J. Austral. Mat. Soc. (*Series A*) **30** (1980), 107–128

SOME REMARKS ON A ONE-DIMENSIONAL SKIP-FREE PROCESS WITH REPULSION

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(Received 23 July 1979; revised 20 February 1980)

Communicated by R. L. Tweedie

Abstract

We extend the results obtained by Hines and Thompson for a Markov chain which has a single reflecting barrier at the origin, nearest neighbour transitions and which moves from $\{j\}$ to $\{j+1\}$ with probability $j/(j+1)$. Martingale limit theorems are used to work out an asymptotic theory for a more general class of such chains for which the probability above has the form $1-\lambda(j)$, $0 < \lambda(j) < 1$ ($j \in \mathbf{N}$), $\lambda(j) \to 0$ ($j \to \infty$) and $\sum \lambda(j) = \infty$. We discuss the case where the last sum is finite and some alternative versions of the general case.

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 60 J 15; secondary 60 F 05, 60 G 42.

1. Introduction

Recently Hines and Thompson (1978) considered the following stochastic version of a self-avoiding random walk on $\mathbf{Z}$; see Barber and Ninham (1970). They consider a sequence $\{X_n\}$ of $\mathbf{Z}_+$-valued random variables where $X_0 = 0$ and if $\mathscr{F}_n$ is the σ-algebra generated by $(X_0, \ldots, X_n)$ then

$$P(X_{n+1} - X_n = -1 \mid \mathscr{F}_n) = 1 - P(X_{n+1} - X_n = 1 \mid \mathscr{F}_n) = \lambda(X_n),$$

where $\lambda(j) = (j+1)^{-1}$ if $j \in \mathbf{N}$ and $\lambda(0) = 0$. This is a random walk in the sense of, for example, Harris (1952) or Karlin and McGregor (1959) which has the feature that the drift away from the origin becomes stronger with distance from the origin.

Let $[p_{ij}^{(n)}]$ denote the n-step transition matrix and let

$$F(s,t) = \sum_{n \geqslant 0} t^n E(s^{X_n}(1+X_n)^{-1}).$$

Hines and Thompson obtained a linear differential equation satisfied by $F(\cdot,\cdot)$ but did not completely specify its solution. They also prove that $G_{00} = \sum p_{00}^{(n)} \leqslant 1 + \sqrt{3}$

https://doi.org/10.1017/S1446788700021960 Published online by Cambridge University Press

and $EX_n = n - 2\log n(1+o(1))$. The last result illustrates the essence of the Hines–Thompson model, namely, that although it possesses a classical linear trend, the fluctuations about this trend are very small.

In this paper we refine and extend the results of Hines and Thompson (1978) by making extensive use of generating function techniques. The main goal is to show that $\operatorname{var} X_n \sim 4\log n$ (Section 5) which then yields some limit theorems for $\{X_n\}$ (Section 6). This goal requires a much finer asymptotic development of EX_n than that above and in turn this requires the generating function of $\{E(1+X_n)^{-1}\}$ (Section 4) and of the probability of first return to the origin (Section 2). In Section 3 we show that the $p_{ij}^{(n)}$ decay geometrically fast and in Section 2 we improve the inequality above by showing that $G_{00} = e$. Thus the bound given by Hines and Thompson is very close to the true value

The techniques used in Sections 2–6 do not seem able to yield detailed results on the asymptotic behaviour of $\{X_n\}$ although this may be attainable by suitably modifying the techniques used by Garding (1961). In Section 7 we show that Martingale techniques yield limit theorems for the general random walk with repulsion as defined above but assuming only that $\lambda(0) = 0$, $0 < \lambda(j) < 1$ $(j \in \mathbf{N})$, $\lambda(j) \to 0$ and $\sum \lambda(j) = \infty$. We shall obtain an almost sure convergence result and corresponding central limit and iterated logarithm theorems. The Martingale techniques do not seem capable of yielding the behaviour of $\operatorname{var} X_n$ and the detailed analysis of specific cases such as is executed in Sections 2–6 appears to be necessary for this. Nevertheless the Martingale techniques will yield an expansion for EX_n of the same finesse as that obtained by Hines and Thompson (1978) for the case $\lambda(i) = (i+1)^{-1}$. Finally in Section 8 we shall discuss the case where $\sum \lambda(j) < \infty$ which has as its essential property that the increments are eventually always unity.

In Section 9 we discuss an extension of the general process of Section 7. This extension allows jumps to the right which may exceed unity but are uniformly bounded. The results obtained indicate that if the distribution of the positive part of the increments has positive variance then the associated variability swamps the small deviations about the linear trend obtained in Section 7 and classical central limit results obtain. We shall end Section 9 by briefly discussing the maximum process $\{V_n\}$ where $V_n = \max_{m \leqslant n} X_m$. In particular we shall show that classical renewal theory arguments show that $\{V_n\}$ possesses the same limit behaviour as $\{X_n\}$.

2. Passage probabilities

Let $P_i(\cdot) = P(\cdot \mid X_0 = i)$ and $q_i = P_i(X_n = 0 \text{ for some } n \geqslant 1)$ be the probability of eventual entry into $\{0\}$. Since $p_{01} = 1$, $q_0 = q_1$. The following result gives the q_i.

THEOREM 1. $q_i = e^{-1}\sum_{k \geq i} 1/k!\ (i \in \mathbf{N})$, $q_0 = 1 - e^{-1}$.

PROOF. Let

$$\rho(0) = 1 \quad \text{and} \quad \rho(j) = \prod_{i=1}^{j} \lambda(i) \Big/ \left(\prod_{i=1}^{j} (1-\lambda(i)) \right).$$

In our case $\rho(j) = 1/j!$ In general

$$q_i = \sum_{k \geq i} \rho(k) / \sum_{k \geq 0} \rho(k)$$

and the theorem follows immediately; see Chung (1967), p. 76.

Since $G_{00} = (1-q_0)^{-1}$, see Chung (1967), p. 55, we obtain

COROLLARY 1. $G_{00} = e$.

It follows that $\{X_n\}$ is *transient*, a comment relevant to the sentence containing equation (3.5) in Hines and Thompson (1978).

In the sequel we shall require an expression for the generating function $P(t) = \sum_{n \geq 0} p_{00}^{(n)} t^n$.

THEOREM 2. $P(t) = t^{1-t^2}[(1-t^2)\int_0^t e^{-yt}(t-y)^{-t^2}\,dy]^{-1}$ $(0 \leq t < 1)$.

PROOF. Let $f(n)$ be the probability of a first return to $\{0\}$ at time n. If

$$F(t) = \sum_{n \geq 1} f(n) t^n$$

then

$$P(t) = (1 - F(t))^{-1}, \tag{2.1}$$

see Chung (1967), p. 55, and $F(\cdot)$ is calculated as follows.

