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PERTURBATIONS OF NORM-ADDITIVE MAPS BETWEEN CONTINUOUS FUNCTION SPACES

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Abstract Let X, Y be two locally compact Hausdorff spaces and $T : C_0(X) \to C_0(Y)$ be a standard surjective ε-norm-additive map, i.e.

$$\big|\|T(f)+T(g)\| - \|f+g\|\big| \le \varepsilon, \text{ for all } f, g \in C_0(X).$$

Then there exist a homeomorphism $\varphi : Y \to X$ and a continuous function $\lambda : Y \to \{\pm 1\}$ such that

$$|T(f)(y) - \lambda(y) f(\varphi(y))| \le \frac{3}{2}\varepsilon, \text{ for all } y \in Y,\ f \in C_0(X).$$

The estimate '$\frac{3}{2}\varepsilon$' is optimal. And this result can be regarded as a new nonlinear extension of the Banach–Stone theorem.

Keywords: Banach–Stone theorem; norm-additive maps; linear isometries; continuous function spaces; stability

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1. Introduction

Let X be a locally compact Hausdorff space. The space $C_0(X)$ will stand for the Banach space of all continuous real-valued functions which vanish at infinity on X (i.e. $\{x \in X : |f(x)| \ge \varepsilon\}$ is compact in X for every $f \in C_0(X)$ and every $\varepsilon > 0$) equipped with the supremum norm. The following result is well-known as the Banach–Stone theorem (see [2, 3, 21]).

Theorem 1.1. *Let X, Y be two locally compact Hausdorff spaces and $T : C_0(X) \to C_0(Y)$ be a linear surjective isometry. Then there exists a homeomorphism $\varphi : Y \to X$ and a continuous function $\lambda : Y \to \{\pm 1\}$ such that*

$$T(f)(y) = \lambda(y)f(\varphi(y)), \text{ for all } y \in Y, \ f \in C_0(X).$$

The Banach–Stone theorem describes a deep fact that the linear metric structure of $C_0(X)$ determines the topology of X. And it has found a large number of generalizations and variants in many different contexts (see [15] for a survey of corresponding results). The classical Mazur–Ulam theorem [18] states that every standard surjective isometry between two real Banach spaces must be linear. Thus, the existence of a standard surjective isometry between $C_0(X)$ and $C_0(Y)$ can also guarantee that X and Y are homeomorphic. Instead of isometries, Amir and Cambern investigated the linear isomorphisms between $C_0(X)$ and $C_0(Y)$, where X, Y are compact Hausdorff spaces or locally compact Hausdorff spaces ([1, 5–7]) . They showed that if the linear isomorphism $T : C_0(X) \to C_0(Y)$ satisfies that $\|T\| \cdot \|T^{-1}\| < 2$, then the underlying spaces X and Y are homeomorphic, and the universal constant '2' is optimal (see [9]).

In another direction, the nonlinear extension of the Banach–Stone theorem has attracted a large number of mathematicians' attention (see [11–14, 17, 23]). Recently, Galego and Porto da Silva [14] studied the bijective coarse quasi-isometries between $C_0(X)$ and $C_0(Y)$ and they obtained an optimal nonlinear extension of the Banach–Stone theorem.

Let E, F be two Banach spaces. A map $T : E \to F$ is said to be a coarse quasi-isometry (or coarse (M,ε)-quasi-isometry) for some constants $M \geq 1$ and $\varepsilon \geq 0$ provided

$$\frac{1}{M}\|u - v\| - \varepsilon \leq \|T(u) - T(v)\| \leq M\|u - v\| + \varepsilon,$$

for all $u, v \in E$. T is called an ε-isometry when $M = 1$ and an isometry when $M = 1$ and $\varepsilon = 0$. If $T(0) = 0$, then T is called standard.

Theorem 1.2. (Galego-Porto da Silva). *Let X, Y be two locally compact Hausdorff spaces and $T : C_0(X) \to C_0(Y)$ be a standard bijective map such that both T and T^{-1} are coarse (M, ε)-quasi-isometries with $M < \sqrt{2}$. Then there exists a homeomorphism $\varphi : Y \to X$ and a continuous function $\lambda : Y \to \{\pm 1\}$ such that*

$$|MT(f)(y) - \lambda(y)f(\varphi(y))| \leq (M^2 - 1)\|f\| + \Delta\varepsilon, \text{ for all } y \in Y, \ f \in C_0(X),$$

where Δ does not depend on f and y.

The upper bound $\sqrt{2}$ on M is optimal even in the linear case when T is a linear isomorphism [9]. When $M = 1$, i.e. $T : C_0(X) \to C_0(Y)$ is a bijective standard ε-isometry, Theorem 1.2 yields that

$$|T(f)(y) - \lambda(y)f(\varphi(y))| \leq 2\varepsilon, \text{ for all } y \in Y, \ f \in C_0(X).$$

As it follows from a result of Omladič and Šemrl [20], T can be weaken as a standard surjective ε-isometry and the estimate '2ε' is optimal (see, also, [4, p. 360]).

In view of geometry, the isometry T from Banach space E to another Banach space F preserves the length of one diagonal of the parallelogram generated by two vectors. But, one may ask what happens if T preserves the length of another diagonal of the parallelogram instead, that is,

$$\|T(u) + T(v)\| = \|u + v\|, \text{ for all } u, v \in E.$$

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By letting $g = -f$ in the above equation, it is clear that T is an isometry with $T(-f) = -T(f)$ and $T(0) = 0$. Thus, the Banach–Stone theorem (Theorem 1.1) still holds when T is surjective. Such transformations are called norm-additive maps and have stronger properties than isometries when the domain is symmetric. And these maps have been studied recently in [8, 10, 16, 19, 22].

