemma 1 the last sum tends to 0 in H^p.

(18) and (19) imply

(20) $$\lambda(F(\cdot + (0,\eta))) = (-1)^m 2^m[(m-1)!]^{-1} \lim_{R\to\infty} \int_0^R \int \Phi^{(m)}(x,y)F_0(x,y+\eta)y^{m-1}dxdy.$$

https://doi.org/10.4153/CJM-1973-058-6 Published online by Cambridge University Press

By a result of Flett [5, Theorem 3] generalizing an inequality of Hardy and Littlewood

$$\int_0^\infty ||F(\cdot\,,y)||_1 y^{n(1/p-1)-1}dy \leqq C_p||F||[H^p]. \tag{21}$$

Hence by dominated convergence as $\eta \to 0$ the right-hand side of (20) approaches the right-hand side of (1). Moreover since $\lim_{\eta\to 0} F(\cdot + (0, \eta)) = F$ in H^p it follows that any $\lambda \in (H^p)'$ can be represented by (1) where $\varphi = \Phi(\cdot\,, 0)$ and by (11), $||\varphi||[\Lambda^{n(1/p-1)}] \leqq C_p||\lambda||[(H^p)']$.

Conversely for any $\varphi \in \Lambda^{n(1/p-1)}$ (1) defines a continuous linear functional on H^p. First observe that since

$$(\partial/\partial y)^2 P = -\sum_{j=1}^n (\partial/\partial x_j)^2 P \quad \text{and} \quad (\partial/\partial y)P = -\sum_{j=1}^n (\partial/\partial x_j)Q_j$$

it follows that

$$(\partial/\partial y)^m P = \sum_{|\beta|=N} c_\beta P^{(\beta,m-N)} \quad \text{or} \quad = -\sum_{|\beta|=N}\sum_{j=1}^n d_{\beta j} Q_j^{(\beta,m-N)}$$

for certain nonnegative constants c_β, $d_{\beta j}$ according as N is even or odd.

It can be shown that if $\varphi \in \Lambda^{n(1/p-1)}$ then

$$\sup_{|\beta|=N} ||\varphi^{(\beta)}||[\Lambda^{n(1/p-1)-N}] \leqq C_p||\varphi||[\Lambda^{n(1/p-1)}]. \tag{22}$$

In the case of the Lipschitz algebras $\Lambda^\alpha \cap L^\infty$ the analogous result is of course well-known (see also [6]). Hence by a previous argument (12) is valid with $||\lambda||[(H^p)']$ replaced by $||\varphi||[\Lambda^{n(1/p-1)}]$. It follows that if N is even

$$\Phi^{(m)}(x, y) = \sum_{|\beta|=N} c_\beta P^{(m-N)}(\cdot\,, y) * \varphi^{(\beta)}(x)$$

while if N is odd

$$\Phi^{(m)}(x, y) = -\sum_{|\beta|=N}\sum_{j=1}^n \varphi_{\beta j} Q_j^{(m-N)}(\cdot\,, y) * \varphi^{(\beta)}(x).$$

Note that since $m > N$, $P^{(m-N)}(\cdot\,, y)$ has mean value 0 on R^n. Hence by use of (12) for even N

$$\Phi^{(m)}(x, y) = (1/2)\sum_{|\beta|=N} c_\beta \int P^{(m-N)}(u, y)\Delta^2(u)\varphi^{(\beta)}(x)du.$$

Thus

$$\begin{aligned}|\Phi^{(m)}(x, y)| &\leqq C_p||\varphi||[\Lambda^{n(1/p-1)}]\int_0^\infty (y+u)^{-n-m+N}u^{n(1/p-1)-N+n-1}du \\ &\leqq C_{p,m}||\varphi||[\Lambda^{n(1/p-1)}]y^{n(1/p-1)-m}.\end{aligned}$$