Let $\{Z_n\}$ be the Markov chain obtained from $\{X_n\}$ by making $\{0\}$ absorbing and

$$q(i,n) = \hat{P}_i(Z_n = 0)$$

where

$$\hat{P}_i(\cdot) = P(\cdot \mid Z_0 = i);\ q(0,n) \equiv 1.$$

Clearly $f(n) = q(1, n-1) - q(1, n-2)\ (n \geq 2)$, $f(1) = 0$ and hence

$$F(t) = t(1-t)\,Q_1(t), \tag{2.2}$$

where $Q_i(t) = \sum_{n \geq 0} q(i,n) t^n$. Finally if $Q(s,t) = \sum_{i \geq 0} s^i Q_i(t)$ then

$$Q_1(t) = (\partial/\partial s)\,Q(s,t)\big|_{s=0}.$$

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Clearly, for $i \in \mathbf{N}$,

$$q(i, n+1) = i(i+1)^{-1} q(i+1, n) + (i+1)^{-1} q(i-1, n)$$

and it follows that

$$(s-t)\partial Q/\partial s + (1 + ts^{-1} - ts) Q = (t+s)/s(1-t)$$

and $Q(0, t) = (1-t)^{-1}$. The integrating factor of this differential equation is

$$I(s, t) = s^{-1} |s-t|^{2-t^2} e^{-st}.$$

Since $I(t, t) = 0$ it follows that if $s \leqslant t$ then

$$I(s,t) Q(s,t) = t(1-t)^{-1} \int_s^t y^{-2} e^{-ty} (t-y)^{1-t^2} \, dy + (1-t)^{-1} \int_s^t y^{-1} e^{-ty} (t-y)^{1-t^2} \, dy. \tag{2.3}$$

We use

$$\begin{aligned} Q_1(t) &= \lim_{s \to 0} s^{-1} (Q(s,t) - (1-t)^{-1}) \\ &= (1-t)^{-1} \lim_{s \to 0} s^{-1} [(I(s,t))^{-1} (tI_1 + I_2) - 1], \end{aligned}$$

where I_1 and I_2 are the first and second integrals, respectively, at (2.3). Partial integration yields

$$I_1 = I(s,t)(t-s)^{-1} - \int_s^t y^{-1} e^{-yt} (t-y)^{1-t^2} (t + (1-t)^2 (t-y)^{-1}) \, dy,$$

whence

$$\begin{aligned} Q_1(t) &= (1-t)^{-1} \lim_{s \to 0} s^{-1} \left\{ s(t-s)^{-1} - (I(s,t)^{-1}(1-t^2) \int_s^t e^{-ty} (t-y)^{-t^2} \, dy \right\} \\ &= (t(1-t))^{-1} - (1+t) t^{-2+t^2} \int_0^t e^{-ty} (t-y)^{-t^2} \, dy. \end{aligned}$$

The assertion of the theorem now follows from (2.2) and then (2.1).

Partial integration yields

Corollary 2. $F(t) = t^2 \int_0^1 (1-y)^{1-t^2} (\exp(-t^2 y)) \, dy.$

3. Geometric ergodicity

Later we shall need to have an estimate of the rate at which the $p_{ij}^{(n)}$ decay to zero. It is known for simple, restricted and self-avoiding walks on most lattices that the

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probability of returning to the origin at time n decreases algebraically fast; see Barber and Ninham (1970). For transient one-dimensional random walks restricted by a reflecting barrier at the origin it is known that this probability behaves, for large n, like $\rho^{n^{-3\,2}}$, where $0 < \rho < 1$ (Veraverbeke and Teugels (1976)), that is, it is ρ^{-1}-transient; see Seneta (1973), pp. 163,4 for the terminology and theory. In the present case we have τ-positivity, namely that $p_{00}^{(2n)}$ behaves, for large n, like τ^{-n} where $1 < \tau < 2$; obviously $p_{00}^{(2n)} = 0$ when n is odd. The following theorem gives full details.

THEOREM 3. *There is a unique number* $\tau \in (1,2)$ *such that*

$$\tau \int_0^1 (1-y)^{1-\tau} e^{-y\tau}\, dy = 1.$$

If $R = \sqrt{\tau}$ *then*

$$R^n p_{ij}^{(n)} \to 2x_j y_i / \sum_k x_k y_k > 0, \tag{3.1}$$

where $n \to \infty$ *through the even integers if* $(i-j)$ *is even and through the odd integers otherwise, and if* $s \leqslant 1/R$ *then*

$$\sum_{j\geqslant 0} x_j s^{j+1}(j+1)^{-1} = R x_0 s^{\tau}(1-Rs)^{-\tau} e^{-R/s} \int_s^{1/R} (1-y^2)(1-Ry)^{\tau-1} e^{R/y} y^{-1-\tau}\, dy \tag{3.2}$$

and

$$\sum_{i\geqslant 0} y_i s^i = R y_0 s(R-s)^{-2+\tau} e^{Rs} \int_s^R y^{-2}(R-y)^{1-\tau} e^{-Ry}\, dy. \tag{3.3}$$

PROOF. The integral representation of $F(\cdot)$ in Corollary 2 is holomorphic in $|t|^2 < 2$ and, moreover, $F(t) \to \infty$ $(t \to \sqrt{2})$. Since $F(1) = 1 - e^{-1}$ it follows that τ exists and $F(R) = 1$. Furthermore, $\sum f(n) n R^n < \infty$ whence $\{X_n\}$ is R-positive; see Seneta (1973), p. 163. It follows that (3.1) holds and also that $\{x_j\}$ and $\{y_i\}$ satisfy the systems

$$R \sum_{i\geqslant 0} x_i p_{ij} = x_j, \quad R \sum_{j\geqslant 0} p_{ij} y_j = y_i, \tag{3.4}$$

$\sum x_k y_k < \infty$ and $\{x_i\}$ and $\{y_i\}$ are the unique solutions, up to constant multipliers, of these systems.

By expressing (3.4) as difference equations and assuming for the moment that the generating functions $V(s)$ and $W(s)$ at (3.2) and (3.3), respectively, both exist, it is easy

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to show that they must satisfy the differential equations

$$(1-Rs)\,V'(s)+(1-s^{-2})\,V(s) = -Rx_0\,s^{-1}(1-s^2)$$

and

$$(R-s)\,W'(s)-(1+Rs^{-1}(1-s^2))\,W(s) = -Ry_0\,s^{-1}.$$

These are solved by the right-hand expressions at (3.2) and (3.3) respectively and since these solutions are both holomorphic at the origin, it is possible to argue back to the existence of $V(\cdot)$ and $W(\cdot)$.

4. The first and second moments

Our aim in this section is to find expressions for the first and second moments of $\{X_n\}$. In principle these can be obtained from the generating function $P(s,t) = \sum_{n\geqslant 0} t^n E_0(s^{X_n})$, where $E_i(\cdot) = E(\cdot \mid X_0 = i)$. This function satisfies a first-order linear differential equation which in turn can be obtained from that satisfied by $F(s,t)$:

$$(4.1)\qquad (1-ts)\,\partial F/\partial s+t(1-s^{-2})\,F = 1+t(s-s^{-1})\,P(t).$$

A partial solution of (4.1) was given by Hines and Thompson (1978). The difficulty in writing down an explicit solution lies in the fact that its integrating factor is not zero in $\{|s|\leqslant 1, |t|<1\}$ whereas that for the equation satisfied by $P(s,t)$ is more tractable in this respect. However, the solution of this equation is not completely straightforward since it requires a preliminary estimate of the rate of convergence of $P_j(t) = \sum p_{0j}^{(n)}\, t^n$: $\lim_{j\to\infty} jt^{-j}\, P_j(t)$ exists and is positive. Even though an explicit expression for $P(s,t)$ can be obtained it is too complicated to be useful and the generating function for $\mu(n) = E_0(X_n)$ which it provides is of no use for our purposes.