Let E, F be two Banach spaces and $T : E \to F$ be a map, $\varepsilon \geq 0$. T is called an ε-norm-additive map provided

$$\big|\|T(u) + T(v)\| - \|u + v\|\big| \leq \varepsilon, \text{ for all } u, v \in E.$$

In this paper, we mainly study the properties of the ε-norm-additive map between $C_0(X)$ and $C_0(Y)$ which is a natural and interesting generalization of norm-additive map to the perturbed case. It is worth noting that although the proof of Theorem 1.2 for the sharp estimate '2ε' of the surjective ε-isometries is very skilful [14], it cannot be applied to ε-norm-additive mappings for hunting the sharp estimate because the ε-norm-additive mapping between $C_0(X)$ and $C_0(Y)$ may be a strictly 2ε-isometry (see Example 2.3).

We mainly prove that if X, Y are two locally compact Hausdorff spaces and $T : C_0(X) \to C_0(Y)$ is a standard surjective ε-norm-additive map, then there exists a homeomorphism $\varphi : Y \to X$ and a continuous function $\lambda : Y \to \{\pm 1\}$ such that

$$|T(f)(y) - \lambda(y) f(\varphi(y))| \leq \frac{3}{2}\varepsilon, \text{ for all } y \in Y,\ f \in C_0(X).$$

The constant '$\frac{3}{2}$' is optimal.

2. Main results

We start this section with the following observation which reveals the relationship between ε-isometries and ε-norm-additive maps on Banach spaces.

Proposition 2.1. *Suppose that E and F are Banach spaces and $T : E \to F$ is an ε-norm-additive map. Then T is a 2ε-isometry.*

Proof. For any $u \in E$, by the definition of T, we have

$$\big|\|T(u) + T(-u)\| - \|u - u\|\big| \leq \varepsilon,$$

i.e.

$$\|T(u) + T(-u)\| \leq \varepsilon.$$

For $u, v \in E$, we obtain that

$$\begin{aligned}\|T(u) - T(v)\| &= \|(T(u) + T(-u)) - (T(-u) + T(v))\| \\ &\leq \|T(-u) + T(v)\| + \|T(u) + T(-u)\| \\ &\leq \|u - v\| + 2\varepsilon,\end{aligned}$$

and

$$\|T(u) - T(v)\| = \|(T(u) + T(-u)) - (T(-u) + T(v))\|$$

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$$\geq \|T(-u)+T(v)\| - \|T(u)+T(-v)\|$$
$$\geq \|u-v\| - 2\varepsilon.$$

Thus, T is a 2ε-isometry and the proof is complete. □

Although every ε-norm-additive map is actually a 2ε-isometry, the converse is not true in general.

Example 2.2. Define $T : c_0 \to c_0$ by $T(u) = (\|u\|, u_1, u_2, \ldots)$ for $u = (u_n)_{n=1}^{\infty} \in c_0$. Then T is a standard $(0-)$isometry, but it is not a δ-norm-additive map for any $\delta \geq 0$.

The following example shows that the constant '2' in Proposition 2.1 is sharp.

Example 2.3. Let $X = \{a\}$, then $C(X) = \mathbb{R}$. Fix $\varepsilon > 0$, define $T : \mathbb{R} \to \mathbb{R}$ by

$$T(u) = \begin{cases} 0 & \text{if } \ u = 0, \\ -\frac{\varepsilon}{2} & \text{if } u = \varepsilon, \\ u + \frac{\varepsilon}{2} & \text{if } u \neq 0, \varepsilon. \end{cases}$$

Then T is a standard ε-norm-additive map. Note that $\big||T(2\varepsilon) - T(\varepsilon)| - |2\varepsilon - \varepsilon|\big| = 2\varepsilon$. This and Proposition 2.1 together show that T is a strictly 2ε-isometry.

Let X, Y be two locally compact Hausdorff spaces and $T : C_0(X) \to C_0(Y)$ be a standard surjective ε-norm-additive map. Combining Proposition 2.1, Theorem 1.2 and the Omladič–Šemrl's theorem [20], we have the following result.

Theorem 2.4. *Let $T : C_0(X) \to C_0(Y)$ be a standard surjective ε-norm-additive map. Then there exists a homeomorphism $\varphi : Y \to X$ and a continuous function $\lambda : Y \to \{\pm 1\}$ such that*

$$|T(f)(y) - \lambda(y) f(\varphi(y))| \leq 4\varepsilon, \text{ for all } y \in Y, \ f \in C_0(X).$$

However, the constant in the estimate above is not the best. Our main goal is to obtain a sharp version of the above theorem by reducing 4ε to $\frac{3}{2}\varepsilon$. And then we will show that $\frac{3}{2}\varepsilon$ is optimal. To begin with, we establish the following useful lemmas.

Lemma 2.5. *Let $x \in X$ and $f_1, f_2, ..., f_n \in C_0(X)$ with $(f_1(x), f_2(x), ..., f_n(x)) \neq 0$. Then there exists a $g \in C_0(X)$ with $g(z) \leq 0$ for all $z \in X$ such that*

$$\|g\| = -g(x) \text{ and } \|g + f_i\| = -g(x) - f_i(x), \ \forall\, i \in \{1, 2, ..., n\}.$$

Proof. Let $I = [-\|f_1\|, \ \|f_1\|] \times [-\|f_2\|, \ \|f_2\|] \times ... \times [-\|f_n\|, \ \|f_n\|] \subset \mathbb{R}^n$. Define $F : X \to \mathbb{R}^n$ by

$$F(z) = (f_1(z), f_2(z), ..., f_n(z)), \ \forall\, z \in X.$$

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Then F is well defined and continuous. Let $\alpha = \max_{1\le i\le n} \|f_i\|$. For every $(u_1, u_2, ..., u_n) \in I$, let

$$h((u_1, u_2, ..., u_n)) = \max_{1\le i\le n} \{f_i(x) - u_i - 3\alpha, -3\alpha\}.$$

Then $h : I \to \mathbb{R}$ is well defined. It is clear that h is continuous and $-3\alpha \le h(F(z)) \le 0$ for every $z \in X$. By the Urysohn lemma, there exists a continuous function $P : I \to [0, 1]$ such that $P((0, 0, ..., 0)) = 0$ and $P(F(x)) = 1$. Define $g : X \to \mathbb{R}$ by

$$g(z) = (P \cdot h)(F(z))(= P(F(z)) \cdot h(F(z))),\ \forall\ z \in X.$$

Then g is well defined. Since P, h, F are continuous, g is also continuous.

