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REMARKS ON REGULAR SEQUENCES

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In this note we exhibit a condition under which a sequence of elements $a_1, \cdots, a_n$ in a commutative noetherian local ring A form an A-sequence, and derive a number of corollaries.

Recall that if M is an A-module, then $a_1, \cdots, a_n$ is an M-sequence if

1) a_{i+1} is a nonzerodivisor on $M/(a_1, \cdots, a_i)M$ for $i = 0, \cdots, n-1$, and

2) $M \neq (a_1, \cdots, a_n)M$.

If $a_1, \cdots, a_n$ is an A-sequence and I is the ideal generated by $a_1, \cdots, a_n$, then I^k/I^{k+1} is free over A/I for all $k \geq 1$.

THEOREM 1. *Let I be the ideal generated by $a_1, \cdots, a_n$ and let $U \subset A$ be the set of nonzerodivisors modulo I. Suppose that*

1) *Some minimal prime containing I has height n, and*

2) *$(I^k/I^{k+1})_U$ is free over A_U/I_U for infinitely many values of k.*

Then $a_1, \cdots, a_n$ is a regular sequence.

Before proving the theorem, we will deduce some consequences. We fix the notations $I = (a_1, \cdots, a_n)$, and U.

COROLLARY 1. *If $a_1, \cdots, a_n$ is an A_U-sequence, then it is an A-sequence.*

Proof. Immediate from the Theorem.

COROLLARY 2. *Let $I = (a_1, \cdots, a_n)$ be an ideal of height n. Suppose that A_P is a Cohen-Macaulay ring for all associated primes P of I.*

Then $a_1, \cdots, a_n$ is a regular sequence.

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Proof. Immediate from Corollary 1 and the fact that in a local Cohen-Macaulay ring, any n elements that generate an ideal of height n form a regular sequence.

Remark. As a consequence of Corollary 2, we see that if $(a_1, \cdots, a_n)$ is a radical ideal of height n, then $a_1, \cdots, a_n$ is a regular sequence. This result was observed, for the case in which $(a_1, \cdots, a_n)$ is prime, in [3].

COROLLARY 3. *Let A be locally Cohen-Macaulay except at the maximal ideal. Suppose that $a_1, \cdots, a_n$ generate an ideal of height n, and depth $A/(a_1, \cdots, a_n) \geq 1$. Then A has depth $\geqq n+1$.*

Proof. Since by the hypothesis U contains a nonunit, A_U is Cohen-Macaulay, and $a_1, \cdots, a_n$ generate an ideal of depth n in A_U. Since $a_1, \cdots, a_n$ are in the radical of A_U, they form a regular sequence, and Corollary 1 applies.

Corollary 1 can also be used to strengthen the result of Levin and Vasconcelos [5] that an ideal I of A is generated by a regular sequence if and only if I/I^2 is a free A/I-module and I has finite projective dimension:

COROLLARY 4. *Suppose that I is an ideal of A, and let U be the set of nonzerodivisors modulo I. Suppose that*

1) *The projective dimension of I_U over A_U is finite, and*
2) *I/I^2 is free over A/I.*

Then I is generated by a regular sequence.

Proof. Let $a_1, \cdots, a_n$ be a minimal set of generators of I. By the freeness of I/I^2, $a_1, \cdots, a_n$ will also be a minimal set of generators for I_U over A_U. By the Theorem of Levin and Vasconcelos, I_U is generated by a regular sequence in A_U. It follows that $a_1, \cdots, a_n$ is an A_U-regular sequence, proving the Corollary.

EXAMPLE. Let k be a field, and set $A = k[[x, y]]/(x^2, xy^n)$. Let $a = y$, and set $I = (a)$. Then (x, y) is a minimal prime of I having height 1, and I^k/I^{k+1} is A/I-free for $k < n$, but a is a zerodivisor. (This shows that the word infinitely is necessary in the Theorem.)

One weakness of the Theorem is that it deals only with an ideal generated by the "right" number of elements, and thus gives no informa-

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tion, for example, about ideals in Cohen-Macaulay rings. (The ideal result might have the form: if U is the set of nonzerodivisors modulo an arbitrary ideal I, and if I_U can be generated by a regular sequence, then so can I. Unfortunately, this is false: I_U might, for example be 0!) It is thus interesting to compare our Theorem (and especially Lemma 2, below) with the following result of Cowsik and Nori [2], in which this weakness is partially overcome. (Cowsik and Nori state their result in less generality than the following, but the ideas in their proof suffice.) To state it we use the notation $\mathrm{gr}_I A$ for the graded ring $A/I \oplus I/I^2 \oplus \cdots$.

THEOREM. *Let A be a local Cohen-Macaulay ring, with maximal ideal M and let $I \subset A$ be an ideal. Set*

$$n = \text{Krull dim } \mathrm{gr}_I A/M \,\mathrm{gr}_I A .$$

Suppose that for every minimal prime ideal P of I, I_P can be generated by an A_P-regular sequence of length n. Then I can be generated by an A-regular sequence.

Proof of the Theorem. Our Theorem follows at once from the following lemmas.

LEMMA 1. *If $I = (a_1, \cdots, a_n)$ has some minimal component of height n, then $\mathrm{gr}_I A/M \,\mathrm{gr}_I A$ is isomorphic to a polynomial ring on n generators over A/m.*

LEMMA 2. *In addition to the hypothesis of Lemma 1, suppose that $(I^k/I^{k+1})_U$ is free over (A/I_U) for infinitely many k. Then I is generated by a regular sequence.*

Proof of Lemma 1. It is enough to show that I^k cannot be generated by fewer elements than the kth power of the ideal generated by the variables in the polynomial ring on n generators. Since the number of generators can only decrease on localization, we may first localize at a minimal prime P of I of height n. But I_P is generated by a system of parameters for A_P, and [1, Prop. 11. 20] shows that I_P^k cannot be generated by too few elements.

Proof of Lemma 2. By [4, 15.1.11.], it suffices to show that if $x_1, \cdots, x_n$ are indeterminates, then the epimorphism

$$\varphi: A/I[x_1, \cdots, x_n] \to \mathrm{gr}_I A$$

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sending x_i to the class of a_i in I/I^2 is an isomorphism. Write F_k for the module of forms of degree k in $A/I[x_1, \cdots, x_n]$.

If k is such that $(I^k/I^{k+1})_U$ is free over $(A/I)_U$, and if P is a minimal prime of I of height n, then Lemma 1 shows that $(I^k/I^{k+1})_P$, which is a localization of $(I^k/I^{k+1})_U$, is free over $(A/I)_P$ on the monomials of degree k in $a_1, \cdots, a_n$. Consequently these monomials form a free basis of $(I^k/I^{k+1})_U$ over $(A/I)_U$. Thus the map

$$(\varphi_k)_U : (F_k)_U \to (I^k/I^{k+1})_U$$

is an isomorphism for each of the infinitely many k for which $(I^k/I^{k+1})_U$ is free over $(A/I)_U$. This implies that $\ker(\varphi_k)_U = 0$ for these values of k. Since U consists of nonzerodivisors modulo I, and since $\ker \varphi_k$ is contained in a free A/I module, we see that φ_k is an isomorphism for infinitely many values of k.

However, $\ker \varphi$ is a homogeneous ideal, and if it contains a nonzero form of a given degree, then it contains nonzero forms of all higher degrees; in particular, if $\ker \varphi \neq 0$, then $\ker \varphi_k \neq 0$ for all but finitely many k. This concludes the proof.

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