Let $M(t) = \sum_{n\geqslant 0} \mu(n)\,t^n$. Hines and Thompson (1978) used (4.1) to show that

$$(4.2)\qquad M(t) = t(1-t)^{-2}+2t(1-t)^{-1}[P(t)-F(1,t)].$$

It is not difficult to use this to obtain the representation

$$(4.3)\qquad n-\mu(n) = 2\sum_{m\leqslant n-1} \eta(m),$$

where $\eta(n) = \theta(n)-p_{00}^{(n)}$, $\theta(n) = E_0((1+X_n)^{-1})$ and hence $\eta(n) = E\lambda(X_n)$. This representation also follows from Proposition 3 below; see (7.2). Equation (4.3) shows that $\{n-\mu(n)\}$ is nondecreasing, which also follows from the fact that $\{n-X_n\}$ is a.s. nondecreasing.

Let $N(t) = \sum_{n\geqslant 0} \eta(n)\,t^n$. We shall obtain an explicit expression for $N(\cdot)$ in

THEOREM 4. *If* $0 \leqslant t \leqslant 1$ *then*

$$N(t) = t\beta(t)\,P(t) \tag{4.4}$$

where

$$\beta(t) = t^{-2+t^2}\int_0^t e^{-yt}(t-y)^{1-t^2}(1-y)^{-1}\,dy \tag{4.5}$$

PROOF. By definition $\eta(n) = \sum_{j\geqslant 1} p_{0j}^{(n)}(j+1)^{-1}$ and

$$p_{0j}^{(n)} = {}_0p_{0j}^{(n)} + \sum_{m=1}^{n} f(m)\,p_{0j}^{(n-m)} \quad (j \geqslant 1),$$

where ${}_0p_{0j}^{(n)}$ is the probability of hitting $\{j\}$ at n without returning to $\{0\}$ in the intervening time. Clearly ${}_0p_{0j}^{(n)} = q_{1j}^{(n-1)}$, where $q_{ij}^{(n)} = \hat{P}_i(Z_n = j)$, and hence

$$P_j(t) = F(t)\,P_j(t) + tq_j(t),$$

where $q_j(t) = \sum_{n\geqslant 0} q_{1j}^{(n)} t^n$ and (4.4) nows follows from (2.1) and with $\beta(t)$ defined as $\sum_{j\geqslant 1} q_j(t)(j+1)^{-1}$.

Let $\beta(i,n) = \sum_{j\geqslant 1} q_{ij}^{(n)}(j+1)^{-1}$. The backward equations for the $q_{ij}^{(n)}$ yield

$$\beta(1,n+1) = \beta(2,n)/2 \quad \text{and} \quad \beta(i,n+1) = (i\beta(i+1,n)+\beta(i-1,n))/(i+1) \quad (i \geqslant 2).$$

If $B(s,t) = \sum_{i,n\geqslant 0} s^i t^n \beta(i+1,n)$ we obtain

$$(t-s)\,\partial B/\partial s - (2-ts)\,B = -1/(1-s).$$

Observing that the integrating factor of this equation vanishes when $s = t$ and that $\beta(t) = B(0,t)$, it is easy to verify (4.5).

We now obtain an expression for $\mu_2(n) = E_0 X_n^2$. This follows by a double differentiation with respect to s of (4.1) and then letting $s = 1$. If

$$M_2(t) = \sum_{n\geqslant 0} \mu_2(n)^{tn} = (\partial^3/\partial s^3)\,F(1,t) + M(t),$$

then

$$(1-t)(M_2(t) - M(t)) - 2tM(t) - 6F(1,t) + 4t(1-t)^{-1} = -2tP(t).$$

Using (4.2) and $F(1,t) = N(t) + P(t)$ we obtain

$$M_2(t) = 2(1-t)^{-1}M(t) - 4M(t) - t(1-t)^{-2} + 4t(1-t)^{-1}P(t).$$

This yields

THEOREM 5.

$$\mu_2(n) = 2\sum_{k\leqslant n}\mu(k) - 4\mu(n) - n + 4G_{00}(n-1), \quad \text{where } G_{00}(n) = \sum_{m\leqslant n} p_{00}^{(m)}.$$

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5. Asymptotic behaviour of the mean and variance

We begin by proving the following refinement of Hines and Thompson's (1978) expansion of $\mu(n)$.

THEOREM 6.

(5.1) $$n-\mu(n)-2H(n) = 2\sum_{m\geqslant 1}(m\cdot m!)^{-1}+o(1) \quad (n\to\infty),$$

where

$$H(n) = \sum_{m=1}^{n} 1/m.$$

PROOF. We begin by determining the asymptotic behaviour of $\{\beta(n)\}$ where $\beta(t) = \sum \beta_n t^n$. By letting $y = tx$ in (4.5) we obtain

(5.2) $$\begin{aligned}\beta(t) &= \int_0^1 (\exp(-xt^2))(1-x)^{1-t^2}(1-tx)^{-1}\,dx \\ &= -(et)^{-1}\log(1-t)-e^{-1}\int_0^1 (1-(1-x)^{1-t^2})(1-tx)^{-1}\,dx \\ &\quad+\int_0^1 (\exp(-xt^2)-e^{-1})(1-tx)^{-1}(1-x)^{1-t^2}\,dx.\end{aligned}$$

The integrand in the last integral at (5.2) is

$$\leqslant e^{-1}[(1-xt^2)/(1-xt)] \leqslant e^{-1}[(1-x^2t^2)/(1-xt)] \leqslant 2e^{-1},$$

and hence dominated convergence shows that this integral converges to $e^{-1}\int_0^1(e^{1-x}-1)/(1-x)\,dx$. Denote the integral by c. Expanding the exponential term as a power series in $(1-x)$ and using Fubini's theorem we find that $c = \sum_{m\geqslant 1}(m\,.\,m!)^{-1}$.

Denote the second integral at (5.2) by $I(t)$, split its range of integration at $1-\varepsilon+\varepsilon t$ and denote the resulting integrals by I_1 and I_2. Now

$$\begin{aligned}0 \leqslant I_1 &\leqslant 1-(\varepsilon(1-t))^{1-t^2}\int_0^{1-\varepsilon+\varepsilon t}(1-tx)^{-1}\,dx \\ &\leqslant [1-(\varepsilon(1-t))^{1-t^2}][-t^{-1}\log(1-t)].\end{aligned}$$

But

$$\begin{aligned}1-(\varepsilon(1-t))^{1-t^2} &= 1-\exp[(1-t^2)(\log\varepsilon+\log(1-t))] \\ &\leqslant K(1-t^2)[-\log\varepsilon-\log(1-t)]\end{aligned}$$

and it follows that $I_1 \to 0$ $(t \to +1)$. In addition

$$I_2 \leqslant \int_{1-\varepsilon+\varepsilon t}^{1} (1-tx)^{-1}\,dx \to \log(1+\varepsilon) \quad (t \to 1).$$

Now let $\varepsilon \to 0$ to obtain $I(t) \to 0$ $(t \to 1)$.