We assert that $g \in C_0(X)$. For any convergent net $\{x_\lambda\}_{\lambda\in\Lambda}$ with x_λ converges to infinity, we have $F(x_\lambda) \to (0, 0, ..., 0) \in \mathbb{R}^n$. Hence $P(F(x_\lambda)) \to 0$. Note that $|h(F(x_\lambda))| \le 3\alpha$ for any $\lambda \in \Lambda$, we have $g(x_\lambda) \to 0$. Thus $g \in C_0(X)$.

For every $z \in X$, $0 \le P(F(z)) \le 1$ and $-3\alpha \le h(F(z)) \le 0$. Thus $-3\alpha \le g(z) \le 0$ for every $z \in X$. Note that $g(x) = -3\alpha$, then $\|g\| = -g(x)$. For every $i \in \{1, 2, ..., n\}$ and every $z \in X$, we have

$$\begin{aligned}\alpha &\ge g(z) + f_i(z) = P(F(z)) \cdot h(F(z)) + f_i(z)\\ &\ge h(F(z)) + f_i(z) \ge f_i(x) - f_i(z) - 3\alpha + f_i(z)\\ &= f_i(x) - 3\alpha = g(x) + f_i(x).\end{aligned}$$

Then

$$|g(z) + f_i(z)| \le 3\alpha - f_i(x) = -g(x) - f_i(x),\ \forall\ z \in X.$$

This implies that $\|g + f_i\| = -g(x) - f_i(x)$ and the proof is complete. □

Similar to Lemma 2.5, we have the following lemma.

Lemma 2.6. *Let $x \in X$ and $f_1, f_2, ..., f_n \in C_0(X)$ with $(f_1(x), f_2(x), ..., f_n(x)) \ne 0$. Then there exists a $g \in C_0(X)$ with $g(z) \ge 0$ for all $z \in X$ such that*

$$\|g\| = g(x) \text{ and } \|g + f_i\| = g(x) + f_i(x),\ \forall\ i \in \{1, 2, ..., n\}.$$

For every $x \in X$ and $f \in C_0(X)$, let

$$P(f, x) = \{(y, \lambda) \in Y \times \{\pm 1\} : \lambda T(f)(y) \ge f(x) - \frac{3}{2}\varepsilon\},$$
$$\bar{P}(f, x) = \{(y, \lambda) \in Y \times \{\pm 1\} : \lambda T(f)(y) \le f(x) + \frac{3}{2}\varepsilon\}.$$

Put

$$P_+(x) = \cap_{f\in C_0(X),\ f(x)\ge 0} P(f, x),\ P_-(x) = \cap_{f\in C_0(X),\ f(x)\le 0} \bar{P}(f, x).$$

Lemma 2.7. *For every $x \in X$, $P_+(x)$ is non-empty.*

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Proof. Fix $x \in X$, the proof is divided into three steps.

Step I. We prove that for $f \in C_0(X)$ with $f(x) > \frac{3}{2}\varepsilon$, $P(f,x)$ is a non-empty compact set. Let

$$P_1 = \{y \in Y : T(f)(y) \geq f(x) - \frac{3}{2}\varepsilon\}, \;\; P_2 = \{y \in Y : -T(f)(y) \geq f(x) - \frac{3}{2}\varepsilon\}.$$

Since T is an ε-norm-additive map,

$$2\big|\|T(f)\| - \|f\|\big| = \big|\|T(f) + T(f)\| - \|f + f\|\big| \leq \varepsilon,$$

i.e.

$$\|f\| - \frac{1}{2}\varepsilon \leq \|T(f)\| \leq \|f\| + \frac{1}{2}\varepsilon.$$

This yields that at least one of the P_1, P_2 is non-empty. Note that $f(x) > \frac{3}{2}\varepsilon$, P_1 and P_2 are two compact sets. Since $P(f,x) = P_1 \times \{1\} \cup P_2 \times \{-1\}$, $P(f,x)$ is a non-empty compact subset of $Y \times \{\pm 1\}$.

Step II. Fix a $f_0 \in C_0(X)$ with $f_0(x) > \frac{3}{2}\varepsilon$, we prove that $P(f_0,x) \cap P(f,x)$ is a non-empty compact set for every $f \in C_0(X)$ with $f(x) \geq 0$. It is clear that $P(f,x)$ is closed in $Y \times \{\pm 1\}$ and hence $(P(f_0,x) \cap P(f,x)) \subset P(f_0,x)$ is compact. It remains to prove that $P(f_0,x) \cap P(f,x)$ is non-empty. By Lemma 2.5, there exists a $g \in C_0(X)$ such that

$$\|g\| = -g(x), \; \|g + f\| = -g(x) - f(x), \; \|g + f_0\| = -g(x) - f_0(x).$$

Since T is an ε-norm-additive map, one has

$$\|T(g)\| \geq \|g\| - \frac{1}{2}\varepsilon, \; \|f + g\| + \varepsilon \geq \|T(f) + T(g)\|, \; \|f_0 + g\| + \varepsilon \geq \|T(f_0) + T(g)\|.$$