It follows that for $F \in H^p$

$$|\langle\varphi, F\rangle| \leqq C_{p,m}||\varphi||[\Lambda^{n(1/p-1)}]\int_0^\infty ||F(\cdot\,, y)||_1 y^{n(1/p-1)}dy. \tag{23}$$

Hence by (21) for even N

(24) $$|\langle \varphi, F\rangle| \leqq C_{p,m}||\varphi||[\Lambda^{n(1/p-1)}]||F||[H^p].$$

If N is odd

$$\int \Phi^{(m)}(x, y)F_0(x, y)dx = \sum_{|\beta|=N} \sum_{j=1}^{n} d_{\beta j} \int P^{(m-N)}(\cdot\, , y) * \varphi^{(\beta)}(x)F_j(x, y)dx$$

and (23), (24) follow as in the case of even N. (Use of the functions F_j for $1 \leqq j \leqq n$ could have been avoided by noting that the results of [**6**] and Remark 1 imply that the Riesz transform preserves Λ^α.)

To see that the right-hand side of (1) does not depend on m note that for $\lambda = \langle \varphi, \cdot\rangle$ defined by (1) and any other $m' > n(1/p - 1)$ the corresponding function Φ_λ defined by $\Phi_\lambda^{(\beta,l)}(-x, y) = (-1)^{|\beta|}\lambda(S^{(\beta,l)}(\cdot + (x, y)))$ for $|\beta| + 1 = m'$ satisfies

$$\Phi_\lambda^{(m')}(-x, y)$$
$$= (-1)^m 2^m[(m-1)!]^{-1} \int_0^\infty \int_{R^n} P^{(m')}(u + x, t + y)P^{(m)}(u, t) * \varphi(u)t^{m-1}dudt$$

hence by changes of the variables of integration and integration by parts $\Phi_\lambda^{(m')}(x, y) = P^{(m')}(\cdot\, , y) * \varphi(x)$. Thus by the first part of the proof λ is also represented by (1) with m' in place of m.

Remark 2. Let $H^p(\Pi_+)$ denote the space of functions f holomorphic in the upper half-plane $\Pi_+ = \{z : \operatorname{Im} z > 0\}$ such that

$$\sup_{y>0} \int |f(x + iy)|^p dx = ||f||[H^p]^p < \infty.$$

Note that the mapping τ from $H^p(\Pi_+)$ to $H^p(R_+^2)$, the space of R^2-valued functions in $H^p(R_+^2)$ as defined in the Introduction which sends f to $(F_0, F_1) = (\operatorname{Re} f, \operatorname{Im} f)$ is a linear isometry between $H^p(\Pi_+)$ and $H^p(R_+^2, R)$ over the real numbers. If a complex structure J is defined on $H^p(R_+^2)$ by means of $J(F_0, F_1) = (-F_1, F_0)$ then τ is linear over the complex numbers. A complex linear functional λ on $H^p(\Pi_+)$ gives rise to a complex linear functional $\lambda \circ \tau^{-1}$ on $H^p(R_+^2)$ if and only if the real linear mapping $\lambda \circ \tau^{-1}$ from $H^p(R_+^2, R)$ to C satisfies $\lambda \circ \tau^{-1}(JF) = i\lambda \circ \tau^{-1}(F)$. Hence by the Proposition for some complex valued function φ satisfying

$$||\varphi||[\Lambda^{n(1/p-1)}] \leqq C_{p,m}||\lambda||[(H^p)'],$$

$$(-1)^m 2^{-m}(m-1)!\lambda(f) = \int\int_{\Pi_+} \operatorname{Re} f(x + iy)\Phi^{(m)}(x, y)y^{m-1}dxdy$$
$$= i\int\int_{\Pi_+} \operatorname{Im} f(x + iy)\Phi^{(m)}(x, y)y^{m-1}dxdy.$$

As a result any bounded complex linear functional λ on $H^p(\Pi_+)$ is of the form

$$\lambda(f) = (-1)^m 2^{m-1}[(m-1)!]^{-1} \int\int_{\Pi_+} f(x+iy)\Phi^{(m)}(x,y)y^{m-1}dxdy.$$