It follows from these results that

$$b(n) = \sum_{m \leqslant n-1} \beta(m) = e^{-1} H(n) + e^{-1} c + o(1) \quad (n \to \infty) \tag{5.3}$$

Let $d(n) = \sum_{m \leqslant n} \eta(m)$. Then

$$d(n) = \mathscr{C}_n\{[\beta(t)/(1-t)]\,[tP(t)]\}, \tag{5.4}$$

where $\mathscr{C}_n f(t)$ denotes the coefficient of t^n in $f(t)$. We have seen (Corollary 1) that $P(1) = e$ and Theorem 3 shows that the power series expansion of $tP(t)$ has a radius of convergence exceeding unity. Finally, it follows from (5.3) that $b(n+1)/b(n) \to 1$ $(n \to \infty)$ and $b(n) = \mathscr{C}_n \beta(t)/(1-t)$. It is now clear that the conditions of the following result of Bojanic and Lee (1974) are fulfilled and hence Theorem 6 follows from (5.2) to (5.4):

PROPOSITION 1. *If $\{a(n)\}$ is a positive sequence,*

$$a(n+1)/a(n) = 1 + O(\Delta(n)), \quad \Delta(n) \to 0 \quad (n \to \infty), \quad A(t) = \sum a(n) t^n$$

and $\Lambda(t) = \sum \zeta(n) t^n$ has a radius of convergence exceeding unity, then

$$\mathscr{C}_n(A(t)\,\Lambda(t)) = a(n)\,\Lambda(1)\,(1 + O(\Delta(n))).$$

Later we shall use the expansion

$$H(n) = \log n + \gamma + (2n)^{-1} + O(n^{-2}), \tag{5.5}$$

where γ is Euler's constant. This yields

COROLLARY 3. $\mu(n) = n - 2\log n - 2(\gamma + c) + o(1)$ $(n \to \infty)$.

We shall now establish the following result:

THEOREM 7. $\operatorname{var} X_n \sim 4 \log n$ $(n \to \infty)$.

The proof depends on a further refinement of Theorem 6. Let $\delta(n) = n - \mu(n) - 2H(n) - 2c$ and $D(t) = \sum \delta(n) t^n$. It follows from (4.3), (4.4) and (5.2)

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that

$$(5.6)\quad D(t) = -2t\varphi(t)(\log(1-t))/(1-t)+2(\log(1-t))/(1-t)+2ct^2\varphi(t)/(1-t) \\ -2c/(1-t)+2t^2\varphi(t)(J(t)-I(t))/(1-t),$$

where $\varphi(t) = P(t)/e$, which is a probability generating function, $I(t)$ is the second integral at (5.2) and

$$J(t) = \int_0^1 \left[\frac{(\exp(1-xt^2))-1}{1-tx}(1-x)^{1-t^2} - \frac{(\exp(1-x))-1}{1-x}\right]dx.$$

We have seen above that $I(t)$, $J(t)\to 0$ $(t\to\infty)$. The key step in the proof is to establish

PROPOSITION 2. $\delta(n) = -4n^{-1}\log n - 4n^{-1}(1+\gamma+c)+o(n^{-1})$.

The proof makes repeated use of Proposition 1. The first two terms on the right at (5.6) can be written as $K(t) = 2[\varphi(t)+(1-\varphi(t))/(1-t)]\log(1-t)$. Now $\varphi(1) = 1$ and using (2.1) and Corollary 2 we find that $\varphi'(1) = 2(1+c)$. Proposition 1 then yields

$$(5.7)\quad \mathscr{C}_n K(t) = -2(3+2c)n^{-1}+o(n^{-1}).$$

The next two terms in $D(t)$ together equal $-2c(1-\varphi(t))/(1-t)-2c(1+t)\varphi(t)$ and hence, by Theorem 3, the coefficient decays geometrically fast.

Write the integrand of $J(t)$ as

$$(5.8)\quad \frac{\exp(1-xt^2)-1}{1-tx} - \frac{\exp(1-x)-1}{1-x} - \frac{\exp(1-xt^2)-1}{1-tx}(1-(1-x)^{1-t^2}).$$

Denote the last term by $\lambda(x,t)$. Observing that

$$(5.9)\quad 1-(1-x)^{1-t^2} = O[(1-t^2)(-\log(1-x))],$$

dominated convergence shows that $\lim_{t\to 1}(1-t)^{-1}\int_0^1 \lambda(x,t)\,dx$ is finite. Rewrite the first two terms at (5.8) as

$$\left[\frac{\exp(1-xt^2)-1}{1-tx} - \frac{\exp(1-x)-1}{1-x}\right] - (\exp(1-x)-1)((1-x)^{-1}-(1-xt)^{-1}) \\ \equiv a(x,t)-b(x,t).$$

Partial integration yields

$$\int_0^1 a(x,t)\,dx = (-\log(1-t))(\exp(1-t^2)-1)/t \\ +t^{-1}\int_0^1 (\log(1-xt))(\exp(1-x)-t^2\exp(1-xt^2))\,dx \\ = (-2\log(1-t))(1-t)(1+o(1))$$

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whence

$$\mathscr{C}_n(1-t)^{-1}\int_0^1 a(x,t)dx = 2n^{-1}(1+o(1)).$$

Now

$$\mathscr{C}_n(1-t)^{-1}\int_0^1 b(x,t)\,dx = \int_0^1 (e^{1-x}-1)\,x^{n+1}(1-x)^{-1}\,dx$$
$$= \int_0^1 x^{n+1}\,dx + O\left(\int_0^1 (1-x)\,x^{n+1}\,dx\right)$$
$$= n^{-1}(1+o(1)).$$

We thus obtain

$$\mathscr{C}_n(1-t)^{-1}J(t) = n^{-1}(1+o(1)).$$

Turning now to $I(t)$, we write

(5.10)
$$(1-t)^{-1}I(t) = (1-t)^{-1}\int_0^1 [1-(1-x)^{1-t^2}+(1-t^2)\log(1-x)]/[1-tx]\,dx$$
$$-(1+t)\int_0^1 (\log(1-x))/(1-xt)\,dx.$$

Now

$$\mathscr{C}_n\left(-\int_0^1 (\log(1-x))/(1-xt)\,dx\right) = -\int_0^1 x^n\log(1-x)\,dx$$
$$= \sum_{m\geqslant 0} m^{-1}(n+m)^{-1} = n^{-1}(1+2^{-1}+\ldots+n^{-1})$$
$$= n^{-1}\log n + \gamma/n + O(n^{-2}).$$

By refining (5.9) to

$$1-(1-x)^{1-t^2} = -(1-t^2)\log(1-x)+O[(1-t^2)^2(\log(1-x))^2]$$

we infer that the first term on the right at (5.10) $\to 0$ $(t\to 1)$. Using Proposition 1 again we obtain

$$\mathscr{C}_n(1-t)^{-1}I(t) = (2/n)\log n + 2\gamma/n + o(n^{-1}).$$

By putting these results together and using Proposition 1 where appropriate we obtain Proposition 2.