Pick $(y, \lambda) \in Y \times \{\pm 1\}$ such that $\|T(g)\| = \lambda T(g)(y)$. Then

$$\begin{aligned} -g(x) - f_0(x) + \varepsilon = \|g + f_0\| + \varepsilon &\geq \|T(g) + T(f_0)\| \geq \lambda T(g)(y) + \lambda T(f_0)(y) \\ &\geq \|g\| - \frac{1}{2}\varepsilon + \lambda T(f_0)(y) = -g(x) - \frac{1}{2}\varepsilon + \lambda T(f_0)(y). \end{aligned}$$

This implies $-\lambda T(f_0)(y) \geq f_0(x) - \frac{3}{2}\varepsilon$. Similarly, we can get $-\lambda T(f)(y) \geq f(x) - \frac{3}{2}\varepsilon$. Thus $(y, -\lambda) \in P(f_0,x) \cap P(f,x)$ and $P(f_0,x) \cap P(f,x)$ is a non-empty compact subset of $Y \times \{\pm 1\}$.

Step III. We prove that the set family $\{P(f,x) \cap P(f_0,x)\}_{f \in C_0(X),\, f(x) \geq 0}$ has the finite intersection property. Fix $f_1, f_2, ..., f_n \in C_0(X)$ with $f_i(x) \geq 0$ for every $i \in \{1, 2, ..., n\}$, by Lemma 2.5, there exists a $g \in C_0(X)$ such that

$$\|g\| = -g(x) \;\text{ and }\; \|g + f_i\| = -g(x) - f_i(x), \; \forall i \in \{0, 1, 2, ..., n\}.$$

Pick $(y, \lambda) \in Y \times \{\pm 1\}$ such that $\|T(g)\| = \lambda T(g)(y)$. For every $i \in \{1, 2, ..., n\}$, by the same argument as in Step II, we obtain

$$(y, -\lambda) \in P(f_i, x) \cap P(f_0, x).$$

Thus

$$\cap_{f\in C_0(X),\ f(x)\geq 0}(P(f,x)\cap P(f_0,x))\neq\emptyset.$$

Note that

$$P_+(x)=\cap_{f\in C_0(X),\ f(x)\geq 0}P(f,x)=\cap_{f\in C_0(X),\ f(x)\geq 0}(P(f,x)\cap P(f_0,x)).$$

The proof is complete. □

The following Lemma 2.8 is analogous to Lemma 2.7 and we omit its proof.

Lemma 2.8. *For every $x\in X$, $P_-(x)$ is non-empty.*

Remark 2.9. Fix $x\in X$, it is not difficult to verify that if $(y,\lambda)\in P_+(x)$, then $(y,-\lambda)\notin P_+(x)$. Suppose on the contrary that $(y,\lambda),(y,-\lambda)\in P_+(x)$, pick $f\in C_0(X)$ with $f(x)>\frac{3}{2}\varepsilon$, then

$$\lambda T(f)(y)\geq f(x)-\frac{3}{2}\varepsilon>0,\ \ -\lambda T(f)(y)\geq f(x)-\frac{3}{2}\varepsilon>0.$$

This leads to a contradiction. By a similar argument as above, we can see that if $(y,\lambda)\in P_-(x)$, then $(y,-\lambda)\notin P_-(x)$.

From now on, we assume that $T:C_0(X)\to C_0(Y)$ is a standard surjective ε-norm-additive map. For every $y\in Y$ and $g\in C_0(Y)$, define

$$Q(g,y)=\{(x,\lambda)\in X\times\{\pm 1\}:\lambda f(x)\geq g(y)-\frac{3}{2}\varepsilon,\ f\in T^{-1}(g)\},$$
$$\bar{Q}(g,y)=\{(x,\lambda)\in X\times\{\pm 1\}:\lambda f(x)\leq g(y)+\frac{3}{2}\varepsilon,\ f\in T^{-1}(g)\}.$$

Put

$$Q_+(y)=\cap_{g\in C_0(Y),\ g(y)\geq 0}Q(g,y),\ Q_-(y)=\cap_{g\in C_0(Y),\ g(y)\leq 0}\bar{Q}(g,y).$$

Lemma 2.10. *For every $y\in Y$, $Q_+(y)$, $Q_-(y)$ both are non-empty.*

Proof. Let $y\in Y$, we just prove $Q_+(y)$ is non-empty, the case for $Q_-(y)$ is similar. The proof is divided into three steps.

Step I. Assume that $g\in C_0(Y)$ with $g(y)>\frac{3}{2}\varepsilon$. We first prove $Q(g,y)$ is a non-empty compact set. By Lemma 2.5, there exists $\bar{g}\in C_0(Y)$ such that

$$\|\bar{g}\|=-\bar{g}(y)\ \text{ and }\ \|\bar{g}+g\|=-\bar{g}(y)-g(y).$$

Since T is surjective, there exists $h\in C_0(X)$ such that $T(h)=\bar{g}$. Find $(x,\lambda)\in X\times\{\pm 1\}$ such that $\|h\|=\lambda h(x)$. For every $f\in T^{-1}(g)$,

$$\begin{aligned}\|\bar{g}\|-\frac{1}{2}\varepsilon+\lambda f(x)&\leq\|h\|+\lambda f(x)=\lambda(h(x)+f(x))\\&\leq\|h+f\|\leq\|T(h)+T(f)\|+\varepsilon\end{aligned}$$

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$$= \|\bar{g} + g\| + \varepsilon = -\bar{g}(y) - g(y) + \varepsilon$$
$$= \|\bar{g}\| - g(y) + \varepsilon.$$