Remark 3. In case $n \geqq 2$ and $p = (n-1)/n$ the proof of the Proposition can be modified to show that if λ is a bounded linear functional on $H^{(n-1)/n}$ then its restriction to the subspace consisting of those $F \in H^{(n-1)/n}$ which satisfy

$$\lim_{y_1, y_2 \to 0} ||F(\cdot, y_1) - F(\cdot, y_2)||_{(n-1)/n} = 0$$

and

$$\lim_{y\to\infty} ||F(\cdot, y)||_{(n-1)/n} = 0$$

is still given by

$$\lambda(F) = (-1)^m 2^m [(m-1)!]^{-1} \lim_{\eta \to +0, R\to\infty} \int_\eta^R \int_{R^n} F_0(x,y)\Phi^{(m)}(x,y)y^{m-1}dxdy$$

for some φ such that $||\varphi||[\Lambda^{n/(n-1)}] \leqq C_p||\lambda||[(H^{(n-1)/n})']$. For the proof note that by the assumed uniform integrability of $|F(\cdot, y)|^{(n-1)/n}$ the measure μ is absolutely continuous with respect to Lebesgue measure. It follows that $||F(\cdot, y)||_1 = o(y^{n/(n-1)})$ as $y \to 0$.

Define $\Lambda_{\infty,1}{}^\alpha$ to consist of all equivalence classes of measurable functions such that for α, k as before (cf. [**6**])

$$\int_{R^n} ||\Delta^k(h)f||_\infty |h|^{-\alpha-n}dh = ||\varphi||[\Lambda^\alpha{}_{\infty,1}] < \infty.$$

Then any $\varphi \in \Lambda_{\infty,1}{}^{n/(n-1)}$ by virtue of (1) gives rise to a bounded linear functional $\langle\varphi, \cdot\rangle$ on $H^{(n-1)/n}$ such that

$$|\langle\varphi, F\rangle| \leqq C||\varphi||[\Lambda_{\infty,1}{}^{n/(n-1)}]||F||[H^{(n-1)/n}].$$

This is immediate from $||F(\cdot, y)||_1 \leqq C||F||[H^{(n-1)/n}]y^{-n/(n-1)}$ and

$$\int_0^\infty ||\Phi^{(m)}(\cdot, y)||_\infty y^{m-n/(n-1)-1}dy \leqq C||\varphi||[\Lambda^{n/(n-1)}].$$

Remark 4. For $m > \nu > 0$ define $P^{(\nu)}$ by

$$P^{(\nu)}(x,y) = e^{-i\pi(\nu-m)}\Gamma(m-\nu)^{-1}\int_0^\infty P^{(m)}(x, y+t)t^{m-\nu-1}dt \tag{24}$$

or equivalently $P^{(\nu)}(\cdot, y)^\wedge(x) = e^{-i\pi\nu}|x|^\nu e^{-y|x|}$ where for $g \in L^1$ the Fourier transform $\hat{g}$ is defined by $\hat{g}(x) = \int e^{-ixy}g(y)dy$. (The corresponding inverse Fourier transform of $h \in L^1$ is denoted $h^\vee$) (24) implies $P^{(\nu)}(x,y) = o(|x|^{-n-\nu-1})$ as $|x| \to \infty$ for fixed $y > 0$ hence $\Phi^{(\nu)}$ can be defined by

$$\Phi^{(\nu)}(x,y) = P^{(\nu)}(\cdot, y) * \varphi(x). \tag{25}$$

https://doi.org/10.4153/CJM-1973-058-6 Published online by Cambridge University Press