Using the refined expression for $\mu(n)$ obtained from Proposition 2 we obtain

(5.11) $$(\mu(n))^2 = n^2 + 4(\log n)^2 - 4n\log n - 4(\gamma+c)\,n + 8(1+\gamma+c)\log n + O(1)$$

To calculate $\sum_{k=1}^{n} \mu(k)$ it is convenient to express $\mu(n)$ as

$$\mu(n) = n - 2H(n) - 2c + 4H(n)/n + 4(1+c)/n + o(n^{-1}).$$

Now

$$\sum_{k=1}^{n} H(k) = \sum_{i=1}^{n} \sum_{k=i}^{n} i^{-1} = (n+1)\,H(n) - n$$

$$= n \log n + (\gamma - 1)\,n + \log n + O(1),$$

where we have used (5.5) and

$$\sum_{k=1}^{n} H(k)/k = \sum_{i=1}^{n} i^{-1} \sum_{k=i}^{n} k^{-1} = \sum_{i=1}^{n} i^{-1}(H(n) - H(i-1))$$

$$= (H(n))^2 - \sum_{i=1}^{n} i^{-1}(H(i) - i^{-1})$$

whence

$$\sum_{k=1}^{n} H(k)/k = (H(n))^2/2 + O(1) = (\log n)^2/2 + \gamma \log n + O(1).$$

It follows that

$$\sum_{k=1}^{n} \mu(k) = n(n+1)/2 - 2n \log n - 2(\gamma - 1 + c)\,n + 2(\log n)^2$$
$$+ 2(1 + 2c + 2\gamma) \log n + O(1).$$

Theorem 7 now follows from the last expression, (5.11) and Theorem 5.

6. Limit theorems

It is intuitively clear that X_n is asymptotically proportional to n. Theorem 7 enables us to prove this.

THEOREM 8. $X_n/n \xrightarrow{\text{a.s.}} 1$ $(n \to \infty)$.

PROOF. $E((n^{-1} X_n - 1)^2) = n^{-2}(\operatorname{var} X_n + (\mu(n) - n)^2) \sim 4(\log n)^2/n^2$ and hence Chebychev's inequality yields

$$\sum_{n \geq 1} P\{|n^{-1} X_n - 1| > \delta\} < \infty$$

for each $\delta > 0$. Thus the sequence $\{n^{-1} X_n - 1\}$ converges completely to zero, and hence almost surely; see Lukacs (1975), p. 51.

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Using Theorems 6 and 7 again it is not hard to show that $E\{[(2\log n)^{-1}(n-X_n)-1]^2\}\to 0$ whence the following result:

THEOREM 9. $(n-X_n)/2\log n \xrightarrow{P} 1 \ (n\to\infty)$.

We may conjecture that this result can be strengthened to almost sure convergence and that $(n-X_n-2\log n)/2(\log n)^{\frac{1}{2}} \Rightarrow N(0,1)$ and we now show that these results are true by using a different tool kit.

7. A generalized skip-free walk with repulsion

We now let $\{X_n\}$ be as defined in Section 1 but assume only that for $j\in\mathbf{N}$, $0<\lambda(j)<1$ and $\lambda(j)\to 0\ (j\to\infty)$. Let $\Lambda(n)=\sum_{i=1}^n\lambda(i)$. The essence of the Hines–Thompson model is that $\Lambda(n)\to\infty\ (n\to\infty)$, which we assume throughout this section. As above we always assume that $X_0=0$.

Let $W_0=0$ and

$$W_n = X_n - n + 2\sum_{m=1}^{n}\lambda(X_{m-1}) \quad (n\in\mathbf{N}).$$

PROPOSITION 3. *$\{W_n, \mathscr{F}_n\}$ is a zero-mean martingale.*

PROOF. Let $Z_n = W_n - W_{n-1} = X_n - X_{n-1} - 1 + 2\lambda(X_{n-1})$. Then

$$E(Z_n\,|\,\mathscr{F}_{n-1}) = (1-\lambda(X_{n-1})) - \lambda(X_{n-1}) - (1-2\lambda(X_{n-1})) = 0$$

and $W_1=0$ (a.s.), thus proving the proposition.

We first observe that the state space Z_+ is transient. To see this observe that if $a(n)=\prod_{i=1}^n\lambda(i)/(1-\lambda(i))$ then $a(n+1)/a(n)\to 0$ and hence $\sum a(n)<\infty$, which condition is necessary and sufficient for transience; see Karlin and McGregor (1959), p. 71. Thus $X_n\xrightarrow{\text{a.s.}}\infty$ and hence

$$n^{-1}\sum_{m=1}^{n}\lambda(X_{m-1})\xrightarrow{\text{a.s.}} 0 \quad (n\to\infty) \tag{7.1}$$

Proposition 3 yields

$$EX_n = n - E\left(\sum_{m=1}^{n}\lambda(X_{m-1})\right) \tag{7.2}$$

and (7.1) with dominated convergence yields $n^{-1}EX_n\to 1$. We shall refine this result below; see Corollary 4.

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Since the sequence of martingale differences $\{Z_n\}$ is orthogonal and $W_1 = 0$ it follows that $EW_n^2 = \sum_{m=1}^n EZ_m^2$ and hence that

$$EW_n^2 - EW_{n-1}^2 = EZ_n^2.$$

By working as in the proof of Proposition 3 it is easy to demonstrate

LEMMA 2. $E(Z_n^2 \mid \mathscr{F}_{n-1}) = 4\lambda(X_{n-1})(1-\lambda(X_{n-1}))$.

Now $\{W_n^2, \mathscr{F}_n\}$ is a submartingale and

$$0 \leqslant EW_n^2 - EW_{n-1}^2 \leqslant 4E\lambda(X_{n-1}),$$

and hence if $\{A(n)\}$ is any positive sequence such that the series $\sum_{n\geqslant 1} \lambda(X_{n-1})/(A(n))^2$ is a.s. convergent then the conditional version of Chow's strong law for martingales, due to Stout (1974), p. 156, shows that $W_n/A(n) \xrightarrow{\text{a.s.}} 0$. Taking $A(n) = n$ and using (7.1) we immediately obtain

PROPOSITION 4. $X_n/n \xrightarrow{\text{a.s.}} 1$.

A regularity condition on the $\lambda(\cdot)$ allows us to obtain the following refinement of this result.

THEROEM 10. *If*

$$\lambda(n) = n^{-\delta} L(n), \tag{7.3}$$

where $0 \leqslant \delta \leqslant 1$ and $L(\cdot)$ *is slowly varying at infinity, then*

$$\frac{n - X_n}{\Lambda(n)} \xrightarrow{\text{a.s.}} 2.$$

REMARKS.

(i) The condition $\delta \leqslant 1$ is necessary since $\Lambda(\infty) = \infty$.