Thus $-\lambda f(x) \geq g(y) - \frac{3}{2}\varepsilon$. This implies that $(x, -\lambda) \in Q(g, y)$ and $Q(g, y)$ is non-empty. For every $f \in T^{-1}(g)$, let

$$R(f)_1 = \{x \in X : f(x) \geq g(y) - \frac{3}{2}\varepsilon\}, \ \ R(f)_2 = \{x \in X : -f(x) \geq g(y) - \frac{3}{2}\varepsilon\},$$
$$R(f) = \{(x, \lambda) \in X \times \{\pm 1\} : \lambda f(x) \geq g(y) - \frac{3}{2}\varepsilon\}.$$

By the above argument, $R(f) \neq \emptyset$ and $R(f) = R(f)_1 \times \{1\} \cup R(f)_2 \times \{-1\}$. Since $g(y) > \frac{3}{2}\varepsilon$, $R(f)_1$, $R(f)_2$ are compact. By the Tychonoff theorem, $R(f)$ is a non-empty compact subset of $X \times \{\pm 1\}$. Note that $Q(g, y) = \cap_{f \in T^{-1}(g)} R(f)$, this implies that $Q(g, y)$ is a non-empty compact set.

Step II. Fix $g_0 \in C_0(Y)$ with $g_0(y) > \frac{3}{2}\varepsilon$, we will show that for every $g \in C_0(Y)$ with $g(y) \geq 0$, $Q(g_0, y) \cap Q(g, y)$ is a non-empty compact set. Note that $Q(g, y)$ is closed in $X \times \{\pm 1\}$. Hence $(Q(g_0, y) \cap Q(g, y)) \subset Q(g_0, y)$ is compact. By Lemma 2.5, there exists a $\bar{g} \in C_0(Y)$ such that

$$\|\bar{g}\| = -\bar{g}(y), \ \|\bar{g} + g\| = -\bar{g}(y) - g(y), \ \|\bar{g} + g_0\| = -\bar{g}(y) - g_0(y).$$

Pick $h \in C_0(X)$ such that $T(h) = \bar{g}$. Find $\{x, \lambda\} \in X \times \{\pm 1\}$ such that $\|h\| = \lambda h(x)$. For every $f \in T^{-1}(g)$,

$$\|\bar{g}\| - \frac{1}{2}\varepsilon + \lambda f(x) \leq \|h\| + \lambda f(x) = \lambda(h(x) + f(x))$$
$$\leq \|h + f\| \leq \|T(h) + T(f)\| + \varepsilon$$
$$= \|\bar{g} + g\| + \varepsilon = -\bar{g}(y) - g(y) + \varepsilon$$
$$= \|\bar{g}\| - g(y) + \varepsilon.$$

This implies that $-\lambda f(x) \geq g(y) - \frac{3}{2}\varepsilon$ and $(x, -\lambda) \in Q(g, y)$. Similarly, we can show $(x, -\lambda) \in Q(g_0, y)$. Therefore, $Q(g_0, y) \cap Q(g, y)$ is a non-empty compact set.

Step III. We check that the set family $\{Q(g, y) \cap Q(g_0, y)\}_{g \in C_0(Y),\, g(y) \geq 0}$ has the finite intersection property. Fix $g_1, g_2, ..., g_n \in C_0(Y)$ with $g_i(y) \geq 0$ for every $i \in \{1, 2, ..., n\}$, by Lemma 2.5, there exists a $\bar{g} \in C_0(Y)$ such that

$$\|\bar{g}\| = -\bar{g}(y), \ \|\bar{g} + g_i\| = -\bar{g}(y) - g_i(y), \ \forall i \in \{0, 1, 2, ..., n\}.$$

Pick $h \in C_0(X)$ such that $T(h) = \bar{g}$. Find $\{x, \lambda\} \in X \times \{\pm 1\}$ such that $\|h\| = \lambda h(x)$. By the same argument as in Step II, we have

$$(x, -\lambda) \in Q(g_0, y) \ \text{ and } \ (x, -\lambda) \in Q(g_i, y), \ \forall i \in \{1, 2, ..., n\}.$$

Thus

$$\cap_{g\in C_0(Y),\, g(y)\geq 0}(Q(g,y)\cap Q(g_0,y))\neq\emptyset.$$

Note that

$$Q_+(y)=\cap_{g\in C_0(Y),\, g(y)\geq 0}Q(g,y)=\cap_{g\in C_0(Y),\, g(y)\geq 0}(Q(g,y)\cap Q(g_0,y)).$$

Then $Q_+(y)$ is non-empty and the proof is complete. □

Lemma 2.11. *For every $x\in X$, $P_+(x)$ is a singleton.*

Proof. Let $x\in X$, by Lemma 2.7, $P_+(x)$ is non-empty. Let $(y,\lambda)\in P_+(x)$, by Lemma 2.10, $Q_+(y)$ and $Q_-(y)$ are non-empty. For every $f\in C_0(X)$ with $f(x)>3\varepsilon$, we have

$$\lambda T(f)(y)\geq f(x)-\frac{3}{2}\varepsilon.$$

If $\lambda=1$, then $T(f)(y)\geq f(x)-\frac{3}{2}\varepsilon>0$. For every $(x',\mu)\in Q_+(y)$ and every $f\in C_0(X)$ with $f(x)>3\varepsilon$, one has

$$\mu f(x')\geq T(f)(y)-\frac{3}{2}\varepsilon\geq f(x)-3\varepsilon>0. \tag{2.1}$$

We assert that $x=x'$. Suppose on the contrary that $x\neq x'$, by the Urysohn lemma, there exists a $f\in C_0(X)$ with $f(x)>3\varepsilon$ and $f(x')=0$. This contradicts to (2.1). And hence we have $x=x'$ and $\mu=1$. Thus $Q_+(y)=\{(x,\lambda)\}$.