With this notation (1) can be generalized to

$$(26)\qquad \langle \varphi, F\rangle = 2^{\nu} e^{i\pi\nu}\Gamma(\nu)^{-1}\int_0^\infty \int_{R^n} F_0(x, y)\Phi^{(\nu)}(x, y)y^{\nu-1}dxdy$$

for $\nu > n(1/p - 1)$. Note that $P(\cdot\,, y)^\wedge(x) = e^{-y|x|}$, hence $F(\cdot\,, 0)^\wedge$ is well-defined by $F(\cdot\,, 0)^\wedge(x) = F(\cdot\,, y)^\wedge(x)e^{y|x|}$. One way to establish (26) is by observing that for $\varphi \in \check{\mathcal{O}}$, the space of Fourier transforms of functions which are infinitely often differentiable and vanish near 0 and infinity the right-hand side of (26) equals

$$\begin{aligned} 2^\nu\Gamma(\nu)^{-1}\int_0^\infty \int_{R^n} & F_0(\cdot\,, 0)^\wedge(x)|x|^\nu\varphi^\vee(x)e^{-2y|x|}y^{\nu-1}dxdy \\ &= \int_{R^n} F_0(\cdot\,, 0)^\wedge(x)\varphi^\vee(x)dx = \lim_{y\to 0}\int F_0(\cdot\,, 0)^\wedge(x)e^{-y|x|}\varphi^\vee(x)dx \\ &= \lim_{y\to 0}\int F_0(x, y)\varphi(x)dx. \end{aligned}$$

For general $\varphi \in \Lambda^{n(1/p-1)}$ note that $\check{\mathcal{O}}$ is dense in $\Lambda^{n(1/p-1)}$ with respect to the weak* topology of $\Lambda^{n(1/p-1)}$ as the dual of $(\lambda^{n(1/p-1)})'$ (see [**6**] and Remark 1). Also by (24) and similar arguments as in the proof of the main result the inner integral over R^n on the right-hand side of (26) is bounded by

$$C_{p,\nu}||\varphi||[\Lambda^{n(1/p-1)}]||F(\cdot\,, y)||_1 y^{n(1/p-1)-1}.$$

Hence again by the result of Flett used above for any fixed $F \in H^p$ the right-hand side of (26) defines a continuous linear functional on $\Lambda^{n(1/p-1)}$. It follows that (26) holds for any $\varphi \in \Lambda^{n(1/p-1)}$.

References

1. A. P. Calderón, *Intermediate spaces and interpolation, the complex method,* Studia Math. *24* (1964), 113–190.

2. A. P. Calderón and A. Zygmund, *On higher gradients of harmonic functions*, Studia Math. *24* (1964), 211–226.

3. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on H^p spaces with* $0 < p < 1$, J. Reine Angew. Math. *238* (1969), 32–60.

4. C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. *77* (1971), 587–588.

5. T. M. Flett, *On the rate of growth of mean values of holomorphic and harmonic functions,* Proc. London Math. Soc. *20* (1970), 749–768.

6. C. S. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech. *18* (1968), 283–323.

7. L. Hörmander, *An introduction to complex analysis in several variables* (van Nostrand, Princeton, 1966).

8. J. L. Kelley and I. Namioka, *Linear topological spaces* (van Nostrand, Princeton, 1963).

9. K. de Leeuw, *Banăch spaces of Lipschitz functions*, Studia Math. *21* (1961) 55–66.

10. E. M. Stein, *On the theory of harmonic functions of several variables, II. Behaviour near the boundary*, Acta Math. *106* (1961), 137–174.

11. E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables, I. The theory of H^p spaces*, Acta Math. *103* (1960), 25–62.
12. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space*, J. Math. Mech. *13* (1964), 407–479.
13. G. Weiss, *Analisis armonico en varias variables. Teoria de los espacios H^p*, Cursos y Seminarios de Matematica, fasc. *9*, Universidad de Buenos Aires, 1960.
14. A. Zygmund, *Trigonometric series*, Vol. I, 2nd ed. (Cambridge University Press, New York, 1959).

University of Florida,
Gainesville, Florida