(ii) The Hines–Thompson model satisfies (7.3) with $\delta = 1$ and $\Lambda(n) = \log n + O(1)$.

(iii) It follows from Seneta (1976), p. 47, that $L(\cdot)$ can be regarded as a function with domain $\mathbf{R}_+$.

PROOF. Proposition 4 and the regular variation hypothesis show that for each ω in a set of probability one we have $\lambda(X_{k-1}) \leqslant (1+\varepsilon)\lambda(k)$ for each $k > K(\varepsilon, \omega)$ and hence if $\sum_{k\geqslant 1} \lambda(k)/(\Lambda(k))^2 < \infty$ we can conclude from Stout's theorem that $W_n/\Lambda(n) \xrightarrow{\text{a.s.}} 0$. If $k \geqslant 2$ the terms of the series are bounded above by the terms of the telescoping series

$$\sum_{k\geqslant 2} \lambda(k)/\Lambda(k)\,\Lambda(k-1) = \sum_{k\geqslant 2} [1/\Lambda(k-1)-1/\Lambda(k)] = 1/\lambda(1).$$

Our assertion will follow once we show that

$$U_n = (\Lambda(n))^{-1} \sum_{m=1}^{n} \lambda(X_{m-1}) \xrightarrow{\text{a.s.}} 1.$$

Let ω be such that $n^{-1} X_n(\omega) \to 1$. The uniform convergence theorem for regularly varying functions, see Seneta (1976), p. 2, shows that

$$1-\varepsilon < \lambda(X_{m-1})/\lambda(m) < 1+\varepsilon \quad \text{if } m \geqslant \mu(\varepsilon).$$

It follows, for example, that

$$\liminf_{n\to\infty} U_n(\omega) \geqslant (1-\varepsilon) \liminf_{n\to\infty} (\Lambda(n))^{-1} \sum_{k=\mu(\varepsilon)}^{n} \lambda(m) = 1-\varepsilon.$$

Similarly the lim sup is bounded above by $1+\varepsilon$ and the assertion now follows.

Minor changes to the proof above show that

$$(7.4) \qquad (4\Lambda(n))^{-1} \sum_{j=1}^{n} E(Z_j^2 \mid \mathscr{F}_{j=1}) \xrightarrow{\text{a.s.}} 1.$$

We shall now use a classical truncation argument to prove the following companion result.

Proposition 5. *Under the conditions of Theorem 10,*

$$(4(\Lambda(n)) \sum_{j=1}^{n} E(Z_j^2 \mid \mathscr{F}_{j-1}) \xrightarrow{L_1} 1.$$

Proof. Let $d > 1$ and $Z_j(n) = Z_j I(V_j^2 \leqslant 4d\Lambda(n))$ where V_n^2 is the sum at (7.4). Since V_n^2 is $\mathscr{F}_{n-1}$ – measurable we have

$$E(Z_j^2(n) \mid \mathscr{F}_{j-1}) = 4\lambda(X_{j-1})(1-\lambda(X_{j-1}))\, I(V_j^2 \leqslant 4d\Lambda(n)).$$

Now

$$0 \leqslant Q_n = (\Lambda(n)^{-1} \sum_{j=1}^{n} \lambda(X_{j-1})\, I(V_j^2 > 4d\Lambda(n)) \leqslant I(V_n^2 > 4d\Lambda(n))\, U_n \xrightarrow{\text{a.s.}} 0$$

and hence

$$(4\Lambda(n))^{-1} \sum_{j=1}^{n} E(Z_j^2(n) \mid \mathscr{F}_{j-1}) \xrightarrow{\text{a.s.}} 1.$$

Next

$$(7.5)\qquad \sum_{j=1}^{n} E(Z_j^2(n)\,|\,\mathscr{F}_{j-1}) = \sum_{j=1}^{n} (V_j^2 - V_{j-1}^2)\, I(V_j^2 \leqslant 4d\Lambda(n))$$
$$= V_n^2\, I(V_n^2 \leqslant 4d\Lambda(n)) + \sum_{j=1}^{n-1} V_j^2\, I(V_j^2 \leqslant 4d\Lambda(n) < V_{j+1}^2).$$

Let $a(n)$ denote the last sum. Then

$$0 \leqslant (Ea(n))/4\Lambda(n) \leqslant d \sum_{j=1}^{n-1} P(V_j^2 \leqslant 4d\Lambda(n) < V_{j+1}^2)$$
$$= d[P(V_1^2 \leqslant 4d\Lambda(n)) - P(V_n^2 \leqslant 4d\Lambda(n))] \to 0.$$

The penultimate term at (7.5) is bounded above by $4d\Lambda(n)$ whence from (7.4) and dominated convergence we can conclude that

$$E\left|(4\Lambda(n))^{-1} \sum_{j=1}^{n} E(Z_j^2(n)\,|\,\mathscr{F}_{j-1}) - 1\right| \to 0$$

Now

$$0 \leqslant \lim_{n\to\infty} E\left\{\sum_{j=1}^{n} E(Z_j^2 - Z_j^2(n)\,|\,\mathscr{F}_{j-1})\right\}$$
$$= E\left\{\lim_{n\to\infty} \sum_{j=1}^{n} (V_j^2 - V_{j-1}^2)\, I(V_j^2 > 4d\Lambda(n))\right\}$$
$$= E\left\{\lim_{n\to\infty} I(V_n^2 > 4d\Lambda(n))\, V_n^2\right\} = 0,$$

since the indicator function is zero for all sufficiently large n. This completes the proof.

It follows from the working above that $\sum_{m=1}^{n} E\lambda(X_{m-1})/\Lambda(n) \to 1$ and hence we have the following generalisation of Hines and Thompson's result quoted in the Introduction; see (7.2)

COROLLARY 4. $E\left(\dfrac{n - X_n}{2\Lambda(n)}\right) \to 1.$

We now turn to the question of asymptotic normality.

PROPOSITION 6. *Under the conditions of Theorem 10,*

$$W_n/2(\Lambda(n))^{1/2} \Rightarrow N(0, 1).$$

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PROOF. We verify that the conditions of Brown's (1971) martingale central limit theorem are fulfilled. Proposition 5 shows that $EW_n^2/4\Lambda(n) \to 1$ and hence Brown's conditions can be expressed in terms of $\Lambda(n)$ instead of EW_n^2. In particular, in addition to (7.4), we need only verify the conditional Lindeberg condition in the form

$$(\Lambda(n))^{-1} \sum_{m=1}^{n} E(Z_m^2 I(|Z_m| > \varepsilon\sqrt{\Lambda(m)}) \mid \mathscr{F}_{m-1}) \xrightarrow{P} 0. \tag{7.6}$$

The conditional expectation is

$$\begin{aligned} &4\lambda(X_{m-1})(1-\lambda(X_{m-1}))\, I(2\lambda(X_{m-1}) > \varepsilon\sqrt{\Lambda(m)}) \\ &\quad + (1-\lambda(X_{m-1}))\, I(2(1-\lambda(X_{m-1})) > \varepsilon\sqrt{\Lambda(m)}) \\ &\quad \leqslant 5\lambda(X_{m-1})\, I(2 > \varepsilon\sqrt{\Lambda(m)}). \end{aligned}$$

Since the upper bound is zero for all sufficiently large m condition (7.6) is satisfied, whence the proposition.