If $\lambda=-1$, then $T(f)(y)\leq-(f(x)-\frac{3}{2}\varepsilon)<0$. For every $(x',\mu)\in Q_-(y)$ and every $f\in C_0(X)$ with $f(x)>3\varepsilon$, one has

$$\mu f(x')\leq T(f)(y)+\frac{3}{2}\varepsilon\leq-(f(x)-\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon=-f(x)+3\varepsilon<0.$$

By the same argument as above, we have $x=x'$ and $\mu=-1$. Thus $Q_-(y)=\{(x,\lambda)\}$.

We assert that $P_+(x)$ is a singleton. Suppose on the contrary that there exist $(y_1,\lambda_1)\neq(y_2,\lambda_2)\in P_+(x)$. By Remark 2.9, $y_1\neq y_2$. Without loss of generality, we assume that $\lambda_1=1$. Then $Q_+(y_1)=\{(x,1)\}$. By the Urysohn lemma, there exists $g\in C_0(Y)$ such that $g(y_1)>3\varepsilon$ and $g(y_2)=0$. Then for every $f\in C_0(X)$ with $T(f)=g$, we have

$$f(x)\geq T(f)(y_1)-\frac{3}{2}\varepsilon=g(y_1)-\frac{3}{2}\varepsilon>0.$$

Thus

$$0=\lambda_2 g(y_2)=\lambda_2 T(f)(y_2)\geq f(x)-\frac{3}{2}\varepsilon\geq g(y_1)-3\varepsilon>0.$$

This leads to a contradiction. Hence $P_+(x)$ is a singleton and the proof is complete. □

Similar to Lemma 2.11, we have the following result.

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Lemma 2.12. *For every $x \in X$, $P_-(x)$ is a singleton.*

Lemma 2.13. *For every $x \in X$, $P_+(x) = P_-(x)$.*

Proof. Fix $x \in X$, by Lemmas 2.11 and 2.12, there exist $y_1,\ y_2 \in Y$ and $\lambda_1,\ \lambda_2 \in \{\pm 1\}$ such that $P_+(x) = (y_1, \lambda_1)$ and $P_-(x) = (y_2, \lambda_2)$. According to the proofs of Lemma 2.11 and 2.12, we have

$$(x, \lambda_1) = \begin{cases} Q_+(y_1), \text{ if } \lambda_1 = 1, \\ Q_-(y_1), \text{ if } \lambda_1 = -1, \end{cases} . \text{ and } (x, \lambda_2) = \begin{cases} Q_-(y_2), \text{ if } \lambda_2 = 1, \\ Q_+(y_2), \text{ if } \lambda_2 = -1. \end{cases}$$

We assert that $y_1 = y_2$. Suppose on the contrary that $y_1 \neq y_2$, by the Urysohn lemma, there exists a $g \in C_0(Y)$ such that

$$\lambda_1 g(y_1) > \lambda_2 g(y_2) + 3\varepsilon > 6\varepsilon. \tag{2.2}$$

For every $f \in C_0(X)$ with $T(f) = g$,

$$\begin{cases} f(x) \geq g(y_1) - \frac{3}{2}\varepsilon & \text{if } \lambda_1 = 1, \\ -f(x) \leq g(y_1) + \frac{3}{2}\varepsilon & \text{if } \lambda_1 = -1. \end{cases}$$

Thus

$$f(x) \geq \lambda_1 g(y_1) - \frac{3}{2}\varepsilon. \tag{2.3}$$

On the other hand, for every $f \in C_0(X)$ with $T(f) = g$,

$$\begin{cases} f(x) \leq -g(y_2) + \frac{3}{2}\varepsilon & \text{if } \lambda_2 = 1, \\ -f(x) \geq -g(y_2) - \frac{3}{2}\varepsilon & \text{if } \lambda_2 = -1. \end{cases}$$

Thus

$$f(x) \leq -\lambda_2 g(y_2) + \frac{3}{2}\varepsilon. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we have

$$-\lambda_2 g(y_2) + \frac{3}{2}\varepsilon \geq f(x) \geq \lambda_1 g(y_1) - \frac{3}{2}\varepsilon > \lambda_2 g(y_2) + \frac{3}{2}\varepsilon.$$

This implies that $\lambda_2 g(y_2) < 0$. By (2.2), $\lambda_2 g(y_2) > 0$. This leads to a contradiction and hence $y_1 = y_2$.

Next we prove that $\lambda_1 = \lambda_2$. Choose $f \in C_0(X)$ such that $f(x) > 3\varepsilon$, then

$$\lambda_1 T(f)(y_1) \geq f(x) - \frac{3}{2}\varepsilon > \frac{3}{2}\varepsilon,\ \lambda_2 T(-f)(y_1) \leq -f(x) + \frac{3}{2}\varepsilon < -\frac{3}{2}\varepsilon.$$

If $\lambda_1 \neq \lambda_2$, then

$$\varepsilon \geq \|T(f) + T(-f)\| \geq |T(f)(y_1) + T(-f)(y_1)| > 3\varepsilon.$$

This leads to a contradiction. Hence $\lambda_1 = \lambda_2$ and $P_+(x) = P_-(x)$. The proof is complete. □

By a similar argument as Lemmas 2.11, 2.12 and 2.13, we can show the following result. To simplify this article, we omit its proof.

Lemma 2.14. *For every $y \in Y$, $Q_+(y) = Q_-(y)$ is a singleton.*

Remark 2.15. For every $x \in X$, $y \in Y$ and $\lambda \in \{\pm 1\}$, by Lemmas 2.11, 2.12, 2.13 and 2.14, one has

$$\{(x, \lambda)\} = Q_+(y) \iff \{(y, \lambda)\} = P_+(x).$$

Now we are ready to show the main result of this paper.