For the next result we shall assume that there is a function $\tilde{\lambda}(\cdot)$: $[1, \infty] \to (0, 1)$ such that $\lambda(n) = \tilde{\lambda}(n)$ $(n \in \mathbf{N})$.

THEOREM 11. *Assume that $\tilde{\lambda}(\cdot)$ has an ultimately monotone derivative and that $\tilde{\lambda}(x) = x^{-\delta} L(x)$ where $L(\cdot)$ is slowly varying at infinity. If $1/3 < \delta \leqslant 1$ or $\delta = 1/3$ and $L(x) \to 0$ $(x \to \infty)$ then*

$$\frac{n - X_n - 2\Lambda(n)}{2\sqrt{\Lambda(n)}} \Rightarrow N(0, 1).$$

REMARK. The Hines–Thompson walk satisfies the hypotheses with $\delta = 1$ and we can replace $\Lambda(n)$ by $\log n$.

PROOF. The theorem follows from Proposition 6 and Slutsky's theorem once we show that

$$\left(\Lambda(n) - \sum_{m=1}^{n} \lambda(X_{m-1})\right) \Big/ \sqrt{(\Lambda(n))} \xrightarrow{\text{a.s.}} 0. \tag{7.7}$$

Since $\tilde{\lambda}(\cdot)$ has a monotone derivative it follows that $-\tilde{\lambda}'(x) \sim \delta\tilde{\lambda}(x)/x$ $(x \to \infty)$, see Seneta (1976), p. 60, whence using Theorem 10, the mean value theorem and the uniform convergence theorem for regularly varying functions we obtain a.s.

$$\lambda(X_n) - \lambda(n) \sim 2\delta\Lambda(n)/n \sim \begin{cases} 2\delta(1-\delta)^{-1} n^{-2\delta} L^2(n) & \text{if } \delta < 1, \\ 2\delta n^{-2}\Lambda(n) L(n) & \text{if } \delta = 1 \end{cases}$$

and $\Lambda(\cdot)$ is slowly varying if $\delta = 1$. Thus if $\delta > \frac{1}{2}$ the series $\sum_{m=1}^{\infty}(\lambda(X_m) - \lambda(m))$ converges a.s. and (7.7) holds. If $\delta = \frac{1}{2}$ this series either converges, or its partial sums diverge to infinity in a slowly varying manner and (7.7) again holds. If $\delta < \frac{1}{2}$ then the

nth partial sum of the series behaves like $2\delta(1-\delta)(1-2\delta)n^{1-2\delta}L^2(n)$. But $\sqrt{\Lambda(n)} \sim [(1-\delta)^{-1}n^{1-\delta}L(n)]^{1/2}$ and since $(1-\delta)/2 \geqslant 1-2\delta$ if and only if $\delta \geqslant 1/3$, we see again that (7.7) is valid.

The last lines of the proof yield

COROLLARY 5. *If* $0 < \delta < 1/3$ *and* $a(n) = 2\delta(1-\delta)^{-1}(1-2\delta)^{-1}n^{1-2\delta}L^2(n)$ *then*

$$(n - X_n - 2\Lambda(n))/a(n) \xrightarrow{\text{a.s.}} 1$$

This result simply means that if $\delta < 1/3$ the deterministic centering sequence $\{2\Lambda(n)\}$ used in Theorem 11 is inappropriate and the random centering inherent in Proposition 6 should be retained.

The next result is a law of the iterated logarithm for $\{X_n\}$.

THEOREM 12. *Under the conditions of Theorem 11*

$$X_n = n - 2\Lambda(n) + (2\Lambda(n)\log\log(3\max\Lambda(n)))^{1/2}\zeta_n$$

where $\{\zeta_n\}$ *has for its set of* a.s. *limit points the interval* $[-1, 1]$ *and* $\limsup \zeta_n = -\liminf \zeta_n = 1$, a.s.

PROOF. Consider the following conditions:

$$\sum (\Lambda(j))^{-\frac{1}{2}} E(|Z_j| I(|Z_j| > \varepsilon\sqrt{\Lambda(j)})) < \infty \quad \forall\ \varepsilon > 0$$

and

$$\sum(\Lambda(j))^{-2} E(Z_j^4 I(|Z_j| \leqslant \delta\sqrt{\Lambda(j)})) < \infty \quad \text{for some } \delta > 0.$$

These together with Proposition 5 and the conditional Lindeberg theorem imply that the law of the iterated logarithm of Heyde and Scott (1973) holds for the martingale $\{W_n, \mathscr{F}_n\}$ and this together with (7.7) imply Theorem 12.

The jth term of the first series $\leqslant I(2 > \varepsilon\Lambda(j)^{\frac{1}{2}})$ which is eventually zero. Thus the first series is a finite sum. It is easy to show that the expectation in the jth term of the second series is bounded above by $32\lambda(j-1)$ and we have already seen that $\sum \lambda(j)/(\Lambda(j))^2 < \infty$.

8. The ultimately deterministic case

We shall now consider the case where $\sum \lambda(n) < \infty$. Observe that the proof of Proposition 4 is still valid. Let $B_n = \{X_{n+1} - X_n = -1\}$.

PROPOSITION 7. $P\{B_n \text{ i.o.}\} = 0$.

PROOF. Clearly $P(B_n \mid \mathscr{F}_n) = \lambda(X_n)$ and hence the assertion will follow from the Borel–Cantelli lemma, see Stout (1974), p. 11, once we show that $\sum_{n \geqslant 1} E\lambda(X_n) < \infty$. If $p(n,j) = P_0(X_n = j)$, then Fubini's theorem allows us to rewrite the last sum as

$$\sum_{j \geqslant 1} \sum_{n \geqslant 1} p(n,j)\,\lambda(j) = \sum_{j \geqslant 1} G_{0j}\,\lambda(j).$$

The Green's functions G_{0j} are evaluated as follows.

Clearly

$$G_{0j} = G_{0,j-1}(1-\lambda(j-1)) + G_{0,j+1}\,\lambda(j+1), \quad G_{00} - 1 = \lambda(1)\,G_{01}$$

and $G_{00} = (1-q_0)^{-1}$ where q_0 is the probability of eventual return to zero. By using the general results quoted in connection with Theorem 1 we see that

$$G_{00} = 1 + \sum_{i \geqslant 1} \rho(k).$$

By working inductively it is easy to see that

$$\Gamma_j = G_{0j}\,\lambda(j) = \sum_{i \geqslant j} \prod_{k=j}^{i} \lambda(k)/(1-\lambda(k).$$

Let $Q = \prod_{k \geqslant 1}(1-\lambda(k)) > 0$. Then $\Gamma_j \leqslant Q^{-1}A(j)$ where $A(j) = \sum_{i \geqslant j} \prod_{k=j}^{i} \lambda(k)$. If $\nu(j) = \max\{\lambda(k), k \geqslant j\}$ we see that $A(j) \leqslant \nu(j)/(1-\nu(j)) \to 0 (j \to \infty)$. In addition, $A(j) = \lambda(j)(1+A(j)) \leqslant (1+\varepsilon)\,\lambda(j)$ for some $\varepsilon > 0$ all sufficiently large j. Thus $\sum \Gamma_j < \infty$ and the proof is complete.