Theorem 2.16. *Let $T : C_0(X) \to C_0(Y)$ be a standard surjective ε-norm-additive map. Then there exists a homeomorphism $\varphi : Y \to X$ and a continuous function $\lambda : Y \to \{\pm 1\}$ such that*

$$|T(f)(y) - \lambda(y) f(\varphi(y))| \leq \frac{3}{2}\varepsilon, \text{ for all } y \in Y,\ f \in C_0(X).$$

Proof. For $y \in Y$, by Lemma 2.14, there exist $x \in X$, $\lambda \in \{\pm 1\}$ such that

$$Q_+(y) = \{(x, \lambda)\}. \tag{2.5}$$

Define

$$\varphi(y) = x, \quad \lambda(y) = \lambda,$$

where x, y, λ satisfy (2.5). Then $\varphi : Y \to X$ and $\lambda : Y \to \{\pm 1\}$ are well defined. It follows from Remark 2.15 that φ is bijective. In what follows, we show that

$$|T(f)(y) - \lambda(y) f(\varphi(y))| \leq \frac{3}{2}\varepsilon, \text{ for all } y \in Y,\ f \in C_0(X). \tag{2.6}$$

Given $y \in Y$ and $f \in C_0(X)$. The proof is divided into two cases.

Case I. $f(\varphi(y)) \geq 0$. By Remark 2.15, we have $\{(y, \lambda(y)\} = P_+(\varphi(y))$. Then

$$\lambda(y) T(f)(y) \geq f(\varphi(y)) - \frac{3}{2}\varepsilon. \tag{2.7}$$

We assert that

$$\lambda(y) T(f)(y) \leq f(\varphi(y)) + \frac{3}{2}\varepsilon. \tag{2.8}$$

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Suppose on the contrary that $\lambda(y)T(f)(y) > f(\varphi(y)) + \frac{3}{2}\varepsilon \geq 0$. If $\lambda(y) = 1$, we have $\{(\varphi(y), 1\} = Q_+(y)$ and

$$f(\varphi(y)) \geq T(f)(y) - \frac{3}{2}\varepsilon > f(\varphi(y)).$$

This leads to a contradiction. If $\lambda(y) = -1$, we have $T(f)(y) < 0$ and $\{(\varphi(y), -1\} = Q_+(y) = Q_-(y)$. Then

$$-f(\varphi(y)) \leq T(f)(y) + \frac{3}{2}\varepsilon < -f(\varphi(y)).$$

This is a contradiction. Combining (2.7) and (2.8), we have

$$|T(f)(y) - \lambda(y)f(\varphi(y))| \leq \frac{3}{2}\varepsilon.$$

Case II. $f(\varphi(y)) \leq 0$. By Remark 2.15 again, we have $\{(y, \lambda(y)\} = P_+(\varphi(y)) = P_-(\varphi(y))$. Then

$$\lambda(y)T(f)(y) \leq f(\varphi(y)) + \frac{3}{2}\varepsilon. \tag{2.9}$$

We assert that

$$\lambda(y)T(f)(y) \geq f(\varphi(y)) - \frac{3}{2}\varepsilon. \tag{2.10}$$

Suppose on the contrary that $\lambda(y)T(f)(y) < f(\varphi(y)) - \frac{3}{2}\varepsilon \leq 0$. If $\lambda(y) = 1$, we have $\{(\varphi(y), 1\} = Q_+(y) = Q_-(y)$ and

$$f(\varphi(y)) \leq T(f)(y) + \frac{3}{2}\varepsilon < f(\varphi(y)).$$

This leads to a contradiction. If $\lambda(y) = -1$, we have $Tf(y) > 0$ and $\{(\varphi(y), -1\} = Q_+(y) = Q_-(y)$. Then

$$-f(\varphi(y)) \geq T(f)(y) - \frac{3}{2}\varepsilon > -f(\varphi(y)).$$

This is a contradiction. Combining (2.9) and (2.10), we have

$$|T(f)(y) - \lambda(y)f(\varphi(y))| \leq \frac{3}{2}\varepsilon.$$

Next we prove that φ is a homeomorphism and λ is continuous. Suppose that $\{y_\alpha\}_{\alpha\in\Lambda} \subset Y$ is a convergent net with $y_\alpha \to y$. Choose a compact neighbourhood U of y, without loss of generality, we can assume that $\{y_\alpha\}_{\alpha\in\Lambda} \subset U$. By the Urysohn lemma, there exists

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a $g \in C_0(Y)$ such that $g|_U \equiv 3\varepsilon + 1$. By (2.6), for any $f \in C_0(X)$ with $Tf = g$, we obtain that

$$\begin{aligned}\frac{3}{2}\varepsilon &\geq |T(f)(y_\alpha) - \lambda(y_\alpha)f(\varphi(y_\alpha))| \\ &\geq |g(y_\alpha)| - |f(\varphi(y_\alpha))| \\ &= 3\varepsilon + 1 - |f(\varphi(y_\alpha))|.\end{aligned}$$

This implies that $|f(\varphi(y_\alpha))| \geq \frac{3}{2}\varepsilon + 1$ and $\{\varphi(y_\alpha)\}_{\alpha\in\Lambda}$ is contained in the compact set $\{x \in X : |f(x)| \geq \frac{3}{2}\varepsilon + 1\}$.

By (2.6), for $\alpha \in \Lambda$, we have

$$|Tf(y_\alpha) - \lambda(y_\alpha)f(\varphi(y_\alpha))| \leq \frac{3}{2}\varepsilon, \text{ for all } f \in C_0(X). \tag{2.11}$$

For any convergent subnet $\{\varphi(y_{\alpha'})\}_{\alpha'\in\Lambda'}$ of $\{\varphi(y_\alpha)\}_{\alpha\in\Lambda}$ with $\varphi(y_{\alpha'}) \to x$, let $\{\lambda(y_{\alpha''})\}_{\alpha''\in\Lambda''}$ be a convergent subnet of $\{\lambda(y_{\alpha'})\}_{\alpha'\in\Lambda'}$ with $\lambda(y_{\alpha''}) \to \lambda$. By (2.11), we have

$$|T(f)(y) - \lambda f(x)| \leq \frac{3}{2}\varepsilon, \text{ for all } f \in C_0(X).$$

This implies that $\{(x, \lambda)\} = Q_+(y)$ and hence $\varphi(y) = x$. Thus φ is continuous. By the same argument, we can get φ^{-1} is also continuous. Hence φ is a homeomorphism. For any convergent subset $\{\lambda(y_{\alpha'})\}_{\alpha'\in\Lambda'}$ of $\{\lambda(y_\alpha)\}_{\alpha\in\Lambda}$ with $\lambda(y_{\alpha'}) \to \lambda$, by (2.11) again, we have

$$|T(f)(y) - \lambda f(\varphi(y))| \leq \frac{3}{2}\varepsilon, \text{ for all } f \in C_0(X).$$

This implies that $\{(\varphi(y), \lambda)\} = Q_+(y)$ and $\lambda(y) = \lambda$. Hence $\lambda : Y \to \{\pm 1\}$ is continuous. The proof is complete. □

The following example shows that the estimate '$\frac{3}{2}\varepsilon$' in Theorem 2.16 is optimal.

Example 2.17. Let $X = \{x_1, x_2\}$ with the discrete topology, then $C(X) = \ell_\infty^2$. Let

$$U_1 = \{(a, b) \in C(X) : b \leq -\frac{1}{2}\varepsilon,\ a + b \geq -\frac{1}{2}\varepsilon,\ (a, b) \neq (\varepsilon, -\frac{1}{2}\varepsilon)\}$$

and

$$U_2 = C(X) \setminus (U_1 \cup \{(0, 0), (\varepsilon, -\frac{1}{2}\varepsilon)\}).$$

Define $T : C(X) \to C(X)$ by

$$T((a, b)) = \begin{cases} (0, 0) & (a, b) = (0, 0), \\ (\varepsilon, \varepsilon) & (a, b) = (\varepsilon, -\frac{1}{2}\varepsilon), \\ (a - \frac{1}{2}\varepsilon, b) & (a, b) \in U_1, \\ (a, b - \frac{1}{2}\varepsilon) & (a, b) \in U_2. \end{cases}$$

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It is not difficult to verify that T is a standard surjective map. In what follows, we show that T is an ε-norm-additive map. By the definition of T,

$$\|T(f)-f\| \le \frac{1}{2}\varepsilon,\ \forall\, f\in C(X),\ f\neq(\varepsilon,-\frac{1}{2}\varepsilon).$$

Then for any $f,g\in C(X)$ with $f,g\neq(\varepsilon,-\frac{1}{2}\varepsilon)$, we have

$$\begin{aligned}\big|\|T(f)+T(g)\|-\|f+g\|\big| &\le \|(T(f)+T(g))-(f+g)\|\\ &\le \|T(f)-f\|+\|T(g)-g\|\le\varepsilon.\end{aligned}$$

The left case is $f=(\varepsilon,-\frac{1}{2}\varepsilon)$ and $g\in C(X)$. When $g=(0,0)$ or $(\varepsilon,-\frac{1}{2}\varepsilon)$, it is clear that

$$\|T(f)+T(g)\|=\|f+g\|.$$

Let $g=(a,b)\in U_1$, we have

$$|b+\varepsilon|\le|b|\le a+\frac{1}{2}\varepsilon.$$

Then

$$\begin{aligned}\|T(f)+T(g)\| &=\max\{|a+\frac{1}{2}\varepsilon|,|b+\varepsilon|\}=a+\frac{1}{2}\varepsilon,\\ \|f+g\| &=\max\{|a+\varepsilon|,|b-\frac{1}{2}\varepsilon|\}=a+\varepsilon.\end{aligned}$$

Thus

$$\big|\|T(f)+T(g)\|-\|f+g\|\big|=\frac{1}{2}\varepsilon.$$

Let $g=(a,b)\in U_2$, we have

$$\begin{aligned}\big|\|T(f)+T(g)\|-\|f+g\|\big| &=\big|\|(a+\varepsilon,b+\frac{1}{2}\varepsilon)\|-\|(a+\varepsilon,b-\frac{1}{2}\varepsilon)\|\big|\\ &\le\|(a+\varepsilon,b+\frac{1}{2}\varepsilon)-(a+\varepsilon,b-\frac{1}{2}\varepsilon)\|\\ &=\|(0,\varepsilon)\|=\varepsilon.\end{aligned}$$

Therefore, T is an ε-norm-additive map. The homeomorphism $\varphi: X\to X$ and $\lambda: X\to\{\pm1\}$ which satisfy Theorem 2.16 are

$$\varphi=Id_X,\quad \lambda(x_i)=1,\ for\ i=1,2.$$

When $f=(\varepsilon,-\frac{1}{2}\varepsilon)$, then

$$|T(f)(x_2)-\lambda(x_2)f(\varphi(x_2))|=|\varepsilon-(-\frac{1}{2}\varepsilon)|=\frac{3}{2}\varepsilon.$$

This implies that the estimate $\frac{3}{2}\varepsilon$ in Theorem 2.16 is optimal.

Remark 2.18. The assumption of surjectivity of T in Theorem 2.16 is essential in general. For instance, let $X = \{a\}$, $Y = \{b, c\}$ be two discrete topological spaces, then $C(X) = \mathbb{R}$ and $C(Y) = \ell^2_\infty$. Define $T : \mathbb{R} \to \ell^2_\infty$ by $T(x) = (x, \sin x)$ for all $x \in \mathbb{R}$. Then T is a norm-additive map, but X and Y are not homeomorphic.

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