Proposition 7 states that the random variable

$$N = \sup\{n \mid I(B_{n-1}) = 1\},$$

the last time the random walk moved toward the origin, is nondefective. If $D = N - X_n$, which is non-negative, then

$$X_n = n - D \quad (n \geqslant N).$$

Let $t(n,j) = P_0(N = n, X_N = j)$. Since $\{N = n, X_N = j\}$ occurs if and only if the walk steps back from $j+1$ to j during $(n-1, n)$ and then steps to the right thereafter, we obtain

$$t(n,j) = p(n-1, j+1)\,\lambda(j+1) \prod_{i \geqslant j} (1-\lambda(i)).$$

Explicit determination of even the marginal distributions of N, X_N or D, or their generating functions, presents difficulties. We now prove the following more modest result.

THEOREM 13.

(a) $$\lim_{n\to\infty}(n-EX_n) = 2\sum_{j\geq 1}\Gamma_j = ED;$$

(b) $$EN < \infty \quad \textit{if and only if} \sum j\lambda(j) < \infty.$$

PROOF. The first equality of (a) follows from (7.2) and the proof of Proposition 7. Let $Y_n = n - X_n$. Clearly $\{Y_n\}$ is nondecreasing and

$$EY_n = E(Y_n;\ N \leqslant n) + E(Y_n;\ N > n).$$

On the set $\{N \leqslant n\}$, $Y_n = D$ and hence by monotome convergence

$$ED = \lim_{n\to\infty} E(D;\ N \leqslant n) \leqslant \lim_{n\to\infty} EY_n < \infty.$$

It follows from this that

$$E(Y_n;\ N > n) \leqslant E(Y_N;\ N > n) = E(D;\ N > n) \to 0 \quad (n \to \infty),$$

whence the second equality in (a).

We shall prove (b) by showing that $EX_N < \infty$ if and only if $\sum j\lambda(j) < \infty$. Clearly

$$P(X_N = j) = \left[\prod_{i\geqslant j}(1-\lambda(i))\right]\sum_{i\geqslant j+1}\prod_{k=j+1}^{i}\lambda(k)/(1-\lambda(k))$$
$$\geqslant \left[\prod_{i\geqslant j}(1-\lambda(i))\right]\lambda(j+1)/(1-\lambda(j+1)) \sim \lambda(j+1) \quad (j\to\infty),$$

whence the 'only if' part of our assertion.

Now

$$P(X_N = j) \leqslant \sum_{i\geqslant j+1}\prod_{k=j+1}^{i}\lambda(k) \leqslant \lambda(j+1)\sum_{i\geqslant j+1}(\nu(j))^{i-j+1}.$$

The 'if' part of our assertion now follows.

9. Further comments

We might expect that the spirit of the Hines–Thompson model is still preserved if we allow its increments to take values larger than unity but insist that they be uniformly bounded, by M say. Thus for $i = 1, \ldots, M$ we specify the probabilities

$$P(X_{n+1} - X_n = i \mid \mathscr{F}_n) = \alpha(X_n, i), \quad \sum_{i=1}^{M}\alpha(X_n, i) = 1 - \lambda(X_n)$$

and $\alpha(j, M) > 0$ for some j. By suitably restricting the behaviour of these increment distributions we might hope to carry through a development similar to that in the

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previous section. It transpires that there can be a substantial qualitative difference in the behaviour of the extended walk.

To see this consider the special case where $\alpha(X_n, i) = (1-\lambda(X_n))\alpha(i)$ where $\{\alpha(i)\}$ is a distribution on $(1, \ldots, M)$ having mean $a \geqslant 1$ and variance $v \geqslant 0$. It is easily seen that $\{W_n, \mathscr{F}_n\}$ is a martingale where

$$W_n = X_n - an + (a+1)\sum_{m=1}^{n} \lambda(X_{m-1})$$

and if $Z_n = W_n - W_{n-1}$ then

$$E(Z_n^2 \mid \mathscr{F}_{n-1}) = v(1-\lambda(X_{n-1})) + (1+a)^2\lambda(X_{n-1})(1-\lambda(X_{n-1})).$$

Stout's theorem again shows that $W_n/n \xrightarrow{\text{a.s.}} 0$. Moreover by comparison with the case $M = 1$ we see again that $\{X_n\}$ is transient and we conclude that $X_n/n \xrightarrow{\text{a.s.}} a$. However, if $v > 0$ there can be no result corresponding to Theorem 10, the variability in the distribution $\{\alpha(i)\}$ swamps the effect arising from the decreasing likelihood of stepping to the left. Furthermore, with appropriate conditions on the $\lambda(\cdot)$, Brown's central limit theorem yields the classical behaviour

$$(X_n - an)/\sqrt{(vn)} \Rightarrow N(0,1).$$

If $v = 0$ the postive part of the increment is concentrated at $a = M$ and the random walk behaves similarly to the $M = 1$ case. Thus under the conditions of Theorem 10 we now have $(an - X_n)/\Lambda(n) \xrightarrow{\text{a.s.}} 1 + a$ and corresponding to Theorem 11 we have the result

$$\frac{X_n - an + (a+1)\Lambda(n)}{(a+1)\sqrt{\Lambda(n)}} \Rightarrow N(0,1).$$

We now comment briefly on the maximum process $\{V_n\}$ where $V_n = \max_{m \geqslant n} X_m$ and $\{X_n\}$ is as defined in Section 7. In contrast to the methods used in Section 7, it can be shown by using classical renewal theory methods that $\{V_n\}$ possesses the same limit behaviour as $\{X_n\}$. To see this let N_j denote the first passage time from $\{j\}$ to $\{j+1\}$ and $\varphi_j(s)$ its generating function, It follows that

$$\varphi_j(s) = s(1-\lambda(j)) + \lambda(j)s\varphi_{j-1}(s)\varphi_j(s)$$

and this can be used to show that

$$EN_j = 1 + 2\lambda(j)(1+o(1)), \quad \operatorname{var} N_j \sim 4\lambda(j) \quad (j \to \infty).$$

These results can be used in conjunction with Kolmogorov's criterion for convergence of random series and Kronecker's lemma to show that if

$S(n) = \sum_{j=0}^{n} N_j$ then

$$(\Lambda(n))^{-1}(S(n) - n - 2\Lambda(n)) \xrightarrow{\text{a.s.}} 0.$$

A direct argument shows that

$$(\Lambda(n))^{-\frac{1}{2}}(S(n) - n - 2\Lambda(n)) \Rightarrow N(0, 1).$$

By using the relations $S(V_n) > n \geqslant S(V_n - 1)$ and $\{S(n) \leqslant j\} = \{V_j > n\}$ we can obtain restatements of Theorems 10 and 11, respectively, for $\{V_n\}$